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1 Energy Demand in the World

In the end, the most fundamental need for humans is the food. What is the maximum average population density permitted on Earth given that we would all eat meat? We can count that every human needs 100 W of power in food, and the animals we farm need about the same amount. The grass for feeding the animals might grow at rate about 0.1 W m^{-2}. The animals grow aout 10 g per day, and the caloric value of the animal meat is about 8.2 MJ kg^{-1}. First thing we might want to determine is the efficiency of energy transfer for animals. If one animal uses 100 W of power, it uses

$$E_a = 100\,\text{W} \times 86400\,\text{s} = 8.64\,\text{MJ}$$

per day. It grows 10 g, and therefore the energy usable by us generated by the animal is

$$W_a = 0.01\,\text{kg} \times 8.2\,\text{MJ kg}^{-1} = 0.082\,\text{MJ}$$

And therefore the efficiency of meat generation is

$$\epsilon_m = \frac{W_a}{E_a} \approx 0.0095 \tag{1}$$

which is rather low. The amount of people that can live on meter squared can then be obtained as

$$\rho = \frac{\epsilon_m P_g}{P_h} \approx 9.5 \times 10^{-6}\,\text{person m}^{-2} = 9.5\,\text{person km}^{-2}$$

where P_g is power generated by grass per area and P_h is power used by per human.
Of course, for vegetable diet, we would substitute the efficiency of meat generation with efficiency of vegetable generation, which is still not 1, as most vegetables do not grow as fast as grass does. However, this efficiency is usually bigger than ϵ_m, hence why the agriculture lead to the creation of cities.
However, food is not the only demand we have. Electricity consumption in the world is on average (with big variance) 0.36 kW and when including heating and other uses of energy, the total energy consumption comes up to 2.5 kW. This number will however keep growing, as more and more regions are developing and demand more power.

2 Classical Electrical Power Generation

2.1 Carnot Cycle

The original and perhaps still most common way of generating electricity is through the thermal power plants, which use expansion of heated gases to output mechanical power, which is then transferred to electrical power.
The amount of useful mechanical work that can be extracted from cyclic heating and cooling of a gas is limited by the Carnot cycle, and the theoretical maximum efficiency is given by

$$\epsilon_c = 1 - \frac{T_c}{T_h}$$

where T_c is the temperature of the cooled gas, which is usually the ambient temperature of the power plant, and T_h is the temperature the gas is heated to. It is clear that hotter gas means greater efficiency.
Real cycles are much less efficient, because the Carnot cycle is a very slow cycle - faster but less effecient cycles are used in reality.

2.2 Turbines

Turbines change the expansive work done by the gas to a rotational mechanical motion. Turbines are not terribly efficient - for a big turbine, the efficiency is about 0.5, for smaller turbines, the efficiency is even lower. However, turbines are fast, and theoretically also store some immediate energy in their angular momentum.
The general feature of a turbine is that the enetering gas is at low velocity and at high pressure, while the gas leaving the turbine is at high velocity but low pressure. This is created by decreasing the radius of the turbine as the gas moves through. For a typical turbine, the radius decreases 10 times and the steam speeds up about 10 times, leading to a 1000 times decrease in the density of the steam.

2.3 Imporvements

As a lot of energy is lost during the process of burning and turning the turbine, some improvements are commonly practiced to use some of the excess energy.
For example, instead of a cooling tower, the power plant can use the heating grid to dispose of hot steam, while also heating the homes. This is called cogeneration (generation of both heat and electricity).
Other possible improvement is the combined cycle - instead of expanding heated gas, a jet engine is used to pressure the gas while itself turning a turbine. We then have two turbines - one inside the jet engine and second one using the gas from the jet engine, which slightly increases the efficiency.

2.4 Alternator

Alternator transfers a movement of a magnet (or electromagnet) to the alternating electric current, using the electromagnetic induction. Usually, the alternator is three phase, meaning that there are three inductors symmetrically around a rotating magnet. On each inductor, alternating voltage (and current) is created with relative phase shift of $\frac{2\pi}{3}$. Importantly, the sum of all these voltages together is

$$V_1 + V_2 + V_3 = V_0 \left[\cos(\omega t) + \cos\left(\omega t + \frac{2\pi}{3}\right) + \cos\left(\omega t + \frac{4\pi}{3}\right) \right] =$$

$$= V_0 \left[\cos(\omega t) + \cos(\omega t)\cos\left(\frac{2\pi}{3}\right) - \sin(\omega t)\sin\left(\frac{2\pi}{3}\right) + \cos(\omega t)\cos\left(\frac{4\pi}{3}\right) - \sin(\omega t)\sin\left(\frac{4\pi}{3}\right) \right] =$$

$$= V_0 \left[\cos(\omega t) - \frac{1}{2}\cos(\omega t) - \frac{\sqrt{3}}{2}\sin(\omega t) - \frac{1}{2}\cos(\omega t) + \frac{\sqrt{3}}{2}\sin(\omega t) \right] = 0$$

Therefore, all three voltages always sum to zero. This is usually used to create the so called null wire, which provides reference voltage and also a control that all three phases work properly. The other ends of the phases are left unconnected and these then supply the power to the grid. But in order to be able to do this, they must exactly match the absolute phase shift of the grid. Usually, this phase shift is specific for every grid and has to be matched when a new power station is build.

2.5 Electrical Grid

The electrical power is transported mainly in form of alternating current, because it is convenient and slightly safer than direct current. However, this form of transport creates inductive losses in dense or conductive media - the polarization of the dielectric or induced currents in material surrounding wire cause energy losses. In air, this is not very important but in soil or in water, these losses are rather big. Therefore, for transport through water or soil, direct current is usually used (there is also smaller security risk involved).
Other type of losses are the losses due to Ohmic heating. For a fixed power transport (which is usually the case), the voltage and the current follow

$$P = VI$$

where P is the power, V is the voltage and I is the current. Along the transporting wire, the voltage drops from V to some $V - V_O$, where the voltage drop is given by Ohm's law

$$V_O = RI$$

where R is the resistance of the wire. Hence the power received on the end of the wire is

$$P' = (V - V_0)I = P - RI^2 = P - R\frac{P^2}{V^2}$$

The power received is maximised for minimal resistance R and maximum voltage V. This is why the voltage is transformed to higher values.
To present some values, the voltage from a power station is usually around 25 kV. This is then transformed to 400 kV, which is the very high voltage mode. This is then gradually transformed down to 132 kV, 33 kV (sometimes used by heavy industry), 11 kV (used by light industry) and finally to 400 V or 230 V, which is used by households, but usually only one phase.
For design of the wires, it is worth refreshing the relation bewteen geometry and material of a wire and its resistance. The resistance R of the wire is

$$R = \rho\frac{l}{A}$$

where ρ is the resistivity, l the length and A the area of the wire.

2.6 Transformer

Transformer is the object that transforms the voltages to higher or lower values. It consists of two inductors on the same core, which ensures that the magnetic flux through these inductors is always the same. Then, Faraday's law of induction leads to

$$V_1 = -\frac{d\Phi}{dt} = -\frac{d(N_1 BS)}{dt} = -N_1 \frac{d(BS)}{dt}$$

where V_1 is voltage induced on inductor 1, N_1 is the number of windings on inductor 1, and B is the magnetic induction in the inductor 1 and S is the are of each winding. Since the core is common to both inductors, B and S is the same for both inductors as well. Therefore

$$-\frac{d(BS)}{dt} = \frac{V_1}{N_1} = \frac{V_2}{N_2}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

Therefore, the voltage induced on inductor 1 is in direct proportion to the ratio of number of windings on inductor 1 to number of windings on inductor 2.

3 Nuclear fission

3.1 Nuclear reactions

The nuclei of all elements are held together by the strong force, which interacts with both protons and neutrons. Since the protons also repel each other by electrostatic force, the nucleus is held together effectively by neutrons.

