PX132: Mechanics and Relativity

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Contents

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Based off notes from the lectures given in 2007.

1 Section A: Mechanics

Classical mechanics is the study of long range forces and their effects on the behaviour and measurement of material objects. Mechanics underlies all of the human engineering and perhaps surprisingly, the same laws determine the notion of atoms to galaxies. In this guide we will look at Newton's laws and laws of conservation and see how they combine to underpin all of physics. So let's get started!

1.1 Units

Try to memorise the SI units and dimensions. They will come up in every area of physics!

Physical Quantity	**Unit Name**	**Symbol (SI base)**
Length	Meters	m
Mass	Kilograms	kg
Time	Seconds	s
Electric Current	Ampere	A
Temperature	Kelvin	K
Force	Newton	$N \rightarrow mkgs^{-1}$
Pressure	Pascal	$Pa \rightarrow Nm^{-2} = m^{-1}kgs^{-2}$
Energy	Joule	$J \rightarrow Nm = m^2kgs^{-2}$
Power	Watt	$W \rightarrow Js_{-1} = m^2kgs^{-3}$
Frequency	Hertz	$Hz \rightarrow s^{-1}$
Charge	Coulomb	$C \rightarrow As$
Electric Potential	Volt	$V \rightarrow JC^{-1} = m^2kgs^{-3}A^{-1}$

2 Notation

From now on let a, v, x and t be acceleration, velocity, position and time respectively. Usually these are a function of t so you can write $a(t)$, $v(t)$, $x(t)$. If we are dealing with motion in two or more dimensions then we use vector notation if possible, e.g. $v(\vec{t}) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$. Remember we can separate forces, velocity etc. into orthogonal directions (at right angles!) to simplify expressions.

3 Equations of Motion

Intergration is a powerful tool - consider it to be the inverse of "the rate of change of". So $a(t) = \frac{dv}{dt}$ and $\Delta v = \int_{t_0}^{t_1} adt$. Where Δ is "change of" within the limits t_0 and t_1. We are also able to write:

$$v(t) = v_0 + \int_{t_0}^{t} a(t')dt'$$

Here we have used a dummy variable t' so not to confuse the variable and the limits. But $v(t) = \frac{dx}{dt}$ so $a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$ and so we can put these together to get:

$$\Delta x = \int_{t_0}^{t_1} v_0 dt + \int_{t_0}^{t^1} \left(\int_{t_0}^{t^1} a(t')dt' \right) dt$$

From this we know that v_0 is constant and so this gives us:

$$x(t) = x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t^1} \left(\int_{t_0}^{t^1} a(t')dt' \right) dt$$

We can use these 2 equations to model classical motion of a point. (Do it for yourself!)

4 Newton's Laws

Get ready for the Big 3!

Newton's 1st Law: A body moves at constant velocity unless acted upon by a force (NB. "At rest" is also a constant velocity!)
Newton's 2nd Law: Force equals the rate of change of momentum ($\vec{p}$)

$$\vec{F} = \frac{f}{dt}n(m_i\vec{v}) = \frac{d}{dt}(\vec{p}) = \frac{d}{dt}(m_i\vec{v}) = m_i\frac{d\vec{v}}{dt} + \vec{v}\frac{dm_i}{dt}$$

If the inertial mass (m_i) is constant this will simply be

$$\vec{F} = m_i\frac{d\vec{v}}{dt} = m_i a$$

Newton's 3rd Law: Every action has an equal and opposite reaction.

$$\vec{F_1} = -\vec{F_2}$$

This is very useful in your set up of diagrams but also tells us that momentum is conserved in a closed system. An example of this is Gravity. Each of us pulls the Earth towards our centre of mass as much as the Earth pulls at the each of us.

$$\vec{F_{12}} = -\frac{Gm_1m_2}{r^2}\hat{r} = -\vec{F_{21}}$$

Where $\hat{r}$ is the unit vector ($|\hat{r}| = 1$), the direction of the force from body 1→2.

5 Applications

I've mentioned one of them already - those magic pictures called diagrams. You really should draw a sketch for each of your mechanics problems to define coordinates, forces, etc. And to show that you understand the system. Up next are the classics of standard problems to makes sure they're familiar.

5.1 Centre of Mass

Any system of bodies can be treated like a point mass localised at the systems centre of mass. Hence the total mass M can be written as $M = \Sigma_i m_i = m_1 + m_2 + m_3 + ... + m_i$ and given associated position vectors r_i with respect to our origin. The centre of mass $\vec{R}$ is given by:

$$\vec{R} = \frac{\Sigma_i m_i\vec{r_i}}{M} = \frac{\Sigma_i m_i\vec{r_i}}{\Sigma_i m_i}$$

5.2 Systems of Particles

Consider this a thought experiment - take a system of particles. In total by Newton's 3rd Law all internal forces cancel out. (Consider $\Sigma_i\Sigma_j\vec{F_{ji}} + \Sigma_i\vec{F}_{(ext)i} = \Sigma_i m_i\vec{a_i}$. Where $i \neq j$.)
So under the influence of an external force, the centre of mass of a system moves as if all the mass was concentrated there. This has some great consequences for simplifying problems. For example, take a rocket and it's ejected reaction mass as a system. The thrust is an internal force and the centre of mass does not move unless there are external forces as well.

5.3 Static Equilibrium

There is no net force acting on a body in static equilibrium. Forces balance, some to zero, there is no acceleration and hence remain in a steady state - static to the observer. Consider a suspended mass:

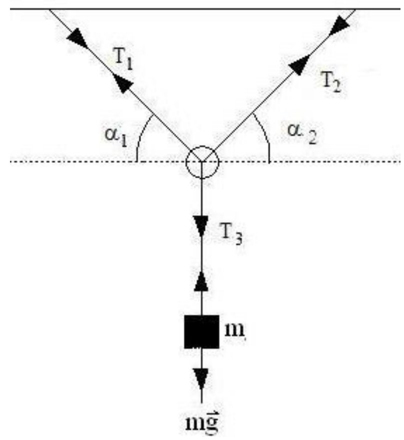

Figure 1: A suspended mass

The centre of mass will allign vertically with the point of contact. To find T_1, T_2, T_3 split the components into orthogonal directions. Looking at the point in the centre we can deduce:
In the horizontal x-direction: $T_1 cos\alpha_1 - T_2 cos\alpha_2 = 0$
In the vertical y-direction: $T_1 sin\alpha_1 + T_2 sin\alpha_2 - T_3 = 0$
In the verticle y-direction at the mass: $T_3 - mg = 0$.
From these simple trigonometric relations, in addition to extra information we can calculate the forces of the system. Remember, there is no restriction on which orthogonal directions you take. Commonly you may take forces perpendicular and parallel to a slope etc.

5.4 Static and Dynamic Friction

There are forces which oppose motion in an immersed body or motion along a surface. In general these forces are called friction, or otherwise drag, air resistance or viscous damping. They usually depend on velocity, fluid density and the materials involved.

