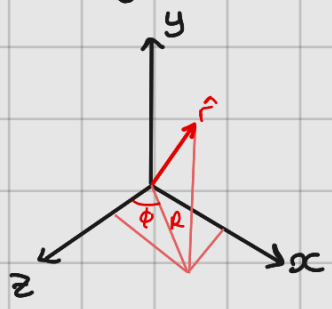
PX157 - Electricity and Magnetism

Part A - Electricity

∿ Cylindrical Coordinates ∿



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$x = R\cos\phi$ $\qquad$ $R = \sqrt{x^2+y^2}$

$y = R\sin\phi$ $\qquad$ $\phi = \tan^{-1}(y/x)$

$z = z$

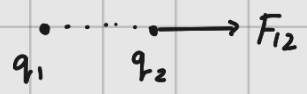
$$dV = r\, d\theta\, dr\, dz$$

∿

1 - Coulomb's Law

$$\boxed{\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{r}_{12}}{|\vec{r}_{12}|^2}}$$

→ $\vec{F}_{12}$ is force of q_1 acting on q_2



- $\hat{r}_{12}$ = unit vector between 2 charges: $\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$

- Principle of Superposition: The total force $\vec{F}$ on charge q_0 is the vector sum of all forces acting on q_0 exerted by every other charge

$$\vec{F}_0 = \sum_{i=1}^{N} \vec{F}_{i0} = \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\hat{r}_{i0}}{|\vec{r}_{i0}|}$$

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2 - Electric Fields

$$\boxed{\vec{E}(\vec{r}) = \lim_{q_0 \to 0} \frac{\vec{F}_0}{q_0}}$$

↳ Electric field at any point in space

$$\boxed{\vec{F}_0 = q_0 \vec{E}(\vec{r})}$$

↳ Force on charge q_0 at r_0 in E

$$\boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_q}{|\vec{r} - r_q|^3}}$$

→ Electric field at point $\vec{r}$ in field of charge q at $\vec{r}_q$

<u>Continuous charge distributions</u>

- When charge is distributed across a line, surface or volume, charge density can be defined
  - Linear charge density $\lambda = \frac{dq}{dl}$
  - Surface charge density $\sigma = \frac{dq}{ds}$
  - Volume charge density $\rho = \frac{dq}{d\tau}$

Then to calculate electric field:

1. Divide distribution into small elements at position $\vec{r}'$
2. Find $d\vec{E}$ at point p at $\vec{r}$ from each element
3. Vector sum all contributions: $\vec{E}(\vec{r}) = \int d\vec{E}$

$$\left(d\vec{E} = \frac{dq}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \frac{dq}{4\pi\varepsilon_0} \frac{\hat{r}_{dq,p}}{r^2_{dq,p}} \right)$$

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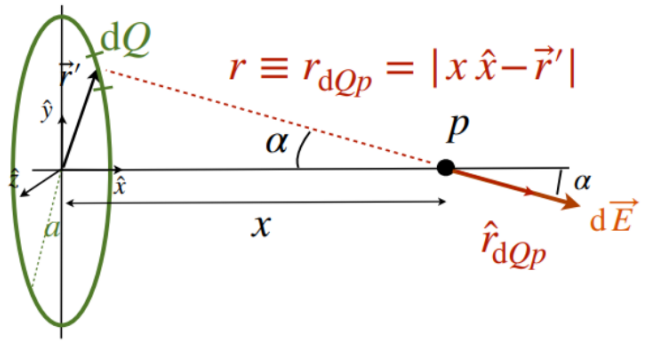
Example: What is the electric field on the x-axis from a ring of charge, centered on origin in yz-plane and with radius a, and with total charge Q?



$$\vec{r} = x\hat{x}$$
$$\vec{r}' = a\cos\phi\,\hat{y} + a\sin\phi\,\hat{z}$$

$$\vec{r}_{dQp} = \vec{r} - \vec{r}' = (x\hat{x} - \vec{r}')$$
$$= x\hat{x} - a\cos\phi\,\hat{y} - a\sin\phi\,\hat{z}$$

$$r = |\vec{r} - \vec{r}'| = (x^2 + a^2\cos^2\phi + a^2\sin^2\phi)^{1/2} = (x^2 + a^2)^{1/2}$$

$$d\vec{E}(x,0,0) = \frac{dQ}{4\pi\varepsilon_0}\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \frac{dQ}{4\pi\varepsilon_0}\frac{x\hat{x} - a\cos\phi\,\hat{y} - a\sin\phi\,\hat{z}}{(x^2 + a^2)^{3/2}}$$

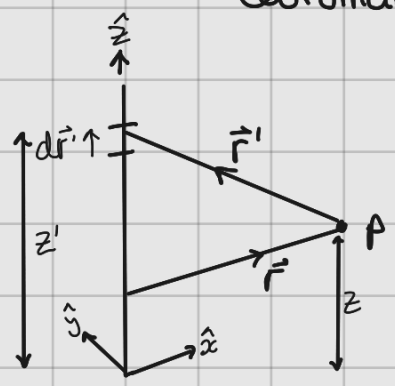
Introduce linear charge density $\lambda = \frac{dQ}{dl} = \frac{dQ}{a\,d\phi}$

$$\int_0^{2\pi} \sin\phi\, d\phi = 0$$

So $$\vec{E}(x,0,0) = \int d\vec{E} = \frac{\lambda a}{4\pi\varepsilon_0 r^3}\int_0^{2\pi}(x\hat{x} - \underbrace{a\cos\phi\,\hat{y}}_{\int_0^{2\pi}\cos\phi\, d\phi = 0} - \underbrace{a\sin\phi\,\hat{z}}_{\int_0^{2\pi}\sin\phi\, d\phi = 0})\, d\phi$$

$$\vec{E}(x,0,0) = \frac{\lambda a}{4\pi\varepsilon_0 r^3}\, 2\pi x\,\hat{x} = \frac{Qx}{4\pi\varepsilon_0 (x^2 + a^2)^{3/2}}\,\hat{x}$$

Example: Electric field from a line of infinite charge - cylindrical coordinates



$$\vec{r}' = z'\hat{z} \qquad \lambda = \frac{dQ}{dz'}$$

Let p lie on the x-axis so that
$$\vec{r} = R\hat{R} + z\hat{z}$$

$$\vec{r} - \vec{r}' = R\hat{R} + (z - z')\hat{z}$$

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$$|\vec{r}-\vec{r}'| = \left(R^2+(z-z')^2\right)^{1/2}$$

$$d\vec{E} = \frac{\lambda\, dz'}{4\pi\varepsilon_0}\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{E} = \int d\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{R\hat{R} + (z-z')\hat{z}}{\left(R^2+(z-z')^2\right)^{3/2}}\, dz'$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[R\hat{R}\underbrace{\int_{-\infty}^{+\infty}\frac{dz'}{\left(R^2+(z-z')^2\right)^{3/2}}}_{=\frac{2}{R}} + \hat{z}\underbrace{\int_{-\infty}^{+\infty}\frac{(z-z')\,dz'}{\left(R^2+(z-z')^2\right)^{3/2}}}_{=0}\right.$$

$$\therefore\ \vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{1}{R}\,\hat{R}$$

3 - Electrical Dipoles

- A pair of point charges $-q$ and $+q$ separated by d.

Electric dipole moment: $\boxed{\vec{p} = q\vec{d}}$ where $\vec{d}$ points from $-q$ to q

- In an external field $\vec{E}$, there is rotation with torque $\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$

- The work done by this torque is equal to the difference in potential energy between 2 'States' where

$$\boxed{U(\phi) = -\vec{p}\cdot\vec{E}}$$

Where ϕ is the angle between $\vec{r}$ and $\vec{E}$ $\left(|\vec{r}| = \frac{d}{2}\right)$

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4 - Electric Flux

> Electric flux – A measure of 'flow' of $\vec{E}$ through a surface

Electric flux through a flat surface $\boxed{\Phi_E = \vec{E} \cdot \vec{S}}$

where $\vec{S}$ points normal to the surface

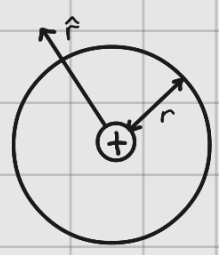
- For an arbitrary surface:
  - Split into N sections that are approximately flat
  - Each section k has $\phi_{E_k} = \vec{E}_{ij} \cdot d\vec{S}_{ij}$

hence for the whole surface:

$$\boxed{\Phi_E = \iint \vec{E} \cdot d\vec{S}} \quad \text{or} \quad \boxed{\Phi_E = \oiint \vec{E} \cdot d\vec{S}}$$

for a closed surface

Example: What is the electric flux through a sphere with central point charge q and radius r?