The repulsion of protons is quite strong and more massive nuclei have generally higher neutron/proton ratio, and hence also more isotopes of varying stability.

What happens when the nucleus is instable? It decays into different nucleus or nuclei. If these products are positively charged, they are then propelled by the electrostatic force, producing kinetic energy. Therefore, if we can break the nucleus and use the kinetic energy of the fragments to heat some coolant, we can connect this heated coolant to turbine and produce electrical power.

But, some of the fragments of the nuclear decays can, upon collision, cause other nuclei to decay as well, causing a chain reaction. Since there are big energies involved (about 200 MeV per reaction, compare to 10 eV per chemical reaction), we must carefully study the processes involved to prevent any uncontrolled reaction.

On Earth, the uranium has several isotopes which are not stable and can decay upon neutron collision. These are most commonly used as fuel in nuclear power plants.

The most common isotope is ${}^{238}_{92}U$. One of its first discovered decays was

$${}^{238}_{92}U + {}^{1}_{0}n \rightarrow {}^{239}_{92}U \xrightarrow{23min} {}^{239}_{93}Np + e^- + \bar{\nu}_e \xrightarrow{2.5days} {}^{239}_{94}Pu + e^- + \bar{\nu}_e$$

where times over the arrows are decay half-lives.

The new plutonium element is very stable (half-live 24 000 years).

However, the fission is only one process that can occure when neutron hits the nucleus. These are some of the processes that can occur

- nucleus elastically scatters the neutron, without changing its structure
- nucleus absorbs and reemits the neutron and some photon, which is an inelastic scattering
- nucleus absorbs the neutron and only emits smalll particles such as electrons or neutrons (does not really decay)
- nucleus splits into nuclei of similar mass about the half of original nucleus - nuclear fission

Each of these processes has certain probability of occuring. This probability is usually incorporated in the form of effective cross-sections.

There is certain chance that neutron will hit the nucleus, which can be approximated by

$$P = N\frac{\sigma}{A}$$

where N is the number of neutrons incident upon part of a plane with area A, and σ is the real cross-section of the nucleus. However, multiple processes can occur independently after the neutron collides. For example, let probability of fission of the nucleus after being hit with a neutron be p_f. The overall probability of nucleus undergoing fission is then product of probability that the neutron hits the nucleus and that the nucleus undergoes fission, i.e.

$$P_f = P \times p_f = N\frac{\sigma}{A}p_f = N\frac{\sigma_f}{A}$$

where $\sigma_f = \sigma \times p_f$ is the effective cross-section. But, many other processes can take place, which leads to the calculation being much more complicated and perhaps even σ_f greater than original σ. However, the final form is still similar to the one above.

This effective cross-section can still depend on several variables, including the energy of the incident neutron.

3.1.1 Uranium reactions

The most common isotope ${}^{238}_{92}U$ usually decays into other nuclei without releasing a neutron, which means that although we can use the energy generated by the fission, the reaction is not sustainable on its own. On the other hand, during fission of ${}^{235}_{92}U$, on average, 2.5 neutrons is released per each reaction, with mean energy of neutron about 2 MeV.

$${}^{235}_{92}U + {}^{1}_{0}n \to X + Y + 2.5n(2\,\mathrm{MeV})$$

This means that ${}^{235}U$ can drive a chain reaction, while ${}^{238}U$ cannot. It should be noted that at high energies of neutrons, even ${}^{238}U$ starts producing neutrons at rate about 2.5 neutrons per reaction with energy about 2 MeV, but the cross-section of the next reaction with these produced neutrons is too small - on average, 0.75 of these produced neutrons have energy high enough to react with the nucleus, but only in about a 0.25 of the cases the reaction causes another fission. Hence, the net total number of neutrons produced per fission and consecutive stimulated fission is $0.75 \times 0.25 \times 2.5 \approx 0.47 < 1$, hence even at these energies, ${}^{238}U$ still cannot sustain the reaction.

This is due to the profile of σ_f for the two uranium isotopes. While for ${}^{238}U$, σ_f grows for increasing neutron energy and starts to be significant only once the neutron energy reaches around MeV, for ${}^{235}U$, σ_f decreases with increasing energy, but is still significant for energies around 1 MeV and very significant for energies around 0.25 eV, which is the energy of thermal neutrons at room temperature. At this energy, $\sigma_f \approx 10^{-21}\mathrm{cm}^2$, which is about 200 times greater than the real cross-section of the uranium nucleus. The reason behind this is essentially due to quantum nature of the reaction - the slower neutron has greater wavelength and therefore has greater probability of at least partially hitting the nucleus, which is enough to trigger the reaction.

Interestingly, plutonium 239 follows similar path, and therefore if ${}^{238}U$ absorbs the neutron and decays to plutonium, this new plutonium can then act as ${}^{235}U$ and drive chain reaction. In fact, over the reactor lifetime, about one third of its power output is created just from the reactions of this generated plutonium.

Finally, it should be noted that between energies from 1 eV to about 10 keV, the profile is σ_f for ${}^{235}U$ is rather chaotic, but still significant.

3.2 Thermal Reactor design

In nature, the concentration of ${}^{235}U$ is about 0.7 %, which is not enough to drive the self-sustained reaction - 2-5 % concentration is needed for that. The concentration is therefore increased in a process called ${}^{235}U$ enrichment. Usually, this means that the uranium oxide U_3O_8, which is mined, is chemically converted to uranium fluoride (UF_6), which is gas and is then centrifuged.

The lighter ${}^{235}U$ is then in the middle of the centrifuge. This has certain security risks, as the enriched uranium can be used to make a nuclear bomb, and therefore uranium centrifuges are strictly regulated.

Other possibilities for enrichment are with mass spectrometer or simply with gaseous diffusion in gravitational field. But, centrifuge is the fastest/cheapest. Once this process is done, uranium is converted to metallic form or to uranium dioxide pellets. These are then used in the reactor.

However, the enriched uranium on its own is usually not dangerous - the neutrons generated by decays are too fast to react with ${}^{235}U$ and just ${}^{238}U$ does not support reaction. To create a steady reaction, neutrons have to be slowed down somehow. This is done using a moderator.

3.2.1 Moderators

Moderator is a material that inelastically scatters the neutrons and thus slows them down. Usually, the material has a light nucleus, as these do well scatter the neutrons inelastically. Most common moderators are water and graphite.

However, water also absorbs neutrons, as the hydrogen is converted to deuterium. If, on the other hand, we use heavy water, which already has deuterium, the neutrons are not lost and the neutrons are moderated well. This means that the uranium can be less enriched, meaning lesser risk of misusing the uranium to build nuclear weapon just from centrifuged uranium, although there are other possibilities which are not covered by this.

The main disadvantage of water is that when evaporated, it looses its moderating properties. Therefore, at high temperatures, it cannot be used effectively as a moderator unless pressurized.

Graphite is worse at slowing down electrons, but remains solid to higher temperatures, which is useful in certain types of reactors.

When the uranium is moderated, the reaction starts to release big amount of energy, which needs to be taken from the fuel using a coolant.

3.2.2 Coolants

Most common type of coolant is water, but it again looses the cooling properties if it evaporates. And when the reactor is moderated by graphite, this means that the reaction is not cooled but still moderated, which leads to big problems. If the reactor uses the same water as moderator and coolant, the cooling is lost along with moderation, and the reaction stops.

Generally, for carbon moderator, it is better to use carbon dioxide as a cooler, because it does not undergo further phase transitions at high temperatures, until it disociates at high temperatures, which can again lead to reaction without cooling.

This can be prevented by using helium instead of carbon dioxide, but helium is much more expensive.

The heated coolant is then used to either directly turn turbines or heat other gas/water that turns the turbines. The whole process is therefore again dominated by Carnot efficiency.