5.4.1 Box on a plane

Define $\vec{n}$ as the normal reaction force - remember Newton's 3rd law. The plane will push back on the box with the same magnitude as the box exerts. Define the magnitude of static friction as $F_s \leq \mu_s \vec{n}$ where μ_s is the coefficient of static friction. At the point of slipping $F_s = \mu_s \vec{n}$. Similarly define the magnitude of dynamic friction as $F_k = \mu_k \vec{n}$ where μ_k is defined as the coefficient of kinetic friction and can be velocity dependent.

In general, $\mu_k \leq \mu_s$. Both $\vec{F_s}, \vec{F_k}$ act along the average line of contact - in this case, parallel to the plane. So now, on to resolving forces into orthogonal components.
$\vec{F_e}$ is the external force on the box.
$x \rightarrow \hat{i}$ is parallel to the plane.
$y \rightarrow \hat{j}$ is perpendicular to the plane.
So, $\vec{n} = (mgcos\alpha)\hat{j}$ and $\vec{F_e}cos\beta - mgsin\alpha)\hat{i}$

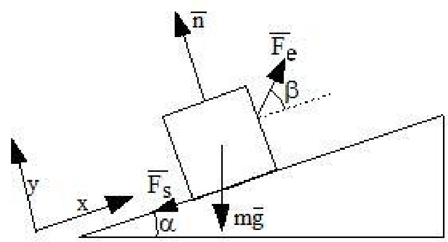

Figure 2: Box on a plane

All that's left is to remember that $F_s \leq \mu_s \vec{n}$ and to use kinetic friction appropriately. Remember if the system isn't in equilibrium it will accelerate up or down the plane, which is just Newton's 2nd Law.

5.4.2 Terminal Velocity

Picture a particle in free fall in a uniform gravitational field. We can write an equation for the drag force acting on the particle as $\vec{F_D} = -Dv^2\hat{v}$. Where D is constant and $\hat{v}$ is a unit vector in the direction of velocity. So from Newton's 2nd Law we are able to write:
$\vec{F} = m\vec{a} = m\vec{g} = m\vec{g} - \vec{F_D}$
Which rearranged is: $\vec{a} = \vec{g} - \frac{D\vec{v^2}}{M}$ and as the velocity increases, the acceleration will deminish to zero, equating to terminal velocity: $\vec{v_T} = \sqrt{\frac{\vec{g}m}{D}}$

5.5 Break Down

In general, follow the following steps:-

- Identify the magnitudes and directions of the forces - i.e draw a diagram!
- Resolve the forces into 2 (or more) orthogonal directions, choosing the directions carefully.
- Draw another diagram if necessary for each direction separately.
- Use Newton's 2nd Law to write net accelerations in the directions you choose and integrate to find velocity and positions as required.

6 Circular Motion

To change the direction of a body moving at constant speed v, we apply a force at right angles to the trajectory. Such a force is called a Centripetal Force. This instantaneous value of velocity is called the tangential velocity. Continuously applying a centripetal force gives circular motion. (See figure 3). Where ds is a line element equal to $rd\theta$.

So for tangential velocity $v = \frac{ds}{dt} = \frac{rd\theta}{dt} = r\omega$, given ω as the angular velocity. Also, centripetal acceleration: $a = \frac{v^2}{r} = r\omega^2$ and so by combining these together and using Newton's 2nd Law we can conclude that the centripetal force is:

$$F = mr\omega^2$$

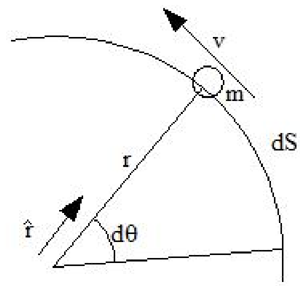

Figure 3: Circular motion of a mass

6.1 Vectors and Circular Motion

As you've realised, vectors are very important and are essential in describing complex systems. By convention we use the right hand rule (the Mathsphys Society logo! See the front cover) to associate our orthogonal unit vectors. (Note that their directions vary continuously!).

6.2 Orbital Acceleration

Recall $\vec{v} = \vec{\omega} \times \vec{r}$ but what about acceleration?

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$\vec{a} = \vec{\omega}\vec{r}(\hat{k} \times \hat{r}) + \omega v(\hat{k} \times \hat{\theta}) \qquad \Rightarrow \qquad \vec{a} = \vec{\omega} r\hat{\theta} - \omega v\hat{r}$$

The first part being the angular acceleration, increasing the speed of the circular motion, and the second term is the centripetal force directed towards the centre of the circle. As you would expect, these terms are perpendicular to eachother.

6.3 Angular Acceleration

Define $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ as the angular acceleration of a body. If $\vec{\alpha}$ is in the same direction as $\vec{\omega}$ then the orbital speed will increase. In the opposite direction it will therefore slow down. Otherwise the direction of $\vec{\omega}$ will cause the orbit plane to tilt. However you should only worry about $\vec{\omega}$ and $\vec{\alpha}$ being aligned for this module.

6.4 Angular Momentum

Remember for linear momentum $\vec{p} = m\vec{v}$. Similarly we define angular momentum to be $\vec{L} = \vec{r} \times \vec{p}$ or $\vec{L} = m\vec{r} \times \vec{v}$. For circular motion we can also use the fact that $\vec{v} = \vec{\omega} \times \vec{r}$ to say that:
$\vec{L} = m\vec{r} \times (\vec{\omega} \times \vec{r}) = m[(\vec{r}\dot{\vec{r}})\vec{\omega} - (\vec{r}\dot{\vec{\omega}})\vec{r}]$, by the vector triple product.
Using the relations we know ($\vec{r}\dot{\vec{r}} = \vec{r^2}$ and $\vec{r}\dot{\vec{\omega}} = 0$) and the right hand rule we can conclude that:

$$\vec{L} = m\vec{r^2}\vec{\omega}$$

Also using the moment of inertia, $I = m\vec{r^2}$ (which we shall see later) we can write angular momentum as:

$$\vec{L} = I\vec{\omega}$$

6.5 Torque

Changing the angular velocity of a body requires a twisting force, or torque. Take a force $\vec{F}$ on the line where point P lies $\vec{r}$ from the origin, then:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Also from Newton's 2nd Law (once again!) we can write:

$$\vec{\tau} = \vec{r} \times \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{L}}{dt}$$

6.6 Rotational Kinetic Energy

For a particle in circular motion with tangential velocity $\vec{v}$ then the kinetic energy is as usual, $E = \frac{1}{2}m\vec{v^2}$. Substituting the angular velocity ($v = r\omega$) and then the moment of inertia ($I = mr^2$) we obtain:

$$E = \frac{1}{2}mr^2\omega^2 \qquad \rightarrow \qquad E = \frac{1}{2}I\omega^2$$

6.7 Moment of Inertia

We have used moment of inertia a couple of times now. Consider two similar discs, only one of which is weighted in the middle and the other on the rim. Clearly one is harder to spin than the other, even with the same mass and shape (think about trying to push a roundabout).