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad d\vec{S} = dS\,\hat{r}$$

as r is constant, $|\vec{E}|$ is constant on the sphere's surface

$$\therefore d\phi_E = \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \cdot dS\,\hat{r}$$

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$$\phi_E = \iint \frac{q}{4\pi\varepsilon_0 r^2}\, ds = \frac{q}{4\pi\varepsilon_0 r^2} \iint ds$$

Using $S = 4\pi r^2 \rightarrow \phi_E = \frac{q}{4\pi\varepsilon_0 r^2} \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$

5 - Gauss' Law

> The total flux through any closed surface is proportional to the enclosed charge

$$\boxed{\Phi_E = \oiint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}}$$

1. Check symmetry (Find $\vec{E}$ and $d\vec{S}$ in terms of 'world' vectors
2. Evaluate $\oiint \vec{E} \cdot d\vec{S}$ over Gaussian
3. Find E and $\vec{E}$ using Gauss Law

——— See worked examples

6 - Conductors

- The electric field <u>in</u> a conductor is zero
  - Excess charges must be on surface

→ Can be used to find $\vec{E}$ from a (charged) conductor using Gauss' law, just knowing $\vec{E} = 0$ inside conductor

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7 - Electric Potential

$$\boxed{W_{a\to b} = \int_a^b \vec{F} \cdot d\vec{l}}$$

$W_{a\to b} = \mathcal{U}(a) - \mathcal{U}(b)$

$$\boxed{\mathcal{U}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r}}$$

→ The potential energy associated with 2 charges separated by r

- If there are multiple charges, the potential energy is the sum of energies

$$\boxed{V(\vec{r}) = \lim_{q_0 \to 0} \frac{\mathcal{U}_{10}}{q_0}}$$

> Electric potential – potential energy per unit charge

→ from the interaction with a charge q_1

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}}$$

→ Electric potential for a charge located at $\vec{r}'$

$$\boxed{V_{ab} = \int_a^b \vec{E} \cdot d\vec{l}}$$

→ This can be used to find the absolute potential of a point at x by setting $a = x$, $b = \infty$

9 - Potential Gradients

$$\boxed{\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}}$$

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10- Capacitors

> Capacitor – a pair of conducting plated separated by d, which, when a voltage is applied across them, can store charge

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}}$$

↳ Q can be found for a capacitor (flat, cylindrical, spherical, etc) using charge density

↳ V can be found using $V_{ab} = \int_a^b \vec{E} \cdot d\vec{l}$

↳ E can be found using Gauss' law

In Series:

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$

In Parallel:

$$\boxed{C_{eq} = C_1 + C_2}$$

$$\boxed{U = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C}}$$

Assuming all energy stored by a capacitor is stored in the electric field between two plates:

$$\boxed{u = \frac{1}{2} \varepsilon_0 E^2}$$

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11 - Field Energy

$$\boxed{U = \sum_{i=1}^{n} \frac{1}{2} q_i V_i}$$

↳ potential energy between n charges

$$\boxed{V_i = \sum_{j=1, j\neq i}^{n} \frac{U_{ij}}{q_i}}$$

↳ potential of ith charge in group of n charges

↳ $U_{ij} = \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|}$

12 - Dielectrics

- In a dielectric all nuclei are in a fixed lattice and electrons are bound to their nuclei - no free charges
- If an electric field is applied across an atom, the centre of the charge distribution moves, producing an electric dipole

> polarisation density - the total local atomic dipole moment per unit volume

$$\vec{P}(\vec{r}) = \frac{1}{\mathcal{V}} \sum_i \vec{P}_i$$

- electric displacement vector combines external electric field and polarisation density

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

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Where ε_r is relative permittivity

$$\boxed{C = C_0 \varepsilon_r}$$

If a capacitor has capacitance C_0 when its plates are separated by a vacuum, C is the capacitance if separated by a dielectric

B - I, R and ε

Current

$$\boxed{I = \frac{dQ}{dt}}$$

$$\boxed{\vec{J} = \frac{I}{A}\hat{z} = nq\vec{v_d}}$$

→ $\vec{J}$ is current density: current per unit cross sectional area, pointing in the direction of $\vec{E}$

n - number of free charges
$\vec{v_d}$ - mean velocity of free charges

Resistance and Resistivity

$$\boxed{\vec{E} = \rho \vec{J}}$$

$$\boxed{\rho = \frac{1}{\sigma}}$$

$$\boxed{\rho = \frac{RA}{L}}$$

ρ = resistivity
σ = conductivity

In Series:

$$\boxed{R_{eq} = R_1 + R_2}$$

In parallel

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

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Energy and Power

$$\boxed{P = IV = I^2R = \frac{V^2}{R}}$$

EMF

Output voltage of a battery with internal resistance, r

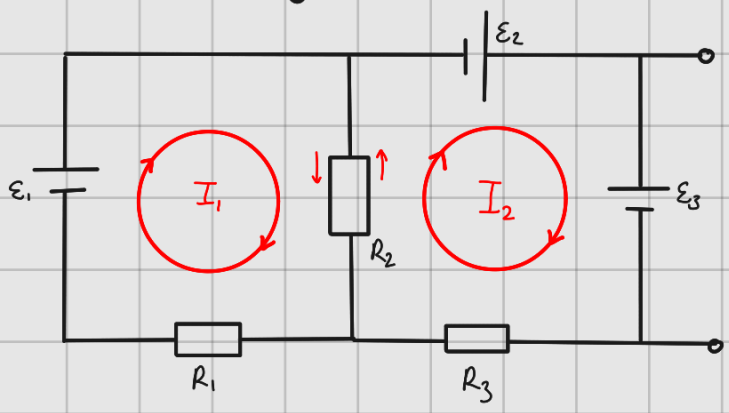
$$\boxed{V = \mathcal{E} - Ir}$$

14 - Circuits

> Kirchoff's Junction Rule - Current going into a junction = current leaving junction

> kirchoff's loop rule - Over a closed circuit loop, the sum of the voltages from emf must equal voltage drop across all other components

Maxwell loops can be used to associate a current to each loop, we add up get the total current



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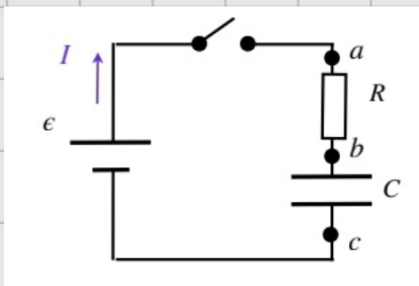
Loop 1: $\mathcal{E}_1 = R_2(I_1 - I_2) + R_1 I_1$

Loop 2: $\mathcal{E}_2 - \mathcal{E}_3 = R_2(I_2 - I_1) + I_2 R_3$

RC Circuits

For charging a capacitor:



at $t=0$ $\quad I = \frac{\mathcal{E}}{R}$ $\quad Q_0 = 0$

at $t=t$ $\quad \mathcal{E} = IR + \frac{Q}{C}$

at $t \to \infty$ $\quad \mathcal{E} = \frac{Q}{C}$ $\quad Q_f = \mathcal{E}C$

↳ Need to solve $\quad R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}$

- Propose solution $Q = \tilde{Q}(t)\, e^{-t/RC}$

For discharging a capacitor, $R\frac{dQ}{dt} + \frac{1}{C}Q = 0$

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Part B - Magnetism

1 - Magnetic Fields

$$\boxed{\vec{F} = q\vec{v} \times \vec{B}}$$

- Magnetic force on a moving charge

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$

→ Lorentz (total) force of moving charge in an electric and magnetic field

> Magnetic Flux - The 'flow' of a magnetic field through a surface

$$\boxed{\Phi_B = \iint_S \vec{B} \cdot d\vec{S}}$$

- Magnetic field lines have no ends - all magnetic flux entering a closed volume must also leave it
  - This gives the Solenoidal condition

$$\boxed{\oiint_S \vec{B} \cdot d\vec{S} = 0}$$

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Motion in a Magnetic field

- Velocity is split into parallel and perpendicular to the magnetic field: $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$