3.2.3 Control

The nature of the chain reaction is such that when more neutrons are suddenly generated, more reactions occur and even more neutrons are generated. To prevent this escalation, some form of control has to be present in the reactor.

First control mechanism is from the reaction itself - some fission fragments (^{87}Kr for example) do not produce neutrons for next reaction immediately, but with some delay (about 10 s in this case). This leads to much slower increase in number of neutrons after some random fluctuation, which helps with reaction control.

Main direct control mechanism is then the use of control rods, usually made from Cadmium or Boron, which can absorb the neutrons without reemission. These control rods are then inserted or retracted from the reactor as needed.

Similarly to control rods, some fission fragments (Xenon for example) can absorb the neutrons without reemission. These fragments tend to build up in the reactor over time and are called reactor poison. As a consequence of this, older reactor fuel needs less control rod area than new fuel.

Ideally, the control rods are manipulated so that the reaction does not increase the overall number of neutrons in the reactor - all extra neutrons are absorbed in the rods. This state is called the criticallity.

3.2.4 After the runtime

After the ^{235}U is depleted in the fuel (returns to natural level of 0.7 %) the reaction is stopped and the fuel is taken out. The composition of the fuel at this point is - 94 % ^{238}U, 0.7 % ^{235}U, 0.9 % plutonium, which is a potential security risk, as it can be chemically separated, 0.6 % ^{234}U and transuranium elements, which generally have quite long half live (around 10 years) which are problematic, and hot fission fragments, which usually have small decay half live (around the order of days), but as they get very hot, this waste has to be cooled down actively for some time before being deposited into a long term storage, where the rest of radioactive material with longer half lives can slowly decay.

3.3 Thermal Reactor examples

Now follows a few examples of typical nuclear thermal reactors

3.3.1 Pressurized water reactor

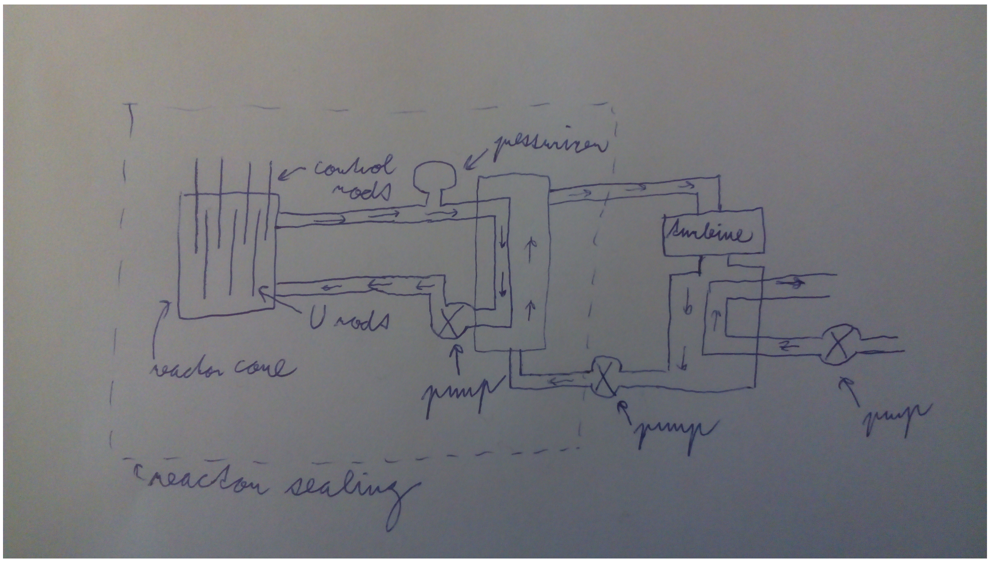

Figure 1: Pressurised water reactor has 3 separate water circuits. The first one both moderates and cools the reactor, and is kept as liquid by the pressurizer. Second circuit heats off this pressurized water and evaporates to steam, which powers a turbine. Then it is cooled down, by the last water circuit, and pumped back.

Pressurized water reactors are quite safe because the water that moderates the reactor is controled by the pressurizer - steam creation in the core means loss of moderation. Also, the radioactively tainted water is kept within the reactor. On the other hand, only the secondary heated water runs the turbine, which always leads to some heat losses.

3.3.2 Boiling water reactor

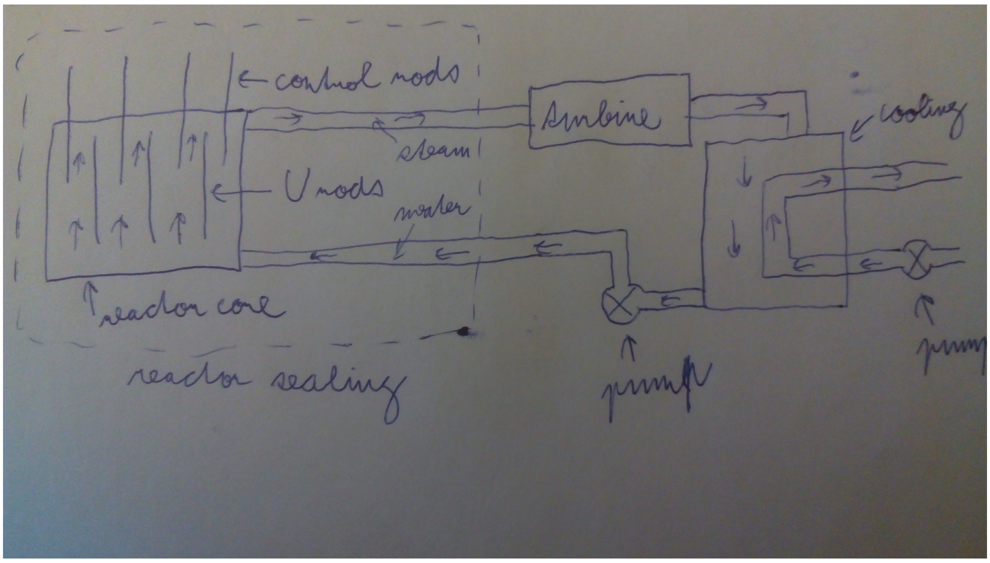

Figure 2: Boiling water reactor uses only two water circuits - first lets the water turn to steam inside the reactor, which then directly powers the turbine. The steam is then cooled down by the second circuit, condensing back to water, which is pumped back to the reactor. The moderator is still water.

Boiling water reactor does not the intermediate water circuit, which leads to radioactively tainted turbine and water outside the reactor, but also to smaller heat losses. The moderation can be also controlled directly by the amount of water that is pumped into the reactor.

3.3.3 Advanced gas reactor

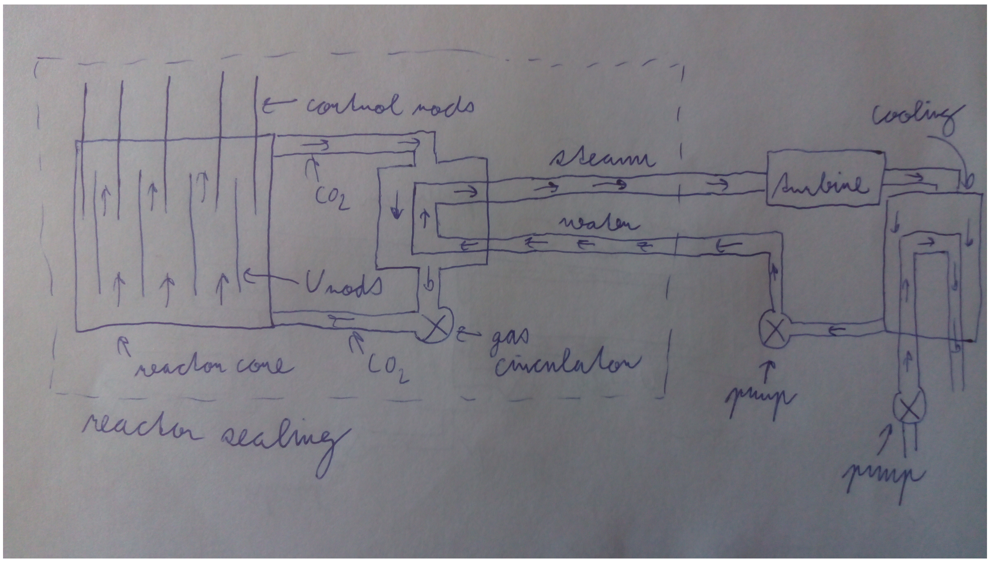

Figure 3: Advanced gas reactor uses graphite moderator in the reactor core and carbon dioxide as a cooling gas. This gas then heats the water that turns into steam and runs the turbine.