The moment of inertia for a point mass is $I = mr^2$ where r is the distance from the axis of rotation. In general we must sum or integrate over the whole mass to find I:

$$I = \Sigma_i m_i (r_i)^2 \qquad \text{or} \qquad I = \int_0^M r^2 dM$$

7 Conservation of Momentum

7.1 Linear Momentum

If no external forces act on a system then the total (linear) momentum of the system is conserved.

$$\frac{d}{dt}\left(\Sigma_i m_i \vec{v_i}\right) = 0$$

7.2 Angular Momentum

If no external torque acts on a system, then the total angular momentum of the system is conserved

$$\frac{d}{dt}\left(\Sigma_i \vec{r_i} \times m_i \vec{v_i}\right) = 0$$

These statements are embodied in Newton's 2nd Law.

It is interesting to note that these conservation laws can be implied from symmetry. In homogenous space, there will be translational invariance, implying conservation of momentum. In isotropic systems (same in all direction), there is rotational invariance, implying conservation of angular momentum.

8 Work, Energy and Power

8.1 Work

Work is measured in Joules (J) and for simple constant forcing in a straight line, is equal to force times the distance travelled in the forcing direction.

In general a force can be resolved into 2 directions - parallel and perpendicular to the path. Only the parallel component changes the kinetic energy, which is work on the body. Parallel forcing which doesn't change the kinetic energy is doing work against friection, or working to gain potential energy.

The force in the direction of the infinitesimal path element at that point is $\vec{F}\dot{d\vec{s}}$. So the work done moving from a point P_1 to P_2 is:

$$W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{s}$$

Taking $\vec{v} = \frac{d\vec{s}}{dt}$ then,

$$W_{12} = m \int_{v_1}^{v_2} \frac{d\vec{v}}{dt} \cdot dt$$

Using the chain rule, $\frac{(dv^2)}{dt} = 2\vec{v} \cdot \frac{d\vec{v}}{dt}$, then

$$W_{12} = \frac{m}{2} \int_{v_1}^{v_2} \frac{d(v^2)}{dt} dt = \frac{m}{2}({v_2}^2 - {v_1}^2)$$

This is the work done on the body and is the difference in kinetic energy as expected.
Note - normal (perpendicular) forcing does not work as it never acts over a distance.

8.2 Energy

A conservative system is where work done is independent of the path taken and only on the end points. Hence $W_{12} = -W_{21}$ and for a closed path $\oint \vec{F} \cdot d\vec{s} = 0$ so energy is conserved.

Remember: The total (kinetic and potential) energy of a conservative system is constant.

8.3 Power

Power is simply "the rate at which you do work" $P = \frac{dW}{dt}$. But since $dW = \vec{F} \cdot d\vec{s}$ then:

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$$

9 Collisions

If a box on a plane wasn't stereotypical enough for you, then get ready for the colliding spheres! Consider two pointlike hard spheres colliding in the second's rest frame (see figure above). We will neglect any spin. By conservation of linear momentum: $m_1\vec{u_1} = m_1\vec{v_1} + m_2\vec{v_2}$. Resolving this into orthogonal directions we obtain:

Horizontal momentum: $m_1\vec{u_1} = m_1\vec{v_1}cos\theta_1 + m_2\vec{v_2}cos\theta_2$

Vertical momentum: $0 = m_1\vec{v_1}sin\theta_1 + m_2\vec{v_2}sin\theta_2$

And if the collision is elastic, then kinetic energy is conserved, which leads us to:

$$m_1\vec{u_1}^2 = m_1\vec{v_1}^2 + m_2\vec{v_2}^2$$

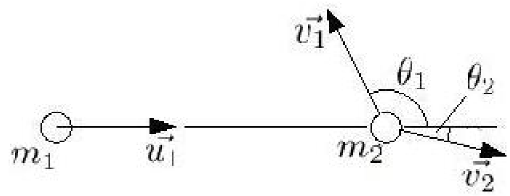

Figure 4: Colliding spheres

Otherwise for inelastic collisions we have:

$$\Delta E = m_1(\vec{v_1}^2 - \vec{u_1}^2) + m_2\vec{v_2}^2$$

We can now solve these equations simultaneously cancelling variables as required - try it! (Hint: squaring equations, and using a trigonometric identity will help). Remember: momentum is always conserved but energy only is in elastic collisions.

A special case is where the spheres stick together. This is even easier to solve as $\vec{v_1} = \vec{v_2}$ and to conserve vertical momentum, clearly $\theta_1 = \theta_2 = 0$. Similarly we can solve for collisions involving angular momentum.

10 Gravitation and Kepler's Laws

In a nutshell, using the equation for gravitational potential you can derive equations for the trajectory of multiple body systems using conservation of angular momentum etc.

10.1 Two Body Problem

Under mutual graviation we can write:

$$m_1\ddot{\vec{r_1}} = -\frac{Gm_1m_2}{r^2}\hat{r} \qquad m_2\ddot{\vec{r_2}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Solving this using the reduced mass: $\mu = \frac{1}{m_1} + \frac{1}{m_2}$ gives:

$$\mu\ddot{\vec{r}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

10.2 Kepler's Laws

Kepler's 1st Law: All planets move in an ellipse with the Sun about a focus.
Kepler's 2nd Law: A line drawn from the Sun to a planet sweeps out equal areas in equal times - this is a consequence of conservation of angular momentum. From figure 5, we are able to calculate the time period:

$$dA = \frac{1}{2}\vec{r} \times d\vec{r}$$

$$\frac{dA}{dt} = \frac{d}{dt}(\frac{1}{2}\vec{r} \times d\vec{r}) = \frac{\vec{L}}{2\mu} = constant$$

$$\frac{\vec{L}}{2\mu} = \frac{Area\ of\ ellipse}{period,\ T} = \frac{\pi ab}{T} \qquad \Rightarrow \qquad T = \frac{2\pi ab}{L}\mu$$

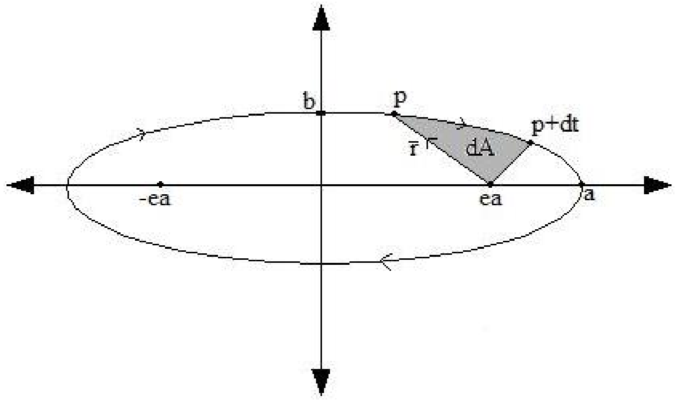

Figure 5: A diagram to of a planet moving from t to dt. Given the eccentricity $e = \sqrt{\frac{1-b^2}{a^2}}$

Which leads to -
Kelpler's 3rd Law: The square of the perioud of revolution of a planet around the Sun is proportional to the cube of the semi-major axis:

$$T^2 = 4\pi^2 \frac{\mu}{Gm_1 m_2} a^3 or T^2 \propto a^3$$

11 Section B: Special Relativity

"It is known that Maxwell's electrodynamics–as usually understood at the present time–when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise–assuming equality of relative motion in the two cases discussed–to electric currents of the same path and intensity as those produced by the electric forces in the former case."