- Magnetic force is perpendicular to the $\vec{B}$ and $\vec{v}$ $\quad \vec{F} = q\vec{v}_{\perp} \times \vec{B}$
  $|\vec{F}| = |q| v_{\perp} B$
  - Balancing this with centripetal force $\quad \dfrac{m v_{\perp}^2}{R_L} = |q| v_{\perp} B$

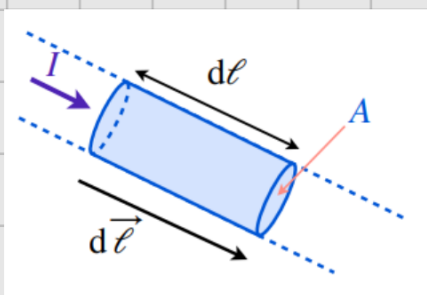
giving the Larmor radius R_L of the charge

$$\boxed{R_L = \frac{m v_{\perp}}{|q| B}}$$

- Parallel speed $\vec{v}_{\parallel}$ is constant, so the charge moves in a helix

- A charge moving in a magnetic field does <u>no</u> work

2 - Force of current loops



A section of wire carrying current I with length $d\ell$ has vector $d\vec{\ell}$ of length $d\ell$ pointing in the direction of I

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The total force on section $d\ell$ is

$$\boxed{d\vec{F} = I d\vec{\ell} \times \vec{B}}$$

The torque on a loop of wire is given by:

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$ $\vec{\mu}$ is the magnetic dipole moment

$$\boxed{\vec{\mu} = NAI\hat{n}}$$ N = number of turns in loop
A = area of loop

The potential energy of a magnetic dipole is

$$\boxed{U = -\vec{\mu} \cdot \vec{B}}$$

3 - Law of Biot-Savart

The magnetic field $\vec{B}$ at position $\vec{r}$ produced by a moving charge q at position $\vec{r}'(t)$ with velocity $v(t)$ at time t is given by:

$$\boxed{\vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi} \frac{\vec{v}(t) \times (\vec{r} - \vec{r}'(t))}{|\vec{r} - \vec{r}'(t)|^3}}$$

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By calculating and comparing the magnetic and electrostatic force on 2 identical particles moving at speed v

$$\boxed{\frac{|\vec{F}_B|}{|\vec{F}_E|} = \left(\frac{v}{c}\right)^2}$$

If the law of Biot-Savart is applied to a wire split into sections $d\vec{r}$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{r}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}$$

(where $\vec{r}'$ is the location of the wire and $\vec{r}$ is where the field is being measured from

For 2 parallel wires:

- If I is in the same direction, the wires attract
- If I is in the opposite direction, the wires repel

3 - Ampères Law

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}}$$

→ Magnitude/direction of I determined using right hand rule

4 - Solenoids

Assume:

- long solenoid → neglect end effects

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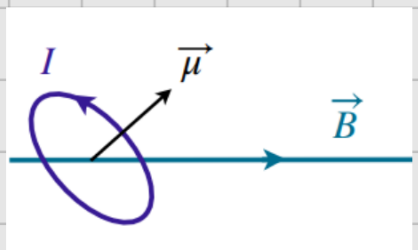
- Number of turns per unit length is large → assume axial symmetry
- No net current along axis

The magnetic field of a solenoid can be calculated using Ampère's law to be (In worked examples)

$$\boxed{\begin{aligned} &\vec{B} = \vec{0} \quad \text{outside solenoid} \\ &\vec{B} = \mu_0 n I \hat{z} \quad \text{inside solenoid} \end{aligned}}$$

5 - Magnetic Materials

A current loop with magnetic dipole moment $\vec{\mu} = IA\hat{n}$ in external magnetic field $\vec{B}$ experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ which acts to align $\vec{\mu}$ and $\vec{B}$



The current also generates a dipole magnetic field parallel to $\vec{\mu}$

- In magnetic materials, atoms behave as magnetic dipoles
  - Many small dipoles create a surface current
- The net current inside a magnetic material is zero

The magnetic field inside a material in an external magnetic field, $\vec{B}_0$ is given by:

$$\boxed{\vec{B} = \mu_r \vec{B}_0}$$

μ_r is relative permittivity

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The 4 types of magnetic materials: Diamagnetic, paramagnetic, ferromagnetic, hard ferromagnetic

Diamagnetic

- $\mu_r < 1$ → $\vec{B}$ is reduced inside material
- no permanent atomic $\vec{\mu}$, only $\vec{\mu}$ induced by $\vec{B}_0$
- μ_r very very close to 1
- Closed shell atomic structures eg. H_2, N_2, NaCl, Si, Ge

Paramagnetic

- $\mu_r > 1$ So $\vec{B}$ is increased inside material
- Permanent atomic $\vec{\mu}$ with random alignment until $\vec{B}$ applied

The magnetic susceptibility can be defined as

$$\boxed{\chi_m = \mu_r - 1}$$

Ferromagnetic

- $\mu_r \gg 1$
- Magnetisation may show hysteresis

Hard Ferromagnetic

- Permanent magnets
- μ_r undefined

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6 - Faraday's Law

$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$

→ $\mathcal{E}$ = induced emf in closed loop

→ $\Phi_B = \iint_S \vec{B}\cdot d\vec{S}$

Lenz's Law

The effect of an induced emf is such that it opposes changing flux which produced it

eg.

making a loop of wire smaller decreases flux through

So current induces to "put more flux in"

Motional EMF

- when charges are moved through a magnetic field in a conductor, a motional emf is generated
  - emf = work done per unit charge moving around a closed loop

$$\boxed{\mathcal{E} = \oint_{loop} (\vec{v} \times \vec{B}) \cdot d\vec{l}}$$

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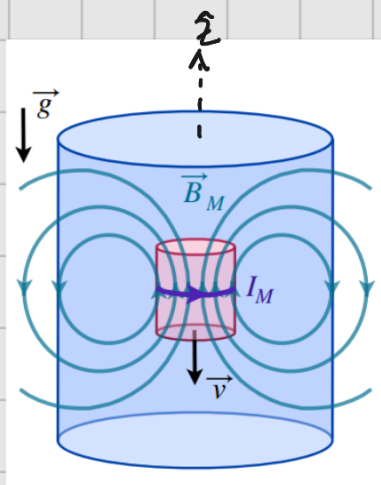
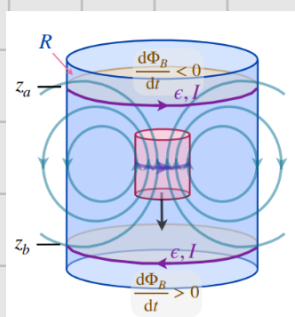
Falling Magnet in a Pipe



Pipe:
- radius a
- paramagnetic

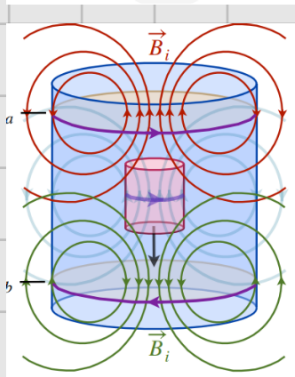
Magnet
- radius a_m
- dipole $\vec{\mu}_m = \mu_m \hat{z}$
- Surface current $\vec{I}_m = I_m \hat{\varphi}$
- Dipolar field $\vec{B}_m$ centred

Magnetic flux:
- As magnet falls, Φ_B at point z_a above magnet decreases
$$\frac{d\Phi_B}{dt}(z_a) < 0$$
- At z_b below magnet, Φ_B increases: $\frac{d\Phi_B}{dt}(z_B) > 0$



Faradays Law:
- At $z = z_a$, $\mathcal{E}$ generated in wall of pipe in φ direction
- At $z = z_b$, $\mathcal{E}$ generated in $-\varphi$ direction

Ohm's Law:
- Eddy current $I = \mathcal{E}/R$ generated in φ direction at $z = z_a$ and $-\varphi$ direction at $z = z_b$



Ampères Law:
- At $z = z_a$, induced wall current produces dipolar magnetic field $\vec{B}_i$ with $\vec{\mu}_i \sim \hat{z}$
- at $z = z_b$, $\vec{\mu}_i$ produced with $\vec{\mu}_i \sim -\hat{z}$

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Force on magnet:

$$F \sim I_m \hat{\varphi} \times \vec{B}_i = I_m \hat{\varphi} \times (B_{ir}\hat{r} + B_{iz}\hat{z})$$
$$= -I_m B_{ir}\hat{z} + I_m B_{iz}\hat{r}$$

F points upwards

7 - Induced Electric Fields

There is an induced emf with a changing magnetic field, even if the circuit is stationary. A force drives a current around a circuit even if there is no motional emf.