As advanced gas reactor uses carbon dioxide as coolant, it can go to higher temperatures and thus have better Carnot efficiency. Also, the water is not radioactively stained. On the other hand, if the coolant for some reason malfunctions, the reaction keeps going, as the graphite does not dissapear.

3.4 Other Reactors

Now, I include a few examples of other types of reactors.

3.4.1 Breeder reactors

Breeder reactors do not use ^{235}U as primary fuel. Instead, they somehow generate a new element in the reaction that then provides most of the power production. I include two types - **Fast breeder reactor** and **Thorium thermal breeder reactor**. Fast breeder reactor uses highly enriched (20 %) ^{235}U core around a ^{238}U shell, which is bombared by non-moderated high energy neutrons, which leads to production of plutonium, which then runs the reactor. However, because the core is so enriched, the reactor gets incredibely hot, and is usually cooled down by liquid metals - lithium, sodium or lead. Lithium and sodium are very dangerous because when in contact with either water or air, they tend to explode. Lead is better and also well absorbs the neutrons that escape the shell.

Thorium thermal breeder reactor uses moderated neutrons to convert thorium 232 to uranium 233, which can be used as a primary source of neutrons in classical reactors as well. The reactions are

$$^{232}_{90}Th + ^{1}_{0}n \rightarrow ^{233}_{90}Th \rightarrow ^{233}_{91}Pa + e^- + \bar{\nu}_e \rightarrow ^{233}_{92}U + e^- + \bar{\nu}_e$$

As thorium is more abundant than uranium, this can solve the problem with shortage of ^{235}U. Also, there is no plutonium created, and hence much smaller security risk.

The thorium is usually present as ThF liquid inside the reactor.

3.4.2 Very high temperature gas reactor

Experimentally, to further increase Carnot efficiency, carbon moderated and helium cooled reactors have been constructed. These can go even higher than 1200 ^{o}C, which is a point when water dissociates to hydrogen and oxygen. These can be used to run one high temperature turbine, and their recombination reaction can run another turbine.

3.4.3 Pebble bed reactor

Instead of uranium rods, this reactor uses small pebbles of uranium that are continuously added and discarded from the reactor, thus removing the need to completly stop the reactor once the fuel is depleted.

3.5 Nuclear Reactor Accidents

3.5.1 Three mile island

Three mile island was a PWR that had a core meltdown in 1979. The sequence of events was as follows

1. The water pump on the secondary water circuit failed, thus the pressurized water in the reactor was not cooled anymore
2. Backup water cooling system was activated (simple circuit directly in the reactor), but the valves were for some reason closed and the pressurized water was therefore still not cooled down.
3. Pressurized water started to overheat - risk of melting down the reactor. To counteract this, control rods were completly inserted. This caused a raise of pressure in the reactor which had to be releived - pressure releive valve was opened, but it did not close after the pressure relief - the water kept escaping.
4. All pressurized water escapes the reactor - fission stops, as there is no moderation. But, because there were already fission fragments present, they continue to decay (for example via β-decays).
5. Fission fragments are not cooled by the pressurized water - reactor core heats further.
6. Reactor core melts.
7. Whole reactor chamber is sealed off and left as is - still standing there up to this day.

As the whole reactor chamber remained sealed, nobody died in the end.

3.5.2 Chernobyl

Chernobyl was a carbon moderated water cooled reactor. This design on its own is dangerous combination. Further, the control rods of the reactor had beginning and the end coated with graphite, possibly so that the moderator is all around the reactor core. Again, the sequence of events is as follows

1. Test for power self-sufficiency scheduled (25.4.1986). The test was supposed to try whether the inertia of turbines could power the water pumps that cooled the reactor in case of electric grid blackout.
2. Reactor power turned down by rods
3. At low power, even less stable isotopes of xenon and other reactor poison remain in the reactor, absorbing neutrons and thus turning the power down even more. These isotopes are however destroyed when the reactor is at high power and release the stored neutrons back.
4. Test post-poned, but the reaction power is kept down.
5. As the reactor poison builds up, control rods are gradually removed until they are completly removed from the reactor.
6. After a new shift comes in, the experiment starts - the water pressure from the pump decreases and the water inside the reactor starts to turn into a steam.
7. Steam does not cool the reactor effectively, but the reactor is still moderated by graphite - situation with positive void coefficient occurs. The more steam is generated in the reactor, the hotter it gets, which leads to more steam generation etc.
8. To fight this, control rods are quickly presented back to reactor
9. But, the rods start with graphite, which increases the reaction rate even more at first, which leads to control rods melting before they can absorb enough neutrons
10. With melted control rods, the reactor keeps heating up until the pressure of the steam breaches the reactor wall.
11. Oxygen from the atmosphere enters the reactor and reaches extremely hot graphite - pressurized vessel explodes.

After the explosion of the pressurized vessel, the radioactive carbon moderator molecules escape to atmosphere, causing a massive radiation doze to everybody close and over the course of next decade to a whole world in fact.
The catastrophe was first kept secret, but increased radiation doses measured in Europe lead to its discovery. It is worth noting that the sequence of last 6 events took about one minute.

4 Renewable Energy Sources

4.1 Solar Power

4.1.1 Solar constant

The Sun is well approximated as a ideal black body with temperature $T_s \approx 6000K$. This means that the intensity of the radiation at the surface of the Sun is

$$I = \sigma T_s^4$$

where $\sigma \approx 5.67 \times 10^{-8}$ W K^{-4}m^{-2}.
The light than radially travels and the intensity drops as inverse square. For distance d away from the Sun (R_s is the radius of the sun)

$$I(d) = \frac{R_s^2}{d^2} I(R_s) = \frac{R_s^2}{d^2} \sigma T_s^4$$

The Earth is at distance $d = 1$ AU $\approx 1.5 \times 10^{11}$ m, so the intensity at Earth I_e is ($R_s \approx 7 \times 10^8$)

$$I_e \approx 1600 \text{ W m}^{-2}$$

In reality, $I_e \approx 1360$ W m^{-2}, which is also called the Solar constant.

4.1.2 Non-focused Absorber

The atmosphere absorbs quite substantial amount of this intensity, leading to surface intensity from direct sunlight

$$I_{se} \approx \frac{I_e}{2}$$

Suppose we want to create an absorber that is simply directed towards the Sun and absorbs this radiation, heats up and then drives a Carnot engine. At the equilibrium temperature of the absorber T, the power radiated by the absorber is equal to the power incident on the absorber. If we assume that the absorber is ideal blackbody (best case for heating)

$$P_r = \sigma T^4 S = I_{se} S_i$$

where S is the total surface of the absorber and S_i is the surface of the absorber upon which the light is incident. Generally, $S_i < S$. Let $\alpha = \frac{S_i}{S} < 1$. Then

$$\sigma T^4 = \frac{I_e}{2}\alpha = \frac{\alpha}{2}\frac{R_s^2}{d^2}\sigma T_s^4$$

Hence

$$T^4 = \frac{\alpha}{2}\frac{R_s^2}{d^2} T_s^4$$

The top Carnot efficiency (assuming that only small heat is required to run the cycle, thus keeping the T the biggest possible) is

$$\eta = 1 - \frac{T_a}{T}$$

where $T_a \approx 293$ K is the ambient temperature. For given numbers, assuming $\alpha = 0.5$ (rectangle facing the Sun), $T \approx 278$ K - less than ambient temperature. Clearly, such absorber will not work.