- A. Einstein, On the Electrodynamics of Moving Bodies, June 30 1905

Special Relativity applies the principal of relativity (that all motion is relative rather than there being a well defined state of rest, proposed by Galileo) to frames in uniform relative motion. So we're just going to brush gravity, curved paths and acceleration under the carpet for now!
This section of the guide will take you through the principles and consequences of Special Relativity and hopefully provide a helpful resource for preparing for your exam.

"The hardest thing in the world to understand is income tax."

- A. Einstein

12 Before Einstein

12.1 A Quick Revision Of Newtonian Physics

You're probably sick to death of this by now, so this sub-section is just intended as a summary of the main principles of mechanics we know (or assume!) and love from classical physics.

- **Newton's First Law:** A body continues in its state of rest or uniform motion unless it is acted upon by an external force.
- **Newton's Second Law:** The rate of change of a body's momentum is equal to the total force acting on it
- **Newton's Third Law:** For every action, there is an equal and opposite reaction.

Definition: Uniform motion is motion in a straight line at constant velocity.
Notice: Newton's Second Law ($\vec{F} = m\vec{a}$) assumes that mass is a constant, which we shall see is not the case.

- Velocity is the rate of change of position in a specified direction: $v_x = \frac{dx}{dt}$
- Acceleration is the rate of change of velocity in a specified direction: $a_x = \frac{dv_x}{dt}$
- Linear momentum $\underline{p} = m\underline{v}$
- Momentum must be conserved.

- Force is the rate of change of momentum $\underline{F} = \frac{p}{dt}$
- Energy cannot be created or destroyed, so must be conserved.
- The work done by a force, F, in moving a body from x=0 to x=x1 (independent of the path chosen):

$$W = \int_{x=0}^{x=x_1} F_x dx = \int_0^v mv\, dv = \Delta E_{KE}$$

Definition: A quantity is INVARIANT if it can be considered the same in any frame of reference. Newton believed that time was invariant, but this is not in fact the case.

12.2 Galilean Transformations

Motion must ALWAYS refer to a frame of reference. We will only consider reference frames at constant velocity - inertial.

Suppose we have two frames of reference, S and S'. Suppose also that S' is moving at a constant velocity, u, in the positive x-direction, and that at time $t = 0$ their origins coincided ($O = O'$). (See Figure 6) It should be clear (treated classically) that at time $t = t$, the distance between the

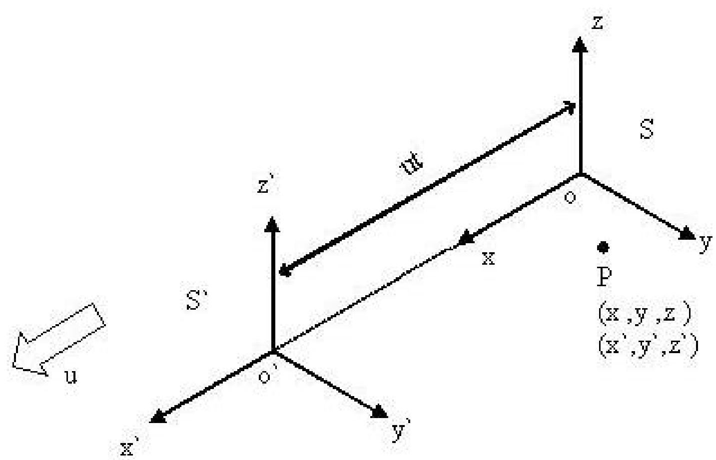

Figure 6: S and S' frames of reference

two frames is ut. So we must have that $x' = x - ut$. Since the motion is entirely in the x-direction, we have that at any time, $y = y$' and $z = z$'.
Now consider a stationary point P observed from both frames (as in Figure 6). We may relate its position relative to frame S to its position relative to S' using the above. These are the intuitive Galilean transformations:

Transform	Inverse Transform
$x' = x - ut$	$x = x' + ut$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = t$	$t = t'$

What if P had a velocity? Suppose it is moving at a constant speed in the x-direction, and its velocity relative to frame S is v (and v' relative to S'). Then simply by differentiating the top terms in the transform equations, we see that:

$$v' = \frac{d}{dt}x' = v - u \qquad v = \frac{d}{dt}v' + u$$

Unfortunately, this is no longer accurate at speeds comparible to the speed of light.
Definition: An inertial frame is a uniformly moving reference frame.
Remember: Motion must always be referred to a frame of reference.

12.3 The Michelson-Morley Experiment

In the nineteenth century, physicists believed in a stationary ether - the thought was that light must travel through some medium, like waves may travel through water or along a string etc. So, supposing there is an ether, it follows that an 'ether wind' would be induced in the laboratory by the motion of the Earth through the ether, and that this would hinder the progress of light travelling against it. In 1887, Michelson and Morley carried out an ingenious experiment to detect the hypothesised drag.
Using the configuration of mirrors shown in Figure 7, they split a beam of light into two perpendicular beams (with the half-slivered mirror) and then rejoined these beams. Now, they reasoned that the beam travelling perpendicular to the ether wind would have to travel further than the parallel beam. Thus there would be a slight delay in one of the beams when it recombined through interference with the other beam - this would result in a predicted fringe shift of about one twenty-fifth of a fringe. (NB The apparatus was free to rotate, allowing any direction relative to the ether wind.)

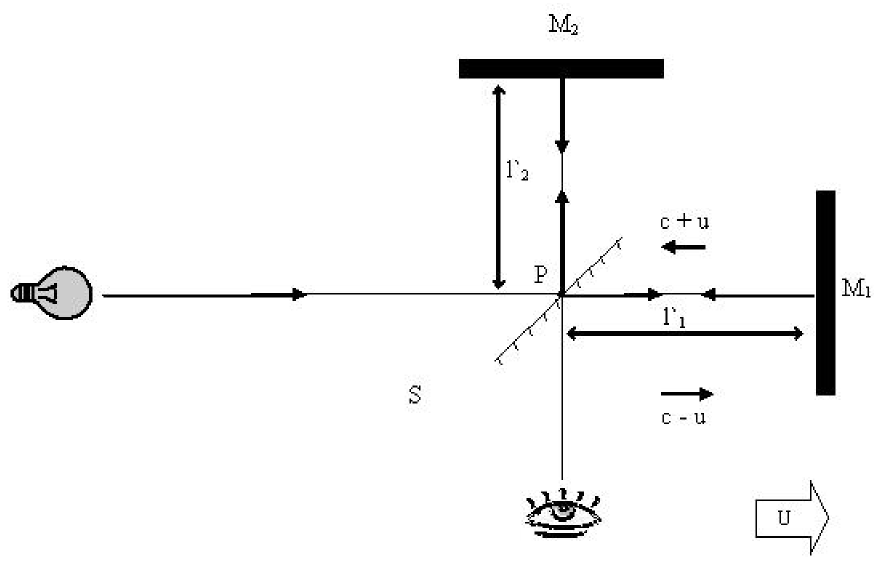

Figure 7: The Michelson-Morley Experiment

However, they did not observe any significant fringe shift, providing strong evidence against the idea of an ether.