For stationary loops:

$$\boxed{\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$

↳ For a closed path $\oint_C \vec{E} \cdot d\vec{l} = 0$, so $\oint_C \vec{E} \cdot d\vec{l} \neq$ if there is a changing magnetic field

- Φ_B generates closed loops of electric field

The energy density of an induced electric field is

$$\boxed{u = \frac{\mathcal{U}}{\mathcal{V}} = \frac{B^2}{2\mu_0}}$$

$\mathcal{U}$ = Potential energy
$\mathcal{V}$ = Volume

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PX157 - Electricity and Magnetism

Part B - Electronics

1 - Inductors

A Solenoid with length ℓ and Φ_B through the coils has induced emf:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -\frac{d}{dt}(N\Phi_B)$$

using $\Phi_B = BA$ and $B = \mu_0 \frac{N}{\ell} I$, the Self inductance can be defined as:

$$\boxed{L = \frac{N\Phi_B}{I}}$$

$$\mathcal{E} = -\frac{d}{dt}N\Phi_B = -\frac{d}{dt}LI = -L\frac{d}{dt}I$$

An inducter has self induced emf $\boxed{\mathcal{E} = -L\frac{dI}{dt}}$

- this opposes changing current

- A Solenoid is the most common form of an inductor with symbol

<u>Solenoid inductance</u>

- $B = \mu_0 nI = \mu_0 \frac{N}{\ell} I \rightarrow \Phi_B = BA = \mu_0 \frac{NIA}{\ell}$

$$\hookrightarrow \boxed{L = \frac{N\Phi_B}{I} = \frac{\mu_0 A N^2}{\ell}}$$

- Varying N, ℓ and A can vary L
- Adding a ferromagnetic core

- The energy stored by an inducter, $\mathcal{U}$ is $\boxed{\mathcal{U} = \frac{1}{2}LI^2}$

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and energy density is $\boxed{u = \frac{\mathcal{U}}{\mathcal{V}} = \frac{B^2}{2\mu_0}}$

In Series $\boxed{L_{eq} = L_1 + L_2}$

In parallel $\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}}$

<u>Transformers</u>

- Two inductors are coupled by a shared magnetic field, enhanced by a ferromagnetic core

2 - AC Circuits

- An alternating voltage can be represented as $V(t) = V_0 \cos(\omega t)$
  - This is the real part of complex voltage $\tilde{V} = V_0 e^{i\omega t}$

For $\tilde{V}_1 = V_0 e^{i\omega t}$ and $\tilde{V}_2 = V_0 e^{i(\omega t + \phi)}$, V_2 leads V_1 by a phase difference ϕ
  - $\tilde{V}_1$ is a phasor

<u>Complex Impedance</u>

> Impedance - the opposition to alternating current presented by the combined effect of resistance and reactance in a circuit

> reactance - the opposition to alternating current presented by inductance and capacitance

$$\boxed{\tilde{V} = \tilde{Z}\,\tilde{I}}$$

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For a resistor:

$$V(t) = V_0 \cos(\omega t)$$

$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R}\cos(\omega t) = I_0 \cos(\omega t)$$

As I and V are in phase, $\tilde{V} = \tilde{Z}_R \tilde{I} \rightarrow \boxed{\tilde{Z}_R = R}$

For a Capacitor

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} \quad \text{using } Q = It \quad C = Q/V$$

If $V(t) = V_0 \cos(\omega t) \rightarrow I(t) = -CV_0\omega \sin(\omega t)$ ← out of phase

$$\tilde{I} = C\frac{d}{dt}\left(V_0 e^{j\omega t}\right) = j\omega C V_0 e^{j\omega t} = j\omega C \tilde{V}$$

$$\tilde{Z}_C = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}}{j\omega C \tilde{V}} \qquad \boxed{\tilde{Z}_C = \frac{1}{j\omega C}}$$

↳ Current leads voltage by $\pi/2$

For an Inductor:

$$V = L\frac{dI}{dt}$$

$V(t) = V_0 \cos(\omega t) \rightarrow I(t) = \frac{V_0}{\omega L}\sin(\omega t)$ ← out of phase

$$\tilde{V} = L\frac{d}{dt}\left(I_0 e^{j\omega t}\right) = j\omega L I_0 e^{j\omega t} = j\omega L \tilde{I}$$

$$\tilde{Z}_L = \frac{\tilde{V}}{\tilde{I}} = \frac{j\omega L \tilde{I}}{\tilde{I}} \rightarrow \boxed{\tilde{Z}_L = j\omega L}$$

↳ Current lags voltage by $\pi/2$

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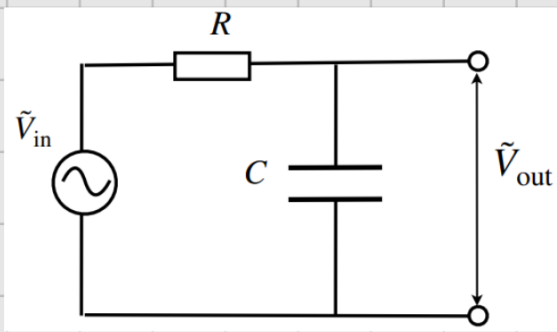
In Series $\tilde{Z}_{eq} = \tilde{Z}_1 + \tilde{Z}_2$

In parallel $\frac{1}{\tilde{Z}_{eq}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2}$

3 - LRC Circuits

Low-Pass filter



1. Inductance

$$\tilde{Z}_{eq} = \tilde{Z}_R + \tilde{Z}_C$$

$$\tilde{Z}_{eq} = R + \frac{1}{j\omega C}$$

2. $\tilde{I}$

$$\tilde{V}_{in} = \tilde{Z}_{eq}\tilde{I} \rightarrow \tilde{I} = \frac{\tilde{V}_{in}}{R + \frac{1}{j\omega C}}$$

3. $\tilde{V}_{out}$

$$\tilde{V}_{out} = \tilde{Z}_C \tilde{I}$$

$$\tilde{V}_{out} = \frac{1}{j\omega C} \frac{\tilde{V}_{in}}{R + \frac{1}{j\omega C}} = \frac{\tilde{V}_{in}}{j\omega RC + 1}$$

4. $|\tilde{V}_{out}|$

$$\frac{|\tilde{V}_{out}|}{|\tilde{V}_{in}|} = \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}$$

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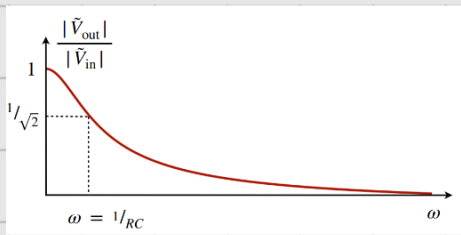

→ High frequency Signals are surpressed as they flow easily through the capacitor

5. Phase

$$\tilde{Z}_{eq} = Z_0 e^{j\phi}$$

$$\phi = \tan^{-1}\left(\frac{Im(Z_{eq})}{Re(Z_{eq})}\right) = \tan^{-1}\left(\frac{-1/\omega C}{R}\right)$$

$\tilde{V}_{in}$ lags behind $\tilde{I}$ as $\phi < 0$

→ This can also be found for a high pass filter (L replaces C)

- A notch filter circuit can be formed (add L with C) to filter out a typical frequency $\omega_c = \frac{1}{\sqrt{LC}}$

4 - AC Power

$$\boxed{I_{rms} = \frac{I_0}{\sqrt{2}}} \quad \boxed{V_{rms} = \frac{V_0}{\sqrt{2}}} \quad \boxed{P = V_{rms} I_{rms}}$$

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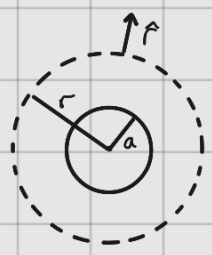
ELECTRICITY

Gauss' Law

Uniform Sphere: $\oint\!\!\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$



Consider a uniform sphere with charge $Q>0$, radius a and volume charge density ρ