4.1.3 Focused absorber

Now, assume instead that we use circular are of radius r to collect the sunlight and focus it to some plane. The light comes at an angle up to $\gamma \approx \frac{R_s}{d}$ (from the edge of Solar disc on the sky). But, this angle is also present between image and focal distance, i. e.

$$\gamma \approx \frac{R'}{f} = \frac{2R'}{r}$$

where R' is the radius of the image of the Sun projected by the mirror onto the focal plane and $f = \frac{r}{2}$ is the focal length of the mirror.

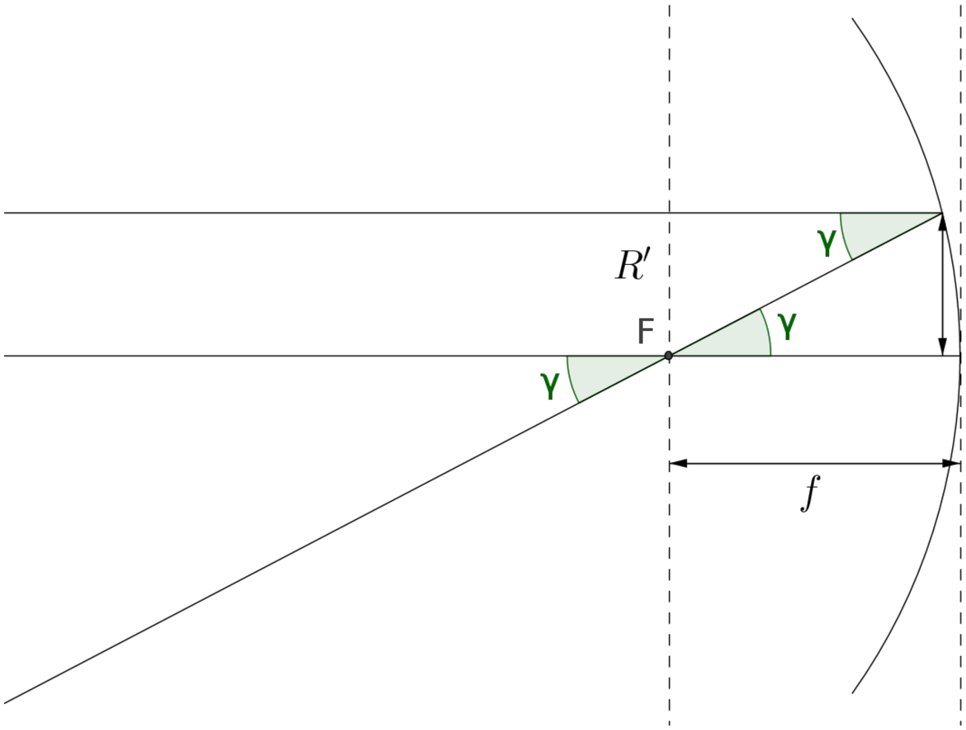

Figure 4: Rays from the Sun are focused by parabolic/spherical mirror. The focal distance $f = \frac{r}{2}$, where r is the radius of the mirror, and R' is the projected size of the Sun on the focal plane.

Therefore

$$R' = \frac{R_s}{2d} r$$

Of course, only the power incident on the mirror can be reflected to the focal plane. Also, some of the light incident on the mirror is blocked by the absorber, which is in the way of light. If we assume that the absorber blocks only by the surface area it also absorbs with, which is αS, then by conservation of energy, the power incident on the image of the Sun in the focal plane, P_i

$$P_i = I_{se}(\pi r^2 - \alpha S)$$

The area of the image is

$$A_i = \pi (R')^2 = \pi r^2 \frac{R_s^2}{4d^2}$$

Hence the intensity of the light incident on the image is

$$I_i = \frac{P_i}{A_i} = \frac{4d^2}{R_s^2} I_{se} \left(1 - \frac{\alpha S}{\pi r^2}\right) = \frac{2d^2}{R_s^2} \frac{R_s^2}{d^2} \sigma T_s^4 \left(1 - \frac{\alpha S}{\pi r^2}\right) = 2\sigma T_s^4 \left(1 - \frac{\alpha S}{\pi r^2}\right)$$

The absorber than absorbs power $P_a = \alpha S I_i$. This absorbed power is partially radiated away and remaining power is used to run an (ideally) Carnot cycle. Then, the energy balance is (Q_h is the heat transferred to Carnot cycle and P_r is the power radiated by the absorber)

$$P_a = P_r + Q_h$$

The work done by Carnot cycle is

$$W = \epsilon Q_h = \left(1 - \frac{T_a}{T}\right) Q_h$$

where T_a is ambient temperature and T is the temperature of the absorber.
The total efficiency of the solar absorber is

$$\epsilon_s = \frac{W}{P_a} = \frac{\left(1 - \frac{T_a}{T}\right) Q_h}{P_a} = \frac{\left(1 - \frac{T_a}{T}\right)(P_a - P_r)}{P_a} = \left(1 - \frac{T_a}{T}\right)\left(1 - \frac{P_r}{P_a}\right) =$$

$$= \left(1-\frac{T_a}{T}\right)\left(1-\frac{\sigma T^4 S}{2\sigma T_s^4 \alpha S\left(1-\frac{\alpha S}{\pi r^2}\right)}\right) = \left(1-\frac{T_a}{T}\right)\left(1-\frac{T^4}{2\alpha T_s^4\left(1-\frac{\alpha S}{\pi r^2}\right)}\right)$$

Taking the derivative with respect to S

$$\frac{\partial \epsilon_s}{\partial S} = \left(1-\frac{T_a}{T}\right)\frac{T^4}{2\alpha T_s^4\left(1-\frac{\alpha S}{\pi r^2}\right)^2}\left(-\frac{\alpha}{\pi r^2}\right)$$

We can see that as S increases, overall efficiency ϵ decreases (there is no maximum efficiency). Therefore, the best idea is to make the absorber negligabely small with respect to the size of the mirror. Then, the efficiency becomes

$$\epsilon_s = \left(1-\frac{T_a}{T}\right)\left(1-\frac{T^4}{2\alpha T_s^4}\right)$$

Assuming $\alpha = \frac{1}{2}$ (for example for any planar absorber)

$$\epsilon_s = \left(1-\frac{T_a}{T}\right)\left(1-\frac{T^4}{T_s^4}\right) \tag{2}$$

The maximum of this is not analytically trivial. But, for example for $T = 1000K$, $\epsilon_s = 0.73$, which is sufficient for energy generation.

Other characteristic we need for solar power is the energy generation per unit area. The total power generated is

$$P = \epsilon_s P_a$$

but, here P_a depends on S, which makes things rather complicated. If we assume that nearly all the power incident on the mirror is in fact incident on the image as well (image size is small enough, which it usually is, as $R' = \frac{R_s r}{2d}$, where $d = 1AU$). Then

$$P = \epsilon_s P_a = \epsilon_s I_{se} \pi r^2$$

So, the power generation per area is (for $T = 1000$ K)

$$P_s = \frac{P}{\pi r^2} = \epsilon_s I_{se} \approx 500\text{W m}^{-2}$$

Furthermore, the power can be generated only during the day. This means that the average power is (in an very ideallized model)

$$\bar{P}_s \approx \frac{1}{2} P_s = 250\text{W m}^{-2}$$

This is very high number, in reality, we have efficiency of around 20W m^{-2}, because our theoretical calculation was very crude, and also because the technology is still in development.