13 After Einstein

13.1 Einsteins Postulates

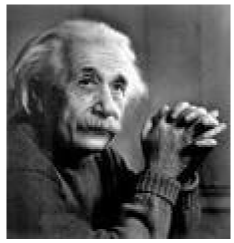

Figure 8: Einstein

You will almost certainly be asked to state these:

1. The laws of physics are the same in all inertial frames.
2. The speed of light in empty space (a vacuum) is the same in all inertial frames and is independent of the motion of its source.

13.2 Minkowski Diagrams And Simultaneous Events

Suppose lightning struck both the front and back of a train (it's a pretty unlucky train), and that both flashes appear simultaneous to an observer on the ground. To an observer on the train, however, lightning struck the rear first how can this be?

Minkowski Diagrams (Space-Time Diagrams) allow us to see what is going on much more easily.

Along the horizontal axis, we have x, the distance from a defined origin. Along the vertical axis, we have ct (notice that this has units of distance, but may be thought of as time increments). So a beam of light emitted from the origin would be represented by the line $x = ct$. Known as World Lines of Light, these are always at 45° to the axes, because they are independent of the motion of the source.

Consider two stationary observers at points A and C (see Figure 13.2). If a beam of light were emitted from point B, it would reach both observers at the same time, t_1. Now suppose that both observers are moving at a speed u in the positive x-direction. Refer to Figure 13.2 - the light beam will now reach A first (at time t_1), and C later (at time t_2). The lesson: events which are simultaneous in one inertial frame are not necessarily simultaneous in another. Time intervals between two events do not have to be the same for observers in relative motion - Relativity of Simultaneity.

13.3 What Does This Mean Spatially?

13.3.1 Lorentz Transformations

To see how the Galilean Transformations must change to accommodate Einstein's postulates, let us consider the following arrangement. Suppose there are two inertial reference frames, S and S', where S' is travelling at a speed u in the positive x-direction as measured in S (see Figure 10). (From now on, whenever you see S and S', assume this definition.)

Let us also suppose that at time $t = 0$, the origins of the two frames (O and O') coincided and that a flash of light is emitted from O. After a time t, an observer at O notices that the beam of light has reached point P, which is a distance r from O and a distance r' from O'. If we know the

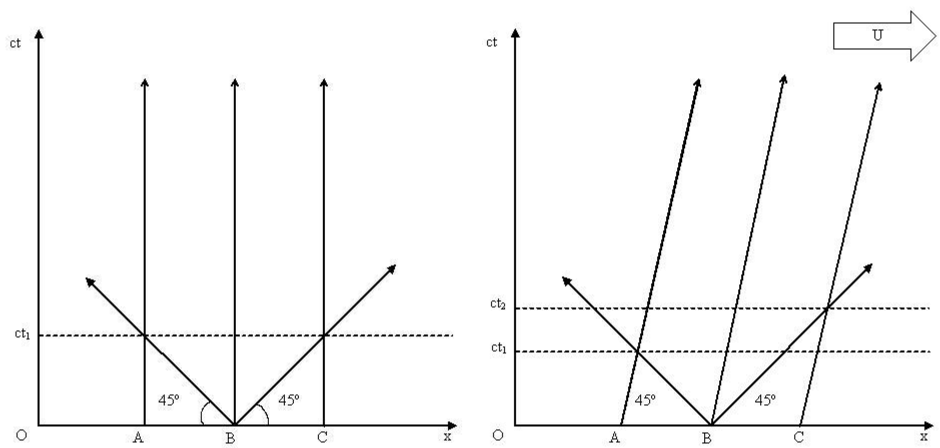

Figure 9: a)Two stationary observers, b) Two moving observers

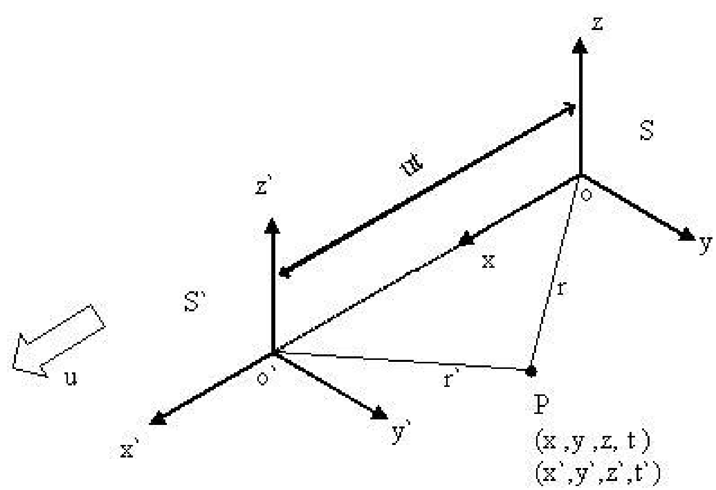

Figure 10: Inertia Frames S and S'

co-ordinates of P in frame S, how can we relate them to the co-ordinates in S'? As before, the y- and z- co-ordinates are easy: frame S' is only travelling in the x-direction, so we may assume that $y = y'$ and $z = z'$. What about the x-co-ordinate? Firstly, note that $r = ct$, where c is the speed of light. But from the Pythagorean Theorem, we also see that:

$$r^2 = x^2 + y^2 + z^2$$

So we must have that:

$$x^2 + y^2 + z^2 = c^2t^2$$

Now, we cannot assume that the same amount of time has passed in frame S' (as S and S' are in relative motion). So let us say that a time t' has passed. Then applying the same principles as above, we have that:

$$(x')^2 + (y')^2 + (z')^2 = (r')^2 = c^2(t')^2$$

How do we proceed? Well, we know that for $u \ll c$, we must be left with our original Galilean

Transformations (what we observe at small velocities!). So let us assume that:

$$x'(x,t) = \gamma(x - ut)$$

for some γ which increases in significance as u approaches c, but approaches 1 as u is decreased, and that:

$$t'(x,t) = \alpha(t - \beta x)$$

for some α with the same conditions as γ and some β which approaches 0 as u is decreased. Substituting $x'(x,t) = \gamma(x-ut)$ and $t'(x,t) = \alpha(t-\beta x)$ into $(x')^2 + (y')^2 + (z')^2 = (r')^2 = c^2(t')^2$ and trawling through some messy algebra (the reader should feel free to try this an exercise!), we arrive at our destination: the Lorentz Transformations, which are as follows:

$$x' = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

and, naturally, the inverse Lorentz Transformations:

$$x = \gamma(x' + ut')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(1 - \frac{ux'}{c^2}\right)$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Remember: u is the velocity of S' as measured in S.
Does γ behave as we dictated? Plotting it as a function of u (see Figure 11), we see that its value only really starts to grow from 1 at around 0.5c this is 1.510^8 m/s! Which is way beyond what we would normally observe.

