MA131A/MA137

Analysis I/Mathematical Analysis Revision Guide

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Introduction

This revision guide for MA131A Analysis I/MA137 Mathematical Analysis has been designed as an aid to revision, not a substitute for it. Analysis I is a hard course to get your head around; once the ideas click in you head, however, it does become easier. Hopefully this guide should clarify any points of confusion you may have, and convince you that there's not quite as much to the course as the large pile of workbooks and notes you have amassed might suggest. For further practice and reference, the questions in R. P. Burn's *Numbers and Functions* are invaluable, and indeed one of the principal sources of this revision guide.

Beware of the proofs provided in this guide: where proofs are included, they are at best sketches of how such a proof might go – they are intended as a aid to remembering the key steps of the proof, not something to be memorised parrot-fashion. An excellent technique of preparing for the exam is simply to read through the revision guide until you come to the first proof, cover it up, and try and prove it without looking. If you get stuck, then (and only then) you should look at the first step in the proof, take it as a hint and try and prove the rest; then repeat this procedure until you have worked through the entire guide.

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1 Numbers and Inequalities

Analysis is the result of two hundred years' work to put calculus on firm footings, and to extend it beyond what could possibly have been imagined in the 1820s, when Cauchy and others started the transformation. First, we recap some key properties of numbers and some key equations and inequalities which are useful over and over again.

The essential difference between calculus and analysis is *proof*: in analysis, we start from the bottom and prove our way up. One key way to prove is that of induction, which you should be familiar with from MA132 Foundations:

Principle of Mathematical Induction. A proposition $P(n)$ relating to a natural number n is valid for all natural numbers n if

1. $P(1)$ is true, i.e. the proposition is valid for $n = 1$; and
2. $P(n) \implies P(n+1)$, i.e. the proposition for n implies the proposition for $n+1$.

Note that in analysis, the natural numbers do ***not*** include zero: i.e. $0 \notin \mathbb{N}$. That said, we could start the induction at 0, 2 or any other number, and it would still work.

Induction is one way of proving the geometric progression formula from A-level. (Try it!) We reproduce the formula here for reference:

Proposition 1.1 (Geometric Progression)**.** Provided $x \neq 1$, $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for all n.

Beware of the proofs in this revision guide: they are very short – please read the disclaimer on the inside front page.

1.1 Inequalities

Vital in analysis is the use of inequalities - relations involving one of $<, \leq, >, \geq$. There are two types of inequalities: those which are true for some values of x, for example $x^2 > 9$ if and only if $x > 3$ or $x < -3$; and those which are always true, such as $x^2 > -1$ is true for all $x \in \mathbb{R}$.

To "solve" an inequality, we manipulate it a bit like we would an equation. However, we must be careful: while adding a number to each side preserves the inequality, and multiplying both sides by a positive number preserves the inequality, multiplying by a ***negative*** number ***reverses*** the inequality: that is, $x > 2$ implies $-x < -2$, not $-x > -2$. So whenever an inequality involves products, quotients, or modulus signs, we must often consider separate cases; this is called ***case analysis***.

We will give an example of case analysis after we formally define the absolute value function:

Definition 1.2. Define $|\cdot| : \mathbb{R} \to \mathbb{R}$ by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Proposition 1.3. For any $x, y \in \mathbb{R}$, $\big||x|\big| = |x|$, $|xy| = |x||y|$, and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

We can use this to express $-c < x < c$ as $|x| < c$. Similarly, we can express $a - c < x < a + c$ as $|x - a| < c$, and we can think of this as "the distance between x and a is less than c".

Example. To solve $|x-3| + |7+x| > 16$, note that $|x-3|$ will change behaviour at 3, and $|7+x|$ will change behaviour at -7, so we consider three cases:

When $x < -7$, we have $(3-x) + (-7-x) > 16 \iff -4-2x > 16 \iff 2x < -20 \iff x < -10$, so it is true for $x < -10$.

When $-7 < x < 3$, the inequality becomes $(3-x) + (7+x) > 16 \iff 10 > 16$, which is false.

When $x > 3$, we get $(x-3) + (7+x) > 16 \iff 2x + 4 > 16 \iff x > 6$, so it is true for $x > 6$.

Thus the inequality holds for $x < -10$ or $x > 6$.

Note: we use $\iff$ to make sure we get the entire solution set, i.e. we can run the argument backwards as well as forwards.

One very useful inequality that holds for all numbers x and y is the Triangle Inequality.

Theorem 1.4 (Triangle Inequality)**.** For any $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$. *(Proof: square both sides.)*

One reason we call this the triangle inequality is because if we substitute $x = a - b$, $y = b - c$, then we get $|a-c| \leq |a-b| + |b-c|$, so the distance from a to c is shorter if you go direct rather than via b.