4.1.4 Photovoltaics

Photovoltaics work on very different principle. They are not much discussed here, however it should be noted that they are much less efficient than the thermal collectors described above, but they do work even in difused sunlight, making the power output from them somewhat more stable.

4.2 Hydroelectric Power

4.2.1 Classical Hydroplants

Classical hydroplants use dams to elevate the water incoming from a river to create a pressure gradient that then powers a turbine at the bottom of the dam. We can easily determine the power generation by such a plant as follows.

The amount of water flowing into the dam per unit time is Q. If the plant is in steady state, the same amount of water Q must leave the dam as well, and the height of the water remains the same.

The pressure at the bottom of the dam is $p = \rho g h$, where ρ is the density of the water, h is the height of the water in the dam and g is acceleration due to gravity.

Due to Bernoulli's equation, the speed of the water entering the pipe leading to the turbine generating the electricity is given by

$$\frac{1}{2}\rho v^2 = \rho g h$$

$$v^2 = 2gh$$

The water at this speed is then slowed to speed v_0 by the turbine and then leaves the dam. Therefore, the change in energy of a small volume of water with mass dm (or ρdV) in this process is

$$dE = \frac{dm}{2}\left(v_0^2 - v^2\right)$$

Dividing both sides by dt

$$\frac{dE}{dt} = \frac{1}{2}\frac{dm}{dt}\left(v_0^2 - v^2\right) = \frac{1}{2}\rho\frac{dV}{dt}\left(v_0^2 - v^2\right) = \frac{1}{2}\rho Q\left(v_0^2 - 2gh\right)$$

Remembering that the energy extracted from the system is $-dE$, the power extracted from the hydroplant is

$$P = -\frac{dE}{dt} = Q\rho g\left(h - \frac{v_0^2}{2g}\right)$$

with $\frac{v_0^2}{2g}$ being an effective drop in the height of the water in the dam.
Hydroelectric power in this form is useful because it has reasonably stable and adjustable power output, which can even be reversed (plant then stores energy in form of potential energy of the water in the dam). The main two problems are the area cost and the fact that as the river carries small dust particles etc. the tube at the bottom of the dam gets eventually silted.
In order to try to determine area effectivity, two parameters need to be determined - h and Q. h is mainly given by the profile of the place where the plant is build - if there is sufficient height gradient, plant can be very tall without taking to much space physically. For start, assume that we can achieve a 100 m height difference. To estimate Q, we use approximate annual rainfall. In UK, this is about 0.8 m^3 per m^2 per year. This is equal to

$$Q_a = 2.5 \times 10^{-8}\ \mathrm{m\ s^{-1}}$$

(this is flow per m^2). So, the stable average power from the plant per unit area should be

$$P_h = Q_a\rho g\left(h - \frac{v_0^2}{2g}\right)$$

Assuming that the exit speed term is negligable compared to 100 m,

$$P_h = 0.025\ \mathrm{W\ m^{-2}}$$

real efficiencies can go up to 0.07 W m^{-2}, as the flow can be bigger at places with higher than average rainfall. However, in both cases this is much smaller power per area efficiency than for solar power.

4.2.2 Tidal Hydroplants

The idea behind tidal plants is to capture a water at high tide and then wait for the low tide to release the water, whilst generating electricity.
This time, the height of the water surface is not fixed, but changes as the water flows from the barrier. Then, the total energy generated from tide of height h is (neglecting the reduction due to exit speed)

$$E = \int_h^0 -dE = \int_0^h \rho dVhg = \int_0^h \rho Shgdh = \frac{1}{2}\rho Sgh^2$$

This energy is gained for a period of 12 h (two low tides and two high tides in one day - one high tide facing the moon and the other at the opposite point of the Earth). Therefore, the average power generated is

$$P = \frac{E}{\frac{1}{2}T_d} = \frac{\rho Sgh^2}{T_d}$$

where $T_d \approx 86400\,\mathrm{s}$ is the time of a day. The height of the tide in the UK is approximately $h \approx 6\,\mathrm{m}$. The surface power efficiency is then

$$P_t = \frac{P}{S} = \frac{\rho gh^2}{T_d} \approx 8\ \mathrm{W\ m^{-2}}$$

The main problem is the technology by which to quickly drop all the water in the reservoir back to ocean at low tide - otherwise the tide starts to rise again, meaning smaller height difference and energy generation. Also, this source is clearly not available to all countries - a coast is needed.

4.3 Wind Power

Wind turbines use the kinetic energy of the wind to create electric current. Again, lets assume that velocity directly before the turbine is v and after the turbine v_2. Furhtemore, the wind decelerates a little bit even before the turbine. Therefore, lets also say that v_1 is the speed of the wind far upstream from the turbine. This time, the pressure before and after the turbine passes is the same, and the height approximately as well and the density also remains constant. But, the continuity of the air must apply, and therefore the air must expand. To better show this, consider an air going through area A (the turbine) at speed v. The amount of air passing per some small time dt is $dm = \rho A v dt$. same amount of air must pass through following area A_2 (behind the turbine) so that

$$dm = \rho A_2 v_2 dt$$

Hence, we have

$$Av = A_2 v_2$$

Similarly

$$Av = A_1 v_1$$

Now, we need to find a kinetic energy of a wind going through area A. This is simply (considering small time dt in which only small amount dm of air passes through the area A)

$$dE = \frac{1}{2} dm v^2 = \frac{1}{2} \rho A v dt v^2$$

And therefore the power of the wind is

$$P = \frac{1}{2} \rho A v^3$$

The power given to the turbine then is

$$P_{turbine} = P_{before} - P_{after} = \frac{1}{2} \rho (A_1 v_1^3 - A_2 v_2^3)$$

Using $A_1 v_1 = Av = A_2 v_2$

$$P_{turbine} = \frac{1}{2} \rho A v (v_1^2 - v_2^2)$$

Now, we can make an assumption about $v = \frac{1}{2}(v_1 + v_2)$ (the average value between the two velocities - for slowly changing speed, this should be valid). Then

$$P_{turbine} = \frac{1}{2} \rho v_1^3 \frac{1}{2} \left(1 + \frac{v_2}{v_1}\right) \left(1 - \frac{v_2^2}{v_1^2}\right) = \frac{1}{2} \rho A v_1^3 C_p$$

$C_p = \frac{1}{2} \left(1 + \frac{v_2}{v_1}\right) \left(1 - \frac{v_2^2}{v_1^2}\right)$ is called the coefficient of performance, and is subject to the turbine design. Interestingly, the functional form of C_p, which is $(1 + x)(1 - x^2)$ has a maximum. Taking a derivative

$$\frac{d}{dx}((1 + x)(1 - x^2)) = 1 - x^2 + (1 + x)(-2x) = 0$$

$$1 - x^2 - 2x - 2x^2 = 0$$

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0$$

Therefore, the only valid critical point (and also a maximum) of $(1 + x)(1 - x^2)$ is at $x = \frac{1}{3}$, which corresponds to $\frac{v_2}{v_1} = \frac{1}{3}$. This means that the theoretical maximum of C_p is

$$C_p \leq \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{9}\right) = \frac{2}{3} \times \frac{8}{9} = \frac{16}{27} \approx 0.59$$

Real world turbine designs are surprisingly close to this value, about 0.4 - 0.5.
However, this calculation is done for a constant value of the wind speed. In reality, the wind speed changes through the time of the day, but also throughout the year. This has to be taken into account both in calculating the average power output and even directly in the design of the turbines. Lets look first at the real output curve of a turbine.