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3}$$

1. Check Symmetry

- $\vec{E}$ is radially outwards, $\vec{E} = E\hat{r}$
- a is constant so $|\vec{E}|$ is constant on surface

2. Choose Gaussian Surface

↳ Sphere of radius r centred on charged sphere

$$\vec{E}\cdot d\vec{S} = E\hat{r}\cdot dS\hat{r} = E\,dS$$

$$\Phi_E = \oint\!\!\oint_S \vec{E}\cdot d\vec{S} = E\oint\!\!\oint_S dS = E\,4\pi r^2$$

3. Find electric field

For $r>a$:

- $Q_{encl} = Q$

↳ $\Phi_E = E\,4\pi r^2 = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi r^2 \varepsilon_0}$ $\quad r>a$

For $r \leq a$:

$$\rho = \frac{Q}{V} \rightarrow Q_{sphere}V_{sphere} = Q_{encl}V_{encl}$$

↳ $Q_{encl} = Q\,\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3} = \frac{Qr^3}{a^3}$

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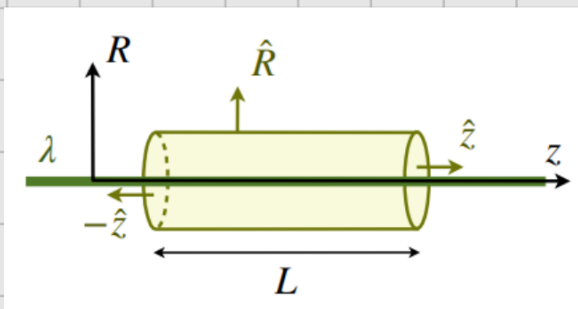
$$\Phi_E = E\,4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0} = \frac{Q r^3}{\varepsilon_0 a^3} \longrightarrow E = \frac{Q r}{4\pi\varepsilon_0 a^3} \qquad r \le a$$

Infinite line of charge



$$\lambda = \frac{Q}{L}$$

1. Symmetry
   - $\vec{E}$ is radially outwards: $\vec{E} = E\hat{R}$
   - $|\vec{E}|$ is constant on a cylindrical surface aligned with the line of charge

2. Choose Gaussian surface
   ↳ choose a cylinder of radius R and length L aligned with the line

Left flat end: $\iint_{Left} \vec{E}\cdot d\vec{s} = \iint_{Left} E\hat{R}\cdot ds(-\hat{z}) = 0$ $\qquad (\hat{R}\cdot\hat{z}) = 0$

Right flat end: $\iint_{right} \vec{E}\cdot d\vec{s} = \iint_{right} E\hat{R}\cdot ds\,\hat{z} = 0$

Curved surface: $\iint_{Curve} \vec{E}\cdot d\vec{s} = \iint E\hat{R}\cdot ds\,\hat{R} = E\iint ds = E\,2\pi RL$

3. Find electric field

$$\Phi_E = E\,2\pi RL = \frac{Q_{encl}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 R}, \quad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 R}\hat{R}$$

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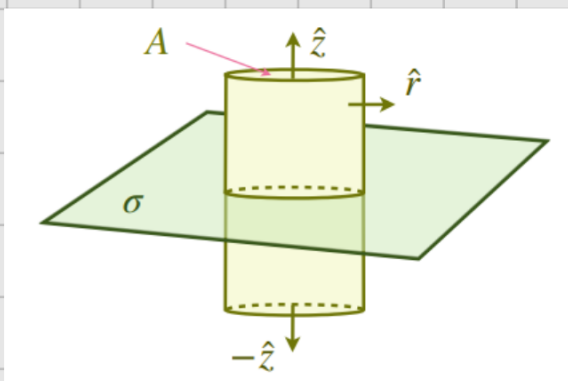
Infinite plane Sheet



Consider uniformly charged infinite Sheet

$$\sigma = \frac{Q}{S}$$

1. Symmetry

- $\vec{E}$ points normal to S if $\sigma > 0$

  $\vec{E} = E\hat{z}$ for $z > 0$

  $\vec{E} = E(-\hat{z})$ for $z < 0$

- $|\vec{E}|$ is constant at fixed distance E from Sheet

2. Choose Gaussian Surface

↳ Consider a cylinder of cross-sectional area A with its axis in the $\hat{z}$ direction

Flat top: $$\iint_{top} \vec{E} \cdot d\vec{s} = \iint_{top} E\hat{z} \cdot ds\,\hat{z} = E \iint_{top} ds = EA$$

Flat bottom: $$\iint_{bot} \vec{E} \cdot d\vec{s} = \iint_{bot} E(-\hat{z}) \cdot ds(-\hat{z}) = E \iint_{bot} ds = EA$$

Curved Surface: $$\iint_{Curve} \vec{E} \cdot d\vec{s} = \iint_{Curve} E\hat{z} \cdot ds\,\hat{r} = 0$$

So $\Phi_E = 2EA$

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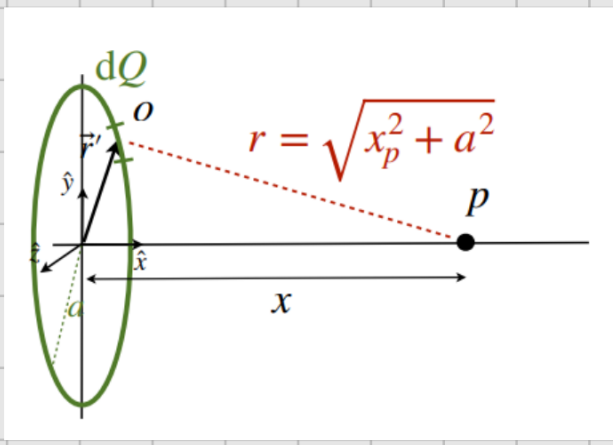
3. Find electric field

$$\Phi_E = 2EA = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \rightarrow \vec{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{z} & z>0 \\ -\frac{\sigma}{2\varepsilon_0}\hat{z} & z<0 \end{cases}$$

Electric Potential

Potential of a ring of charges

$$V_{ab} = \int_a^b \vec{E}\cdot d\vec{l} \qquad V = \frac{Q}{4\pi\varepsilon_0 r}$$



$$dV = \frac{dQ}{4\pi\varepsilon_0 r}$$

(From above: $\vec{E} = \frac{Qx\hat{x}}{4\pi\varepsilon_0 (x^2+a^2)^{3/2}}$
using
$E = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$)

Choose path $d\vec{l} = d\vec{r} = dx\,\hat{x}$

$$\therefore V_{ab} = \frac{Q}{4\pi\varepsilon_0}\int_x^\infty \frac{x}{(x^2+a^2)^{3/2}}$$

$$V_{ab} = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2+a^2)^{1/2}}\right]_x^\infty = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+a^2}}$$

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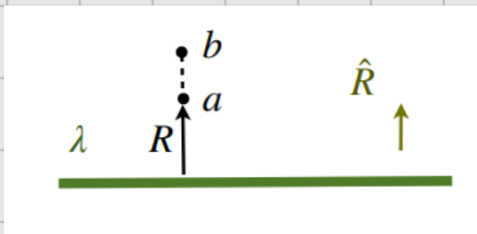
Infinite Line of charge