Definition 1.5. Suppose we have a list of n positive numbers $a_1, a_2, \dots, a_n$. We define
Arithmetic Mean $= \frac{a_1+a_2+\dots+a_n}{n}$ and
Geometric Mean $= \sqrt[n]{a_1a_2a_3\dots a_n}$

Another useful inequality is Bernoulli's inequality:

Theorem 1.6 (Bernoulli's Inequality)**.** When $x > -1$ and $n \in \mathbb{N}$, $(1+x)^n \geq 1+nx$.

Theorem 1.7 (Power Rule)**.** If $x, y \in \mathbb{R}$, $x, y > 0$ then, for each $n \in \mathbb{N}$, $x < y \iff x^n < y^n$

Proof. By induction we prove that $x < y \implies x^n < y^n$ using the transitivity of inequalities in the induction step. The converse can be proven by the use of the contrapositive rule (which you should meet in Foundations) and induction as in the first part. □

2 Sequences

In some sense, analysis is the study of the infinite. However, infinity is not always a nice mistress. Our first way of "taming" the infinite is to consider *sequences* of numbers. A sequence of (real[1]) numbers is just a list of real numbers put in a definite order, that is

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, a_4, \dots).$$

To put it another way, a sequence assigns to each natural number n a real number a_n; thus a sequence is a function $a\colon \mathbb{N} \to \mathbb{R}$ where we write a_n instead of $a(n)$. (Sometimes the sequence starts at a_0; it doesn't really matter, as long as it's clear.)

Note that the n is a dummy variable: $(a_n)_{n=1}^{\infty}$ is the same thing as $(a_j)_{j=1}^{\infty}$ or $(a_\lambda)_{\lambda=1}^{\infty}$. When it is clear what range n should have, we often drop the suffixes and write (a_n) for $(a_n)_{n=1}^{\infty}$.

Not all sequences are useful or interesting, so we have various classes of sequences ("things which sequences can do") so that we can talk about those which *do* interest us with ease.

2.1 Monotonicity and Boundedness

Definition 2.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence. We say that:
1. (a_n) is *increasing* if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$;
2. (a_n) is *strictly increasing* if $a_{n+1} > a_n$ for every $n \in \mathbb{N}$;
3. (a_n) is *decreasing* if $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$;
4. (a_n) is *strictly decreasing* if $a_{n+1} < a_n$ for every $n \in \mathbb{N}$;
5. (a_n) is *monotonic* if it is either increasing or decreasing or both[2];
6. (a_n) is *non-monotonic* if it is neither increasing nor decreasing.

Note that since $a_{n+1} > a_n$ certainly implies that $a_{n+1} \geq a_n$, if a sequence is strictly increasing it is automatically increasing as well. Similarly, if a sequence is strictly decreasing it is automatically decreasing as well. A sequence cannot be *both* strictly increasing and strictly decreasing. However, a sequence *can* be both increasing and decreasing: this happens if and only if (a_n) is a constant sequence – that is, $a_n = c$ for every n and some constant c – since $a_{n+1} = a_n$ is equivalent to $a_{n+1} \geq a_n$ and $a_{n+1} \leq a_n$.

The word "monotonic" is simply a catch-all word to indicate that the sequence is always doing the same thing (it is "monotonous") – namely that it is increasing or decreasing (or both). Thus constant sequences are monotonic, increasing sequences are monotonic, decreasing sequences are monotonic. When the sequence oscillates, that is when it is *not* monotonic, we simply call it non-monotonic.

Example.
1. The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is strictly decreasing, and hence decreasing, and hence monotonic.
2. The sequence $(1, 1, 2, 2, 3, 3, \dots)$ is increasing (but not strictly increasing), and hence monotonic.
3. The sequence $(\sin n)_{n=1}^{\infty}$ is neither increasing nor decreasing; hence it is non-monotonic.

[1] In this course all the sequences will consist of real numbers; however, in subsequent courses you will consider sequences of complex numbers, sequences of vectors, even sequences of sequences and sequences of functions. This corresponds simply to changing the domain of the sequence function $a\colon \mathbb{N} \to \mathbb{R}$ to be $a\colon \mathbb{N} \to \mathbb{C}$ etc.

[2] The mathematical sense of the word "or" is inclusive or, that is either one or other or both.

Definition 2.2. Let $(a_n)_{n=1}^{\infty}$ be a sequence. We say that:

1. U is an ***upper bound*** for (a_n) if $a_n \leq U$ for every $n \in \mathbb{N}$;
2. (a_n) is ***bounded above*** if it has an upper bound;
3. L is a ***lower bound*** for (a_n) if $a_n \geq L$ for every $n \in \mathbb{N}$;
4. (a_n) is ***bounded below*** if it has a lower bound;
5. (a_n) is ***bounded*** if it is bounded above and below, that is if it has both an upper bound and a lower bound.