4.3.1 Power output curve

Due to friction, the turbine only starts to output power once the wind speed reaches some critical speed v_{cin} (cut in, usually about 4 m s^{-1}). Then, the power output rises as the cube of speed, as we calculated. But, because this on its own would create significant power output fluctuations, once the wind speed reaches so called rated speed v_{rated} (about 12 m s^{-1}), the power output becomes constant - so called rated power

$$P_{rated} = \frac{1}{2}\rho A C_p v_{rated}^3$$

This power output remains until some limit speed v_{cout} (cut out, approx. 25 m s^{-1}), when it is no longer safe to run the turbine. Therefore, the average power output of the turbine is

$$\bar{P} = \frac{1}{T}\int_0^T P(v(t))dt \leq P_{rated}$$

where T is the time of period of the wind (assume somewhat periodic behaviour, usually day or a year). We then prescribe a coefficient C_f that characterizes this inequality as

$$\bar{P} = C_f P_{rated}$$

This coefficient depends on the profile of the wind on specific site. For good sites, this is about 0.4. Therefore, the average power output is

$$\bar{P} = \frac{1}{2}C_f C_p A \rho v_{rated}^3$$

Now, we need to discuss the power per area efficiency. The limiting factor here is the requirement for the wind to restore to the original state with speed v_1. In order to do this, for circular turbines with radius a, the turbines should be separated by $10a$. Therefore, power occupied by one turbine is 100 a^2 and the area of the turbine is πa^2. Then

$$P_w = \frac{\bar{P}}{100a^2} = \frac{1}{2}C_f C_p \frac{\pi a^2}{100a^2}\rho v_{rated}^3 = \frac{1}{200}C_f C_p \pi \rho v_{rated}^3$$

For typical values of coefficients($C_f = 0.4$, $C_p = 0.5$, $v_{rated} = 12\text{m}^3\text{s}^{-3}$) and $\rho \approx 1\text{kg m}^{-3}$, $P_w \approx 5\text{W m}^{-2}$.

4.3.2 Wind profile

It should be quickly noted that the wind speed generally rises as we rise from the surface of the Earth up, and does so more quickly on see or in the flat country than for example in the cities. That is why it is worth building off-shore wind facilities - because the wind speed/height of the turbine ratio is favourable.

5 Fusion

Contrary to nuclear fission, fusion creates energy by creating a more massive nuclei from the lighter nuclei. In the case we will study, we create a $^4_2$He nucleus from $^2_1$H and $^3_1$H nucleus, which we alse mark as $^2_1$D (deuterium) and $^3_1$T (tritium).

There is a lot of hope for the fusion, because it is in some sense classical power generation - we heat some form of carnot cycle and generate energy at any time we need. It is not a renewable source of energy, but if the fusion from only deuterium is achieved, the amount of it on Earth (0.0156 % of all of Earth's hydrogen) will be enough to produce energy feeding current energy demand for a few billion years.

However, there are still many obstacles to overcome.

5.1 Reactions

The basic two reactions in a fusion are the following two. Many other reactions and some intermediate reactions also take place, but these are the main exotermic reactions

$$^2_1D + {}^2_1D \rightarrow {}^3_1T + {}^1_1H(3\,\text{MeV})$$

$$^3_1T + {}^2_1D \rightarrow {}^4_2He(3.5\,\text{MeV}) + {}^1_0n(14.5\,\text{MeV})$$

Thus, the overall output is about 21 MeV per 3 deuterium atoms.

It should be noted that the probability of the second reaction occuring is about 20 times higher than that of the first one. That is why right now most experiments with fusion use D+T mixture.

The fusion is also the power source of the Stars, but there it follows slightly different way and is very mass inefficient (1 kg of solar mass produces less energy on average than human body). But, if done properly, fusion is still much more energy rich than any chemical reactions.

Deuterium on Earth is pretty plentiful, however tritium is hard to obtain, and usually is created from lithium

$$^{6}_{3}Li + ^{1}_{0}n \rightarrow ^{4}_{2}He + ^{3}_{1}T$$

Lithium is more abundant, but still quite rare. If we could obtain the lithium from the oceans, the reserves would increase, but we cannot do that now.

Other problem we can see right now are the reaction products. While proton and helium are both reasonably containable, hot neutrons escaping the fusion gradually destroy the walls of the confinement chamber of the reactor.

Last problem is of course that in order to achieve efficient fusion at reasonable pressures, we need temperature about 10^8 K, which is very hard to create, let alone sustain.

5.2 Plasma Energy

There are two leading approaches to fusion - tokamak reactor fusion and inertial confinement fusion. First, I will discuss tokamaks. For this, we need to establish energy of a plasma.

We can sufficiently describe the overall behaviour of hydrogen plasma as that of an ideal gas with number of particles being twice the number of atoms involved, as nuclei and electrons disociate. These new particles have simple 3 degrees of freedom, hence the internal energy density is

$$U = 2 \times \frac{3}{2} N k_b T = 3 n k_b T$$

Assume that the plasma somehow dissipates (exactly how will be discussed further down), so that over time τ_e, it looses all its energy (essentially). We call this time the confinement time. Then, the average power loss density in plasma is

$$P_l = \frac{U}{\tau_e} = \frac{3 n k_b T}{\tau_e}$$

The power generated in the plasma is equal to the rate of the fusion reaction times the energy of the outgoing helium (no proton in only D-T reaction), which slows down in the plasma, while neutron escapes. So

$$P_f = f E_{He}$$

The rate of reaction f is equal to

$$f = n_D n_T < \sigma v >$$

where n_D is the number density of deuterium nuclei, n_T is the number density of tritium nuclei, σ is the cross-section of the reaction and v is the speed of the nuclei. $< \sigma v >$ is then the averaged value for different velocities and directions. Here, $n_D + n_T = n$, where n is the total number density of isotopes of hydrogen present in the reactor. The maximum of product $n_D(n - n_D)$ lies at $n_D = 0.5n = n_T$, so we have

$$f = \frac{1}{4} n^2 < \sigma v >$$

and

$$P_f = \frac{1}{4} n^2 < \sigma v > E_{He}$$

If the fusion is to be stable, this power loss has to be less or equal to the power gain from the fusion. Hence

$$P_l \leq P_f$$

$$\frac{3 n k_b T}{\tau_e} \leq \frac{1}{4} n^2 < \sigma v > E_{He}$$

$$n \tau_e \geq \frac{12 k_b T}{< \sigma v > E_{He}}$$

This is called the Lawson criterion.

This can be further developer for temperatures around 10^8 K, when $< \sigma v >= \gamma T^2$, where γ is a proportionality constant. Then, summarizing all constant terms in a constant β (excluding the number of degrees of freedom)

$$n\tau_e \geq \frac{3k_bT}{\beta T^2}$$

$$nT\tau_e \geq \frac{3k_b}{\beta}$$

This is the famous fusion triple product. The limit for 10^8 K is around 3×10^{28} K s m^{-3}, which puts $\beta \approx 1.4 \times 10^{-51}$ kg m^{5}s^{-3}K^{-2}.
In conquering this inequality, tokamak strategy chooses longer τ_e (which is harder) in order to be able to use only lower n. The inertial confinment fusion chooses very high n (which is harder), but can then afford very short τ_e.

5.3 Tokamak

Tokamaks use magnetic fields to confine the plasma. In magnetic fields, charged particles move in spirals, with radius given by the Lorentz force

$$ma = m\frac{v^2}{r} = qvB$$

$$r = \frac{mv}{qB}$$

In an estimate, the speed of the particles is their speed due to temperature, i.e.

$$v = \sqrt{\frac{k_bT}{m}}$$

Therefore

$$r = \frac{\sqrt{mk_bT}}{qB}$$

this means that different particles have different orbits. In plasma, these different particles collide and gradually diffuse - this is what causes the plasma to dissipate over time.
Therefore, a magnetic field along the general direction of plasma confinement can keep the plasma in place for some time. However, this plasma needs to be somehow heated. To heat plasma, we use the fact that it is a conductor and therefore heats by Ohmic heating. Then, if the plasma is kept in a loop (in which we can construct good confining magnetic field and keep the particles moving at high enough speeds while cycling), we can use it as a secondary coil of a transformer. These are the two coils that keep a tokamak going.
Essentially, the bigger the structure, the longer the confinment time, as the particles take longer to diffuse to the walls etc. This is why ITER is now being build.
Again, the hot neutrons hitting the sides of the tokamak are a big problem, as they cause material degradation.