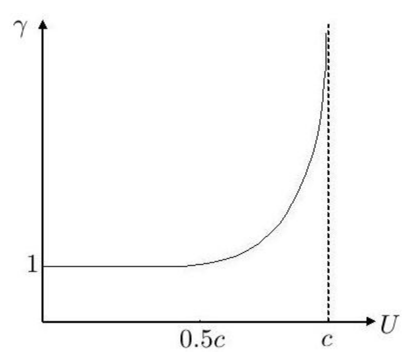

Figure 11: Graph of γ

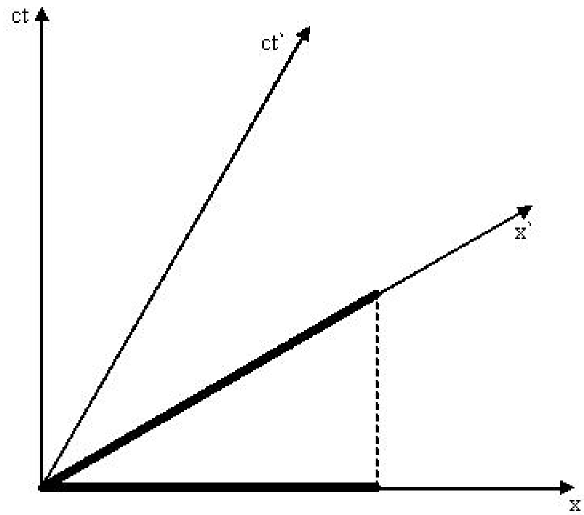

Figure 12: Length Contraction on a Minkowski Diagram TAKE OUT SEEN BETTER LATER ON!!!!!!!!!!!!!!!

13.3.2 Length Contraction

The hint is in the title - the length of a body in motion relative to the observer will be measured to be shorter than if it were at rest. To see this, let us simply consider a straight bar observed from frames S and S' (see Figure 12). The bar lies along the x-axis, with its ends at x_1 and x_2. The bar is at rest relative to S, so we may define the proper length, L_o, to be:

$$L_o = x_2 - x_1$$

What would an observer in S' measure? Remembering that time is always changing, we must measure the position of each of its end points (now located at x_1' and x_2') in one instantaneous measurement at time $t = t'$. Using the inverse Lorentz Transformations, we see that:

$$x_1 = \gamma(x_1' + ut') \qquad x_2 = \gamma(x_2' + ut')$$
$$\Rightarrow x_2 - x_1 = L_0 = \gamma(x_2' - x_1')$$

But $x_2' - x_1' = L'$, where L' is the length of the bar measured in S'. So we have that:

$$L_0 = \gamma L' \qquad \text{or} \qquad L' = \frac{L_0}{\gamma}$$

Remember: Length is greatest in the object's inertial frame.

13.4 What Does This Mean For Time?

13.4.1 Time Dilation

This is a very similar calculation to that for length contraction (but with the opposite result). There is no absolute time. Events may be judged to be at different times in different frames of reference.

Suppose an observer in frame S at position x_0 measures the time for Event 1 to be t_1 and for Event 2 to be t_2. This is the rest frame, and we define the proper time, Δt, to be:

$$\Delta t = t_2 - t_1$$

This much is obvious, but what is the time difference $\Delta t'$ in S', which is travelling at a velocity u relative to S? Let us say that Event 1 occurs at time $t = t'_1$ and Event 2 at time $t = t'_2$. Then applying the Lorentz Transformations, we see that:

$$t'_1 = \gamma\left(t_1 - \frac{ux_0}{c^2}\right) \qquad \text{and} \qquad t'_2 = \gamma\left(t_2 - \frac{ux_0}{c^2}\right)$$

$$\Rightarrow t'_2 - t'_1 = \gamma(t_2 - t_1)$$

$$\Rightarrow \Delta t' = \gamma\Delta t \qquad \text{or} \qquad \Delta t = \frac{\Delta t'}{\gamma}$$

Aside: The travelling light clock is a great example for length contraction and time dilation - look it up!
Remember: The time difference between two events is smaller in the rest frame of the events than in any other frame in relative motion.

13.5 So can we construct a Minkowski Diagram?

Suppose frame S' (co-ordinates (x', ct')) is moving at a speed u relative to frame S (co-ordinates (x, t)). The world line of an observer in S' is the ct' axis. From the Lorentz transformations we can see that this must be the line $ct = (\frac{c}{u})x$. By analogy to $y = mx + c$ we can see that the ct' axis has gradient $\frac{c}{u}$ on the (x, ct) grid.

Similarly, the x' axis is the line $ct' = 0$ so must have the gradient $ct = (\frac{u}{c})x$ Remember figure 5b? (LOOK UP/CHANGE TO THE RIGHT ONE), world lines parallel to the ct' axis are simultaneous in the S' frame - physics is the same in all inertial frames.

Fact: $(ct')^2 - (x)'^2 = (ct)^2 - x^2 = s^2$. s^2 is invariant - it is the same for all observers. We can use this and length contraction calculations to calibrate the x' and ct' axis. The intersection of x'-axis with $(x')^2 - (ct)^2 = 1$ defines $x = 1$.

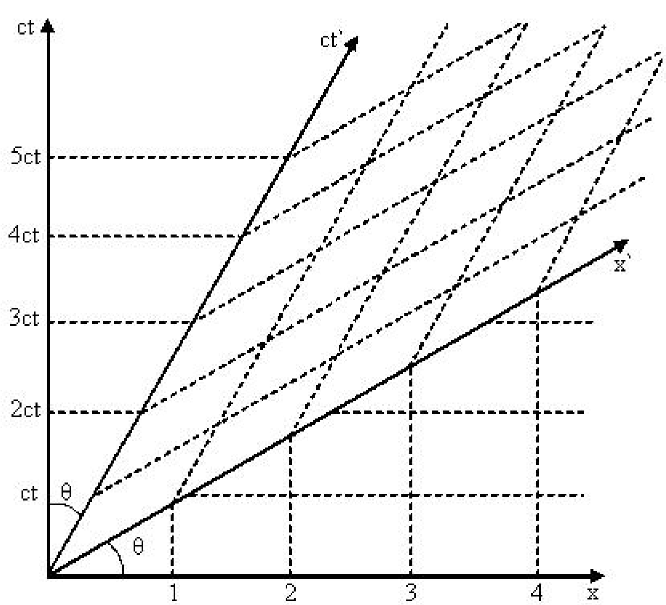

Figure 13: Minkowski diagram calibration

INSERT DIAGRAM- Sort the correct diagrams in each place
Time dilation can be mapped in the same way.