$\lambda = \frac{dQ}{d\ell}$ $\left(\text{from earlier: } \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 R}\hat{R}\right)$

from a to b, $d\vec{\ell} = dR\hat{R}$

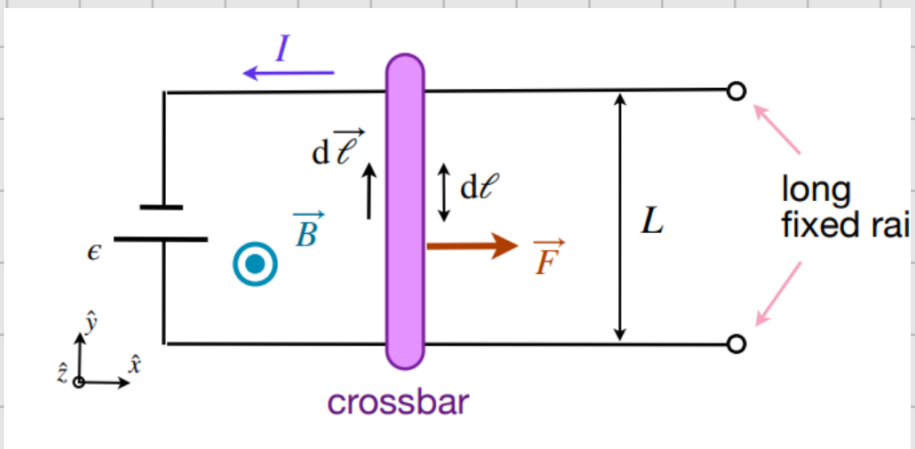
$$\therefore V_{ab} = \int_a^b \vec{E}\cdot d\vec{\ell} = \frac{\lambda}{2\pi\varepsilon_0}\int_{R_a}^{R_b} \frac{1}{R}\hat{R}\cdot dR\hat{R} = \frac{\lambda}{2\pi\varepsilon_0}\Big[\ln|R|\Big]_{R_a}^{R_b}$$

$$V_{ab} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{R_b}{R_a}\right)$$

MAGNETISM

Force of Current Loops

Moveable Crossbar



$\vec{B} = B\hat{z}$

$d\vec{\ell} = d\ell\hat{y}$

$$dF = I\,d\vec{\ell}\times\vec{B}$$

$$dF = I\,d\ell B(\hat{y}\times\hat{z}) = I\,d\ell B\hat{x}$$

$$F = I\int_0^L d\vec{\ell}\times\vec{B} = ILB\hat{x}$$

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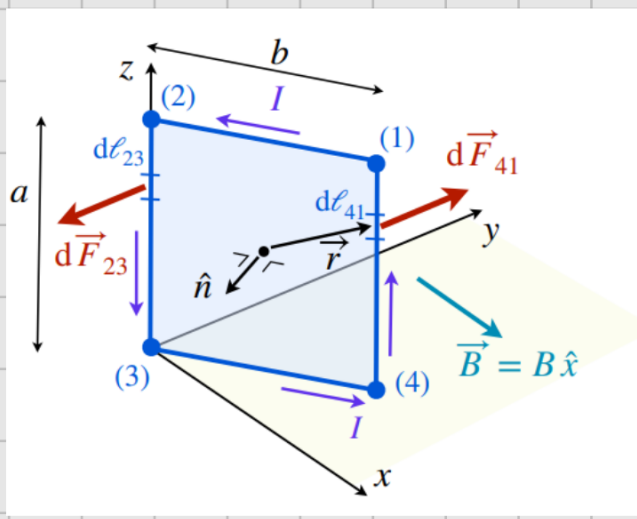
Current loop



Forces on sides:

- $\vec{F}_{12} \sim \hat{z}$
- $\vec{F}_{34} = -\vec{F}_{12}$, So cancel
- $d\vec{F}_{23} = I\, d\vec{\ell} \times \vec{B}$ with $d\vec{\ell} = d\ell\,(-\hat{z})$

$$= I\, d\ell B\,(-\hat{z} \times \hat{x}) = I\, d\ell\,(-\hat{y})$$

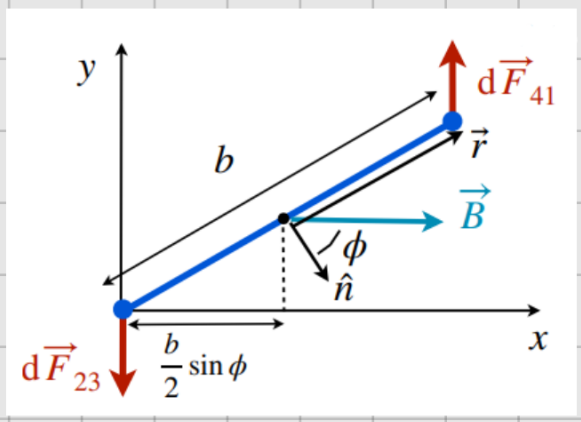
$$\vec{F}_{23} = \int_0^a -I\, d\ell B\,\hat{y} = -IaB\hat{y}$$

- $d\vec{F}_{41} = I\, d\vec{\ell} \times \vec{B}$ with $d\vec{\ell} = d\ell\,\hat{z}$

$$= I\, d\ell B\,(\hat{z} \times \hat{x}) = I\, d\ell B\hat{y}$$

$$\vec{F}_{41} = \int_0^a I\, d\ell B\,\hat{y} = IaB\hat{y}$$

→ Opposite forces cancel so no net force, however $\vec{F}_{41}$ and $\vec{F}_{23}$ cause a torque



→ loop rotates until its plane is parallel with $\vec{B}$

→ The direction of normal vector n is determined by the right hand rule

$\vec{\tau} = \vec{\mu} \times \vec{B}$ $\quad$ $\vec{\mu} = NAI\hat{n}$

$\vec{\mu} = a^2 I \hat{n}$ $\quad$ $\tau = a^2 IB(\hat{n} \times \hat{x}$

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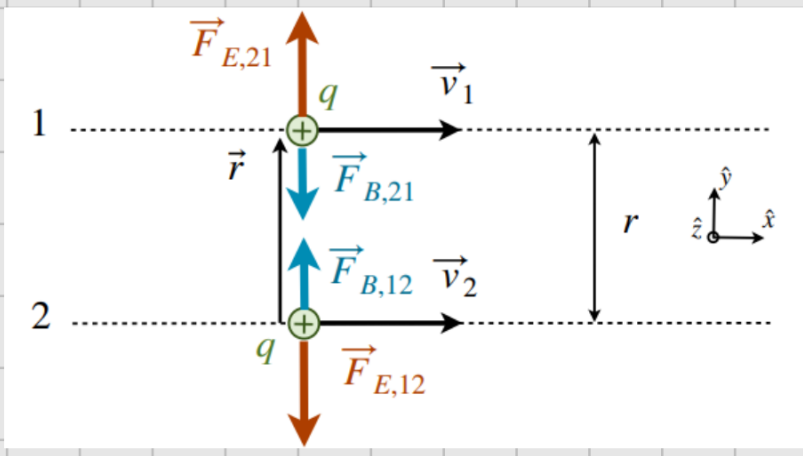
Law of Biot-Savart

Two moving charges



Two protons move at the same speed v in the x direction

$\vec{v}_1 = \vec{v}_2 = v\hat{x}$

$$\vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi} \frac{\vec{v}(t) \times (\vec{r} - \vec{r}'(t))}{|\vec{r} - \vec{r}'(t)|^3}$$

$$\vec{F} = q\vec{v} \times \vec{B}$$

- Electrostatic force on q_1 from q_2:

$$\vec{F}_{21} = \frac{q^2}{4\pi\varepsilon_0 r^2}\hat{y}$$

- Electrostatic force on q_2 from q_1

$$\vec{F}_{12} = -\frac{q_1}{4\pi\varepsilon_0 r^2}\hat{y}$$

- Magnetic field at q_1 from q_2

$$\vec{B}_2 \text{ (at 1)} = \frac{\mu_0 q}{4\pi} \frac{\vec{v}_2 \times \vec{r}}{r^3} = \frac{\mu_0 q}{4\pi} \frac{v}{r^2}\hat{z}$$

- Magnetic field at q_2 from q_1

$$\vec{B}_1 \text{ (at 2)} = -\frac{\mu_0 q}{4\pi} \frac{\vec{v}_1 \times \vec{r}}{r^3} = -\frac{\mu_0}{4\pi} q \frac{v}{r^2}\hat{z}$$

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- Magnetic force on q_1 from q_2

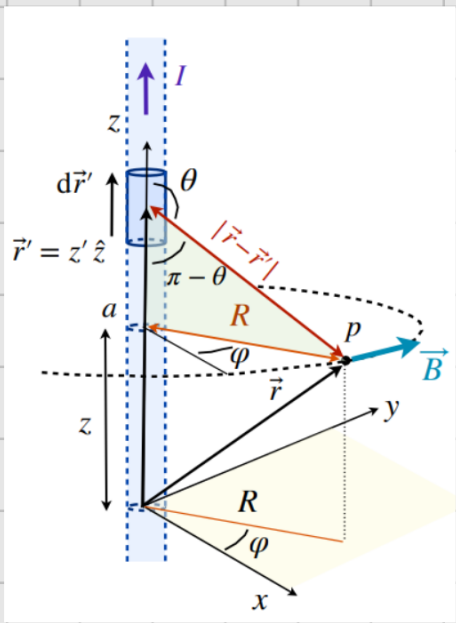
$$F_1 = q\vec{v}_1 \times \vec{B}_{2(at\,1)} = -\frac{\mu_0}{4\pi}\frac{q^2 v^2}{r^2}\hat{y}$$