5.4 Inertial Confinment Fusion

The inertial confinment fusion tries to not confine the plasma anyhow and simply rely on massive density of the particles. To have an idea of what density is required, we need an idea of the time τ_e. We can assume that the reaction, if there was any, stops when the volume of the plasma doubles, that is, radius changes (for initial spherical volume) by $\sqrt[3]{2} \approx 1.25$.
Then, the distance travelled by the plasma is $\frac{R}{4}$, where R is the original radius of the plasma. Hence the natural confinment time is

$$\tau_e \approx \frac{R}{4v} \approx \frac{R}{4\sqrt{\frac{k_bT}{m}}}$$

Hence the nTτ product rule becomes (using always SI units)

$$nT\frac{R}{4\sqrt{\frac{k_bT}{m}}} \geq 3 \times 10^{28}$$

$$nR \geq 4\sqrt{\frac{k_b}{mT}} \times 3 \times 10^{28}$$

Using $T \approx 10^8$ and multiplying both sides by average mass, which is 2.5 m_u (average between deuterium and tritium nucleus)

$$\rho R \geq 3\,\mathrm{kg\ m^{-2}}$$

A simple relation exists for fraction of burned D-T in the process of expansion of plasma - the ratio is

$$\Phi = \frac{\rho R}{\rho R + 60\,\mathrm{kg\ m^{-2}}}$$

The power produced in fusion per mass is $\epsilon_r \approx 10^{14}$ J kg^{-1}. Common achieved ratio is around $\rho R \approx 30\,\mathrm{kg\ m^{-2}}$. The amount of energy that can be very fastly contained is about 10^8 J. Hence the mass of the sample to produce this energy at this ratio is

$$m = \frac{100\,\mathrm{MJ}}{\Phi \times 100\,\mathrm{GJ\ g^{-1}}} \approx 3\,\mathrm{mg}$$

These are very small pieces of fuel, and this has many problems.
The inertial confinement was tried with such small D-T pellets, which are heated by laser drive. These pebbles have to be very symmetric, otherwise they rupture on the unsymmetric fluctuation, where the pressure gradient will peak, before fusion is initiated. Also, the lasers that heat such pallet has to be very symmetric. This is done by indirect laser drive - the lasers only shine upon a case that emits other radiation, which is very isotropic. This case is usually made from gold or similar material, and this process is very secretive, as it is a security risk due to hydrogen bomb. The pellet is also coated with ablator, which upon heating evaporates of and thus quickly compresses the fuel, which can ignite.
Further problem here is that if the ablator gets heated too quickly, the resulting pressure gradient leads to Rayleigh-Taylor instability - the lighter D-T fuel wants to escape before the heavier ablator.
Overall, there are many problems with inertial confinement fusion, but there is not so much plasma physics, so it is still an active area of research.

6 Energy Balance in the World

In the end, we will briefly discuss a few topics otherwise not covered in the module.

6.1 Heat pumps

Reversing a Carnot cycle and using the ground as a cold reservoir is the principle behind heat pumps. These extract heat from cold reservoir and give it to hot reservoir, requiring power. The coefficient of performance for these is

$$k = \frac{Q_h}{W} = \frac{1}{\epsilon_c} = \frac{T}{T - T_c}$$

where T is the temperature of hot reservoir, T_c is the temperature of cold reservoir and ϵ_c is the Carnot efficiency. This can be (and usually is) greater than one.
The problems are two - initial investment as the pump must dig quite deep to avoid oscillations in temperatures due to day and night and summer and winter. Furthermore, the soil can be artificially frozen, which is a problem (erosion and ecosystem problems). This essentially limits the power output of the heat pumps.
This can be partially prevented if the soil is artificially heated over the summer.
Overall in the UK, the average energy used for heating is about 37 kWh per day.

6.2 Transport

The losses in energy due to transport on earth are mainly due to air friction

$$F_{air} = \frac{1}{2} C \rho S v^2$$

where S is the area of the cross-section of the object in the direction of movement, ρ is the density of the air (or water if moving in water), v^2 is the square of speed of the object and C is a constant in order of unity.

The other losses are due to rolling, but they are usually much smaller. These take form

$$F_{roll} = \alpha mg$$

where m is mass of the object, g is acceleration due to gravity and α is constant in order of 10^{-3}.

6.2.1 Planes

Some basic estimates for power required to fly a plane at velocity v. The plane sends some air under itself with speed u. The amount of air it sends is approximately

$$dm = \rho S v dt$$

where S is the area of wings of the plane and v is the speed of the plane. Therefore, the change of momentum of the air downwards is

$$dp = u dm = u \rho S v dt$$

and therefore the lift force on the aircraft is

$$F = \frac{dp}{dt} = u \rho S v$$

this has to be equal to the force of gravity, so we have

$$mg = u \rho S v$$

$$u = \frac{mg}{\rho S v}$$

Hence the average power used to lift the airplane is

$$P_{lift} = \frac{dE_{air}}{dt} = \frac{1}{2} F u = \frac{m^2 g^2}{2 \rho S v}$$

where the factor of $\frac{1}{2}$ ensures the average. Then there is the air drag factor

$$P_{drag} = F_{air} v = \frac{1}{2} C \rho A v^3$$

Hence the overall energy per distance travelled efficiency is

$$\epsilon = \frac{P_{lift} + P_{drag}}{v} = \frac{m^2 g^2}{2 \rho S v^2} + \frac{1}{2} C \rho A v^2$$

This has maximum at

$$v = \frac{mg}{\rho \sqrt{SAC}}$$

which is the optimal flight speed.
The overall transport requirement is about 50 kWh per day per person.

6.3 Everything else

The electronic appliances use about 9 kWh per day per person and industry uses another 50 kWh per day per person. Combined renewables can produce up to 171 kWh per day per person, while we need about 173 kWh per day per person, which means that we can only barely make it and need some other sources to balance the fluctuations.

Appendices

A Common values used for back of the envelope calculations

General	
Total yearly world electricity consumption (2012)	0.8×10^{20} J
Total yearly world energy consumption (2012)	5.6×10^{20} J
Total yearly UK electricity consumption	1.2×10^{18} J
Total yearly UK energy consumption	8.0×10^{18} J
Average power per person (World)	2.5 kW
Average power per person (UK)	4.0 kW
Food energy consumption per person	100 W
Caloric value of meat	8.2 MJ kg^{-1}
Power output of one large coal station	0.6 GW
Power output of one large nuclear station	1.0 GW
Classical Electrical Power Generation	
Turbine efficiency	0.5
Very High voltage grid	400 kV
Household voltage	230 V
Copper resistivity	1.7×10^{-8} Ωm
Nuclear Power	
Average energy output of fission of uranium	200 MeV
Average number of released neutrons from one fission of uranium	2.5
Average energy of released neutrons from fission of uranium	2 MeV
Fission cross-section σ_f for ^{235}U at room temperature neutrons	10^{-21} cm^{-2}
Natural concentration of ^{235}U	0.7 %
Enriched concentration of ^{235}U	2-5 %
Fraction of power generated by secondary plutonium	$\frac{1}{3}$
Renewables	
Power per area - Hydro	0.07 W m^{-2}
Power per area - Biofuel	0.8 W m^{-2}
Power per area - Wind	4.0 W m^{-2}
Power per area - Solar (thermal)	20.0 W m^{-2}
Plant hydrocarbon growth	0.1 W m^{-2}