13.6 What Does This Mean For Velocity?

13.6.1 Lorentz Transformations Of Velocity

Let us return to our S and S' frames, and consider an object moving in the positive x-direction with speed v_x relative to S (and $v'_{x'}$ relative to S') - as in Figure 14. How can we relate v_x and $v'_{x'}$? Firstly, note from the Lorentz Transformations that:

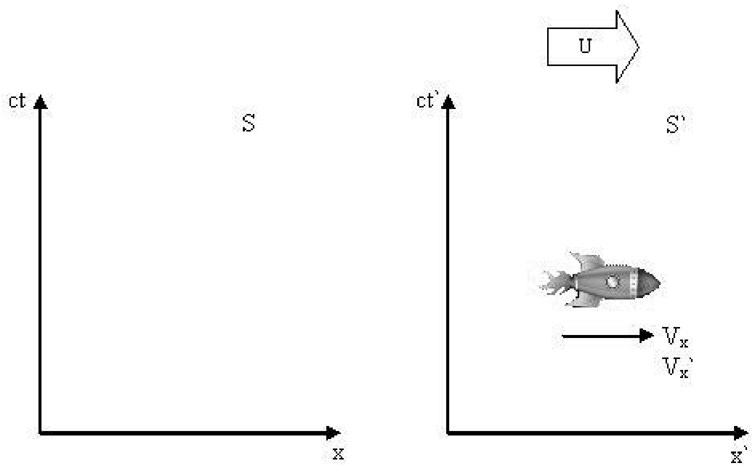

Figure 14: Velocity in S and S' Frames

$$\Delta x' = \gamma(\Delta x - u\Delta t) \qquad \Delta t' = \gamma\left(\Delta t - \frac{u\Delta x}{c^2}\right)$$

Also, by definition:

$$v_x = \frac{dx}{dt} \qquad v'_x = \frac{dx'}{dt}$$

and then please grit your teeth through this abuse of mathematics! Using $\Delta x' = \gamma(\Delta x - u\Delta t)$:

$$dx' = \gamma(dx - udt)$$

$$\Rightarrow dx' = \gamma\left(\frac{dx}{dt} - u\frac{dt}{dt}\right)dt$$

$$dx' = \gamma(v_x - u)dt$$

And using $\Delta t' = \gamma\left(\Delta t - \frac{u\Delta x}{c^2}\right)$:

$$dt' = \gamma\left(dt - \frac{udx}{c^2}\right)$$

$$dt' = \gamma\left(\frac{dt}{dt} - \frac{u}{c^2}\frac{dx}{dt}\right)dt$$

$$dt' = \gamma\left(1 - \frac{uv_x}{c^2}\right)dt$$

So:

$$v'_{x'} = \frac{dx'}{dt'} = \frac{\gamma(v_x - u)dt}{\gamma(1 - \frac{uv_x}{c^2})dt}$$

$$\therefore v'_{x'} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \qquad \text{or} \qquad v_x = \frac{v'_{x'} + u}{1 + \frac{uv'_{x'}}{c^2}}$$

Remember: It's easy to get confused here - u is the speed of S' as measured in S; v_x is the speed of the object relative to S; $v'_{x'}$ is the speed of the object relative to S'.
Notice: Try 'plugging in' v_x=c. What is $v'_{x'}$?

13.6.2 The Relativistic Doppler Effect

You should be familiar with the Doppler Effect, most easily observed in the form of red or blue shifts from stars moving relative to the Earth (see Figure 15). But surely there are relativistic effects here? Let's work it through.

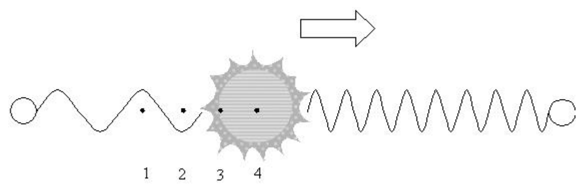

Figure 15: Doppler Shift

Call the rest frame of the star S, and the rest frame of the Earth S' (we shall assume that the star is moving away from Earth, thus S from S', at a relative speed u in the positive x-direction). In frame S, the light emitted is of frequency F_0 and period T_0. This is observed in S' as frequency F' and period T'.
During one period, a time $\Delta t' = \gamma T_0$ passes in frame S' BUT - the star is receding during this time, moving a distance of $\Delta x' = \gamma u T_0$, observed on Earth. This delays the light by:

$$\frac{\Delta x'}{c} = \frac{\gamma u T_0}{c}$$

Thus the period T' observed in S' (Earth) is:

$$T' = \gamma T_0 (1 + \frac{u}{c})$$

$$T' = \frac{T_0(1 + \frac{u}{c})}{[(1 + \frac{u}{c})(1 - \frac{u}{c})]^{1/2}}$$

$$T' = T_0 \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}}$$

Then noting that $F' = \frac{1}{T'}$ and $F_0 = \frac{1}{T_0}$, we conclude that:

$$F' = f_0 \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}$$

Remember: This equation is for a receding star. If the star is approaching, we simply have $F' = f_0\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$.
Note: Often $\frac{u}{c}$ is replaced by β - so watch out!

13.7 Relativistic mass and momentum

In this section, we consider linear momentum and mass (see section 3.8 for more on mass-energy). Can we find an expression for relativistic mass and momentum? Let us make two sensible assumptions: a) Relativistic mass is conserved in all inertial reference frames; and b) Linear momentum is conserved in all reference frames.

Definition: The rest mass of a body, m_0, is simply that - its mass when it is not in motion. The closer to the speed of light a body is travelling, the heavier it appears to be. For clarity, in this section we shall say that a body has mass $m(u)$ if it is travelling at a speed u.

Now, suppose we have two identical bodies, each of rest mass m_0, each travelling towards the other at a speed u. They collide inelasticly and stick together. We need to consider this collision in two frames - frame S will observe the situation described, and frame S' will be the rest frame of body 1 (see Figure 16 for full description).

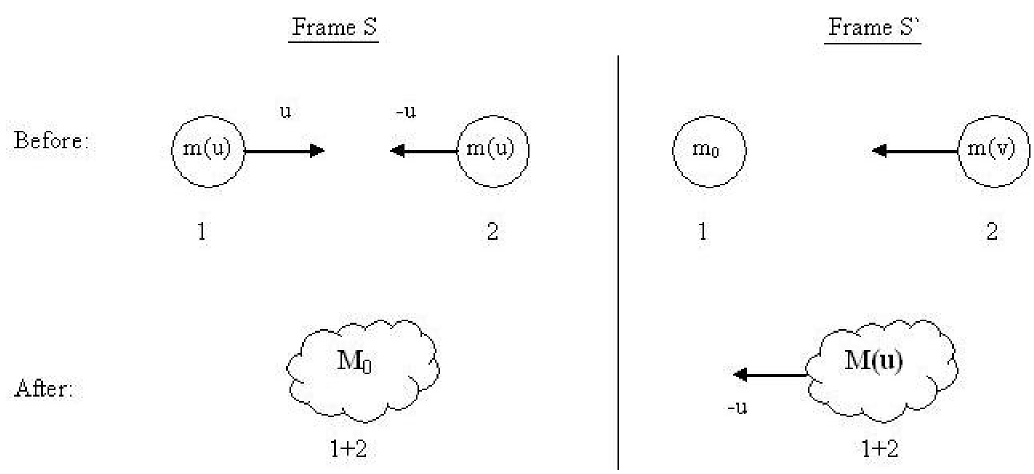

Figure 16:

First thing's first: what is v? (NB We need to be very careful with signs!) We know that in frame S, the speed of body 2 is $-u$. So let's apply the Lorentz Transformation for velocity to find it in frame S:

$$v'_{x'} = \frac{v_x - u}{(1 - \frac{uv_x}{c^2})}$$

But here, $v'_{x'} = -v$ and $v_x = -u$ (warned you about the signs!). So:

$$-v = \frac{-u - u}{1 - \frac{u(-u)}{c^2}}$$

$$\Rightarrow v = \frac{2u}{1 + \frac{u^2}{c^2}}$$

Now we may continue. Applying conservation of relativistic mass, we see that:
Frame S : $M_0 = 2m(u)$
Frame S' : $M(u) = m_0 + m(v)$
Then applying conservation of linear momentum to frame S', we see that:

$$-M(u)u = -m(v)v$$

This is enough information. Try the algebra yourself (remember that $\frac{M(u)}{M_0} = \frac{m(u)}{m_0}$ must hold as variation of mass with speed must be universal) - you should find that:

$$m(u) = \gamma(u)m_0 \qquad \Rightarrow \qquad m = \gamma m_0$$

where m is the relativistic mass of the body. It should be easy to see that relativistic momentum, $\underline{p}_{rel}$, is defined as:

$$\underline{p}_{rel} = \gamma m_0 \underline{v}$$

13.8 Force and Newton's 2nd Law

We have two major problems when considering relativistic forces:

- Acceleration is not invariant (it is in Newtonian mechanics).
- Force is not usually in the same direction as the resultant acceleration (it is parallel in Newtonian mechanics).

Thus there is not much we can do at this stage! We still have the relationship: $\underline{F} = \frac{dp}{dt}$ (Newton's 2nd Law) but what is this now?

$$\underline{F}_{rel} = \frac{d}{dt}(\underline{p}_{rel}) = \frac{d}{dt}(\gamma m_0 \underline{v})$$

$$\Rightarrow \underline{F}_{rel} = \gamma m_0 \frac{d\underline{v}}{dt} + m_0 \underline{v} \frac{d\gamma}{dt}$$

... as both γ and $\underline{v}$ depend on time.

13.9 Energy And Mass

Remember the good old days where $K.E = \frac{1}{2}m_0 v^2$? Well, that's no longer! Here's an idea to make Newton turn in his grave!
The Principle of Mass-Energy: Mass may be created or destroyed, but at the cost of an equivalent amount of energy either vanishing or appearing respectively.

This must mean that we can find a relationship between mass and energy. Let's consider a body (initially at rest) which is being acted on by a force in the positive x-direction. We know from Newtonian mechanics that *work done = change in kinetic energy* (E_{KE}), ie:

$$E_{KE} = \int_0^v \frac{d\underline{p}}{dt} dx = \int_0^v \frac{dx}{dt} d\underline{p} = \int_0^v v d\underline{p}$$

We need to integrate this by parts, as follows:

$$E_{KE} = pv - \int_0^v p \frac{dv}{dp} dp = pv - \int_0^v p dv$$

But $\underline{p}_{rel} = \gamma m_0 \underline{v}$. So:

$$E_{KE} = \gamma m_0 v^2 - \int_0^v \gamma m_0 v dv = \gamma m_0 v^2 - \left[-\frac{m_0 c^2}{\gamma} \right]_0^v$$

$$E_{KE} = \gamma m_0 v^2 + \frac{m_0 c^2}{\gamma} - m_0$$

$$E_{KE} = \gamma m_0 \left[v^2 + c^2 (1 - \frac{v^2}{c^2}) \right] - m_0 c^2$$

$$E_{KE} = \gamma m_0(v^2 + c^2 - v^2) - m_0c^2$$
$$E_{KE} = \gamma m_0c^2 - m_0c^2$$
$$\Rightarrow E_{KE} = (\gamma - 1)m_0c^2$$

In fact, m_0c^2 is the rest mass energy, E_0. So we have that the total energy, E_{tot}, is:

$$E_{tot} = E_{KE} + m_0c^2$$

$$\Rightarrow E_{tot} = \gamma m_0c^2$$

Note: You may be wondering about potential energy - if you want this answering, take the 4th Year course in General Relativity! You do not need to worry about it for this course.
Let us have another look at relativistic mass. We have that:

$$m = \gamma m_0$$

Note: E_0^2 is invariant. So:

$$m_0^2 = m^2(1 - \frac{v^2}{c^2})$$
$$\Rightarrow m_0^2c^2 = m^2c^2 - m^2v^2$$
$$\Rightarrow \frac{E_0^2}{c^2} = \frac{E^2}{c^2} - p^2$$

Therefore the Relatvistic Energy is:

$$E^2 = E_0^2 + p^2c^2 = m_0^2c^4 + p^2c^2$$

13.9.1 Massless particles

Classically a particle must possess mass to have momentum and kinetic energy. However, what about photons travelling at c? Energy and momentum would be infinite! Instead consider the relativistic energy, $E^2 = m_0^2c^4 + p^2c^2$, with $m_0 = 0$ and so $E = pc$. According to Einstein, massless particles still have energy and momentum.

$$p = \frac{h}{\lambda} \qquad c = \lambda f \qquad and \qquad E = hf$$

14 Cheat Sheet

This is by no means everything you need to know, but hopefully a helpful summary of equations you'll need. Our best advice is to practice exam papers and go through worked examples. A travelling light clock, muon lifetimes and the train paradox are great places to start.Good luck with the exam!

Newtonian	**Relativistic**
Galilean Transformations: $x' = x - ut$ $y' = y$ $z' = z$ $t' = t$	Lorentz Transformations: $x' = \gamma(x - ut)$ $y' = y$ $z' = z$ $t' = \gamma(t - \frac{ux}{c^2})$
	$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
$v' = v - u$	$v'_{x'} = \frac{v_x - u}{1-\frac{uv_x}{c^2}}$
Mass invariant	$m(u) = \gamma(u)m_0$
Length $= x_2 - x_1$	$L_0 = x_2 - x_1$ but $L' = \frac{L_0}{\gamma}$
Time invariant, $\Delta t = t_2 - t_1$	$\Delta t' = \gamma \Delta t$
Momentum: $\underline{p} = m\underline{v}$ Force $\underline{F} = \frac{d}{dt}(m\underline{v})$ $= m\underline{a}$	$\underline{p}_{rel} = \gamma m_0 \underline{v}$ $\underline{F}_{rel} = \frac{d}{dt}(\underline{p}_{rel})$ $= \gamma m_0 \frac{d\underline{v}}{dt} + m_0 \underline{v} \frac{d\gamma}{dt}$
$E_{KE} = \frac{1}{2}mv^2$	$E_{KE} = (\gamma - 1)m_0c^2$ Rest Mass Energy $= m_0c^2$ $E_{tot} = \gamma m_0 c^2$
Doppler effect (receeding source): $F' = F_0(1 - \frac{u}{c})$	$F' = F_0\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}}$ $F' = \frac{1}{T'}$; $F_0 = \frac{1}{T_0}$
	$E^2 = E_0^2 + p^2c^2$ $= m_0^2c^4 + p^2c^2$ E_0^2 is invariant
	$c^2t^2 - x^2 = l^2$ l^2 is invariant