- Magnetic force on q_2 from q_1

$$F_2 = q\vec{v}_2 \times \vec{B}_{1}(at\,2) = \frac{\mu_0}{4\pi}\frac{q^2 v^2}{r^2}\hat{y}$$

A Straight wire

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int I\,\frac{d\vec{r}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$



- Using cylindrical coordinates (R, φ, z)
- Section of wire at $\vec{r}'$ with field measured from point p at $\vec{r}$

$$\vec{r} = R\hat{R} + z\hat{z}$$
$$\vec{r}' = z'\hat{z}$$
$$d\vec{r}' = dz'\hat{z}$$

$$\vec{r} - \vec{r}' = R\hat{R} + (z-z')\hat{z}$$

$$d\vec{r} \times (\vec{r}-\vec{r}') = dz'\hat{z} \times \left(R\hat{R} + (z-z')\hat{z}\right)$$

$$= \begin{vmatrix} \hat{R} & \hat{\phi} & \hat{z} \\ 0 & 0 & dz' \\ R & 0 & z-z' \end{vmatrix} = \hat{R}(0-0) - \hat{\phi}(0 - R\,dz') + z'(0-0)$$

$$= R\,dz'\,\hat{\phi}$$

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$$d\vec{r} \times (\vec{r}-\vec{r}') = R\,dz'\,\hat{\phi}$$

$$|\vec{r}-\vec{r}'| = (R^2 + (z-z')^2)^{1/2}$$

$$\therefore d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I R\,dz'\,\hat{\phi}}{(R^2+(z-z')^2)^{3/2}}$$

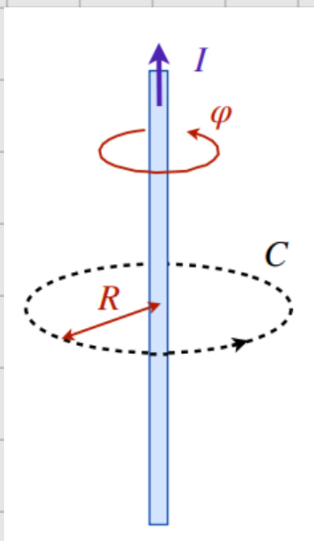
$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{\phi}\int_{-\infty}^{+\infty} \frac{R}{(R^2+(z-z')^2)^{3/2}}\,dz'$$

?

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{\phi}\left[\frac{-(z-z')}{R(R^2+(z-z')^2)^{1/2}}\right]_{-\infty}^{\infty}$$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi R}\hat{\phi}}$$

Ampère's Law



MF of Infinite Straight Wire

1. Symmetry
- $|\vec{B}|$ is constant at fixed radius R from the wire
- As $\vec{B} \perp I d\vec{l}$ So $B_z = 0$
- Solenoidal condition on cylindrical surface $\vec{B}_R = 0$
- Magnetic field is in ϕ direction $\vec{B} = B_\phi \hat{\phi}$

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2. Define path C

Pick circular path centred on the wire in the $\hat{\phi}$ direction

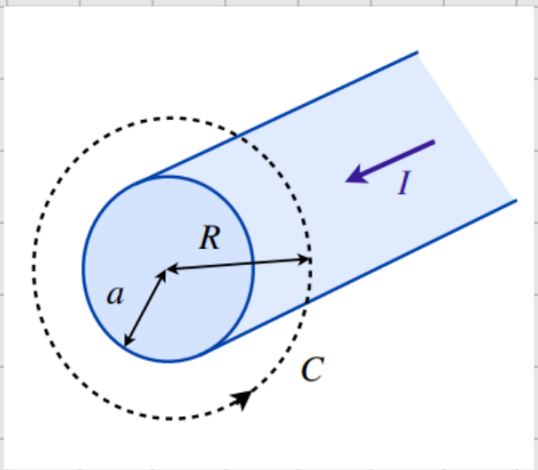
$$\vec{B} = B_\phi \hat{\phi} \quad \text{and} \quad d\vec{l} = dl\,\hat{\phi}$$

3. Evaluate Ampère's law

$$\mu_0 I_{encl} = \oint_C \vec{B} \cdot d\vec{l} = \oint_C B_\phi \, dl \; \hat{\phi} \cdot \hat{\phi} = \oint_C B_\phi \, dl$$

$$= B_\phi \oint_C dl = 2\pi R B_\phi = \mu_0 I$$

$$\hookrightarrow \vec{B} = \frac{\mu_0 I}{2\pi R} \hat{\phi}$$



MF of long cylindrical conductor

$$J = \frac{I}{A} = \frac{I}{\pi a^2}$$

1. Symmetry

- MF is azimuthal $\vec{B} = B_\phi \hat{\phi}$
- $|\vec{B}|$ is constant at fixed radius R from wire axis

2. Pick path

$$d\vec{l} = dl\,\hat{\phi} \quad \rightarrow \quad \vec{B} \cdot d\vec{l} = B_\phi \, dl$$

3. Evaluate Ampère's law

$$\oint_C \vec{B}_\phi \cdot d\vec{l} = B_\phi \oint_C dl = 2\pi R B_\phi$$

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Outside the cylinder: $R \geq a$

$I_{encl} = I$

$\hookrightarrow 2\pi R B_\phi = \mu_0 I \rightarrow \vec{B} = \frac{\mu_0 I}{2\pi R} \hat{\phi}$

Inside the cylinder: $R < a$

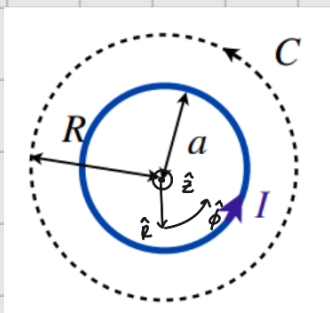
$J = \frac{I_1}{A_1} = \frac{I_2}{A_2} \rightarrow \frac{I}{\pi a^2} = \frac{I_{encl}}{\pi R^2} \rightarrow I_{encl} = I \left(\frac{R}{a}\right)^2$

$\hookrightarrow 2\pi R B_\phi = \mu_0 I \left(\frac{R}{a}\right)^2 \rightarrow \vec{B} = \frac{\mu_0 I R}{2\pi a^2} \hat{\phi}$

4. Summarise

$$\vec{B} = \begin{cases} \frac{\mu_0 I R}{2\pi a^2} \hat{\phi} & R < a \\ \frac{\mu_0 I}{2\pi R} \hat{\phi} & R \geq a \end{cases}$$

Solenoids



MF of a Solenoid

$\vec{B} = B_R \hat{R} + B_\phi \hat{\phi} + B_z \hat{z}$

1. Azmuthal field ($B_\phi \hat{\phi}$)

Chose Concentric circle C with axis of Solenoid with radius R

$\hookrightarrow d\vec{l} = dl\, \hat{\phi}$

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$$\oint_C \vec{B}\cdot d\vec{\ell} = B_\phi \oint_C d\ell = 2\pi R B_\phi = \mu_0 I_{encl}$$

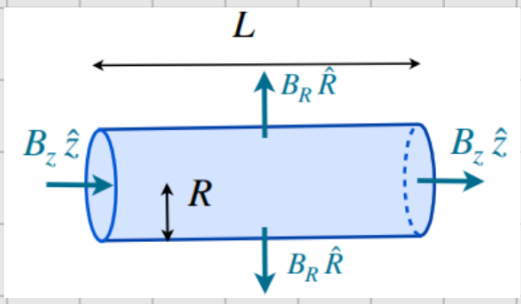
→ There is no axial current so $I_{encl} = 0$

↳ $\vec{J}\cdot\hat{z} = 0$

$\therefore\ 2\pi R B_\phi = 0 \rightarrow B_\phi = 0$, <u>No</u> azimuthal field

2. Radial field

- Consider the solenoid as a cylindrical surface



Need to evaluate $\oint\!\!\oint_S \vec{B}\cdot d\vec{S} = 0$

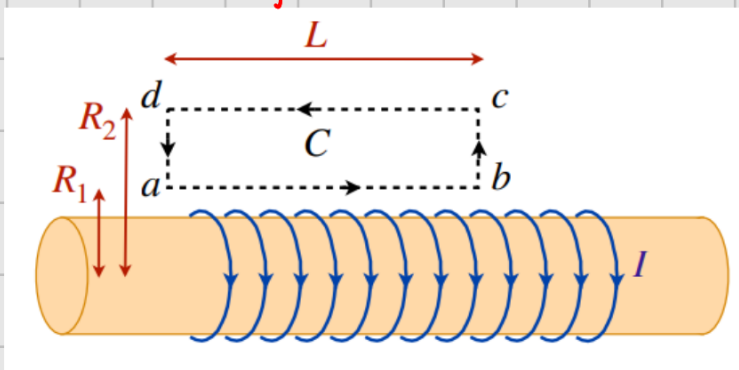
$$= \iint_{left} \vec{B}\cdot(dS(-\hat{z})) + \iint_{right} \vec{B}\cdot(dS\hat{z}) + \iint_{side} \vec{B}\cdot(dS\hat{R})$$

→ MF entering left leaves right, so these terms cancel

$$\therefore\ \oint\!\!\oint \vec{B}\cdot d\vec{S} = \iint_{side} \vec{B}\cdot(dS\hat{R}) = B_R \iint dS = B_R 2\pi R L = 0$$

$B_R = 0 \rightarrow$ <u>No</u> radial field

3. Axial field



Chose rectangular path in plane of constant ϕ outside solenoid and use $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I_{encl}$

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$$\oint_C \vec{B}\cdot d\vec{l} = \int_a^b \vec{B}\cdot dl + \int_b^c \vec{B}\cdot dl + \int_c^d \vec{B}\cdot dl + \int_d^a \vec{B}\cdot dl$$

$$= \int_0^L B_z(R_1)\,dz + \underbrace{\int_{R_1}^{R_2} B_R\,dR}_{\substack{B_R=0 \\ \text{so } \int=0}} + \int_L^0 B_z(R_2)\,dz + \underbrace{\int_{R_2}^{R_1} B_R\,dR}_{\substack{B_R=0 \\ \text{so } \int=0}}$$

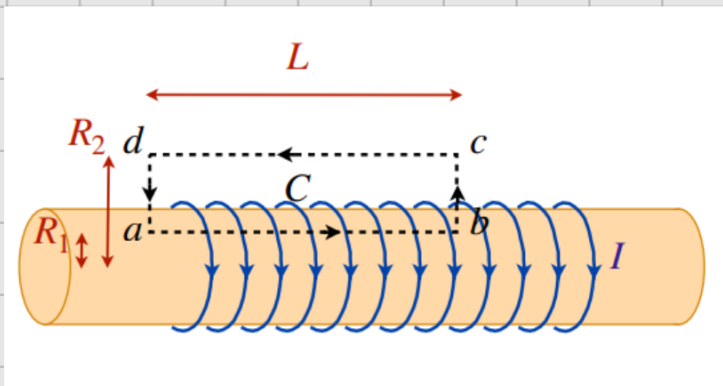
$$\rightarrow \oint \vec{B}\cdot d\vec{l} = \int_0^L B_z(R_1)\,dz + \int_L^0 B_z(R_2)\,dz$$

$$= B_z(R_1)L - B_z(R_2)L \quad = \mu_0 I_{encl}$$

As I_{encl} by rectangle $=0$, $\quad B_z(R_1)L = B_z(R_2)L$

As $R_2 \rightarrow \infty$, $B_z(R_1)=0$ for all values of R_1.
So $B_z = 0$ outside the solenoid

4. Axial Field inside



Chose rectangular path in plane of constant ϕ <u>in and out</u> solenoid and use

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\oint_C \vec{B}\cdot d\vec{l} = \int_a^b \vec{B}\cdot dl + \int_b^c \vec{B}\cdot dl + \int_c^d \vec{B}\cdot dl + \int_d^a \vec{B}\cdot dl$$

$$= \int_0^L B_z(R_1)\,dz + \int_{R_1}^{R_2} B_r\,dR + \int_L^0 B_z(R_2)\,dz + \int_{R_2}^{R_1} B_r\,dR$$

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$$= B_z(R_1)L - B_z(R_2)L$$

as $R_2 > a$, $B_z(R_2) = 0$, So

$$\oint_C \vec{B} \cdot d\vec{\ell} = B_z(R_1)L$$

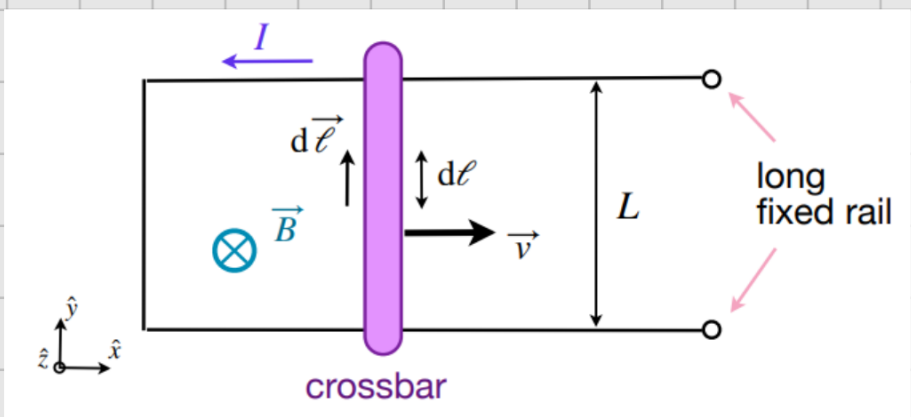
$I_{encl} = NI$, so $B_z(R_1)L = \mu_0 N I = \mu_0 n L I$

number of turns through length L

For $R < a$, $B_z(R) = \mu_0 n I$

$$\vec{B} = \mu_0 n I \hat{z}$$

Faraday's Law



Moveable crossbar

- In Δt, cross bar moves $\Delta x = v\Delta t$
- Area of circuit covered by cross bar $A = L\Delta x = Lv\Delta t$
- Magnetic flux increases by $\Delta\Phi_B = B\Delta A = BLv\Delta t$

$$\mathcal{E} = -\frac{\Delta\Phi_B}{\Delta t} = -BLv$$

//

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Using $|\mathcal{E}| = IR \rightarrow I = \frac{BLv}{R}$

Force on bar: $d\vec{F} = I\, d\vec{\ell} \times \vec{B} = I\, d\ell B\, \hat{y} \times (-\hat{z}) = -I\, d\ell B\, \hat{x}$

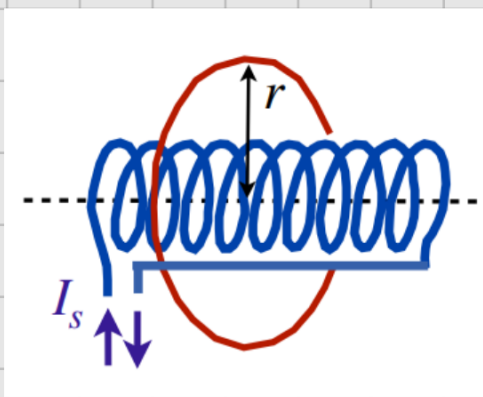
$\hookrightarrow F = -ILB\hat{x}$, so an external force $\vec{F}' = +IBL\hat{x}$ must be applied to keep constant speed v

Work done per second $P = Fv = ILBv = \frac{B^2L^2v^2}{R}$

Power dissipated through ohmic heating $P = I^2R = \frac{B^2L^2v^2}{R}$

energy conserved

Induced emf



A Solenoid with I_s and area A and n turns per metre has B:

- $\vec{B} = \vec{0}$ outside
- $\vec{B} = \mu_0 n I_s \hat{z}$ inside

For a concentric circle of wire with radius r and resistance R, the magnetic flux is:

$$\Phi_B = \iint_A \vec{B} \cdot d\vec{A} = \mu_0 n I_s A$$

As $I_s(t)$, $\vec{B}(t)$ and $\Phi_B(t)$ so

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\mu_0 \frac{NA}{L} \frac{dI_s}{dt}$$

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and a current is induced in the wire I_w.

$$I_w = \frac{\mathcal{E}}{R} = \frac{\mu_0 N A}{L R} \frac{dI_s}{dt}$$

The force on a charge $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ and emf $\mathcal{E} = \oint_{loop} \vec{E} \cdot d\vec{l}$.

$\vec{v} = 0$ so no external $\vec{B} = \vec{0}$.

For stationary loops $\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$

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