Note that we don't care if we have the best possible upper bound – if U is an upper bound for a sequence, then so is any number greater than U. Similarly, if L is a lower bound for a sequence, then so is any number less than L.

Example.
1. The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is bounded, since $0 < \frac{1}{n} \leq 1$ for every $n \in \mathbb{N}$.
2. The sequence $(n)_{n=1}^{\infty}$ is bounded below but not bounded above, since $n \geq 1$ for every $n \in \mathbb{N}$, but for any C there is an n such that $n > C$, so no possible upper bound can work.

2.2 Limits of Sequences

Our main reason for working with sequences is to investigate what can happen to the terms of a sequence as you send n to ∞: does it oscillate, settle down, shoot off to infinity, or maybe something else? Often a sequence seems to be "going somewhere", and we try and encapsulate this in the various notions of a ***limit*** of a sequence.

2.2.1 Tending to Infinity

We first define a sequence tending to infinity[3]:

Definition 2.3. A sequence (a_n) ***tends to infinity***[4] if, for every $C > 0$ there exists $N \in \mathbb{N}$ such that $a_n > C$ for all $n > N$.

There are many shorthands to avoid always writing "(a_n) tends to infinity": $(a_n) \to \infty$ as $n \to \infty$ and $\lim_{n \to \infty} a_n = \infty$ are two that we will use (though we often omit the "as $n \to \infty$" out of laziness).

Example.
1. $(n^2) \to \infty$: given $C > 0$, let $N > \sqrt{C}$; then $n > N \implies n > \sqrt{C} \implies n^2 > C$.
2. $(\log n) \to \infty$: given $C > 0$, let $N > e^C$; then $n > N \implies n > e^C \implies \log n > C$.

How do you figure out what N should be? Very often, it is easiest to work backwards from what you want: so in the example of (n^2), we can see that (as $n > 0$)

$$n^2 > C \iff \sqrt{n^2} > \sqrt{C} \iff n > \sqrt{C},$$

so we pick some natural number $N > \sqrt{C}$ and then $n > N$ gives us $n^2 > C$ by working the sequence of implications backwards (which we can do because they are $\iff$).

Note carefully the following:

> A sequence which tends to infinity is not bounded above; put another way, any sequence which is bounded above cannot tend to infinity: just use the upper bound as your C; then the terms will never exceed it, yet you want them all to exceed it after a certain point. However, not every sequence which is not bounded above tends to infinity: for example, $(-1)^n n = (-1, 2, -3, 4, -5, 6, \dots)$ is unbounded but does not tend to infinity. (If you want an example that is still bounded below, try $(1, 0, 2, 0, 3, 0, \dots)$.)
>
> A sequence which tends to infinity need not be increasing either: $(2, 1, 4, 3, 6, 5, \dots)$ tends to infinity but is not increasing. In addition, an increasing sequence does not necessarily tend to infinity: the sequence $\left(\frac{n}{n+1}\right) = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$ is increasing, but does not tend to infinity, because it is bounded above. However, an increasing sequence that is not bounded above ***does*** tend to infinity.

[3] Note that we do *not* define "infinity"; we only define "tends to infinity" as one unit.

[4] We can write this very concisely in symbols: $\forall C > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies a_n > C$. While this is useful shorthand, it sometimes obscures the meaning of the words.

Just as we can have a sequence getting arbitrarily large in the positive direction, we can also have it getting arbitrarily large in the negative direction, i.e. "tending to minus infinity":

Definition 2.4. A sequence (a_n) *tends to minus infinity* if, for every $C < 0$ there exists $N \in \mathbb{N}$ such that $a_n < C$ for all $n > N$.

We write this as $(a_n) \to -\infty$ (as $n \to \infty$) or $\lim_{n\to\infty} a_n = -\infty$.

We now consider the relationship between different sequences which tend to plus or minus infinity.

Lemma 2.5. Let (a_n), (b_n) be sequences with $a_n \le b_n$ for all n. Then if $(a_n) \to \infty$, then $(b_n) \to \infty$.

We can turn this around and say if $a_n \ge b_n$ and $(a_n) \to -\infty$ then $(b_n) \to -\infty$ as well.

Proposition 2.6. Let (a_n), (b_n) be sequences with $(a_n) \to \infty$ and $(b_n) \to \infty$. Then $(a_n + b_n) \to \infty$ and $(a_n b_n) \to \infty$. Furthermore, if $\lambda > 0$, then $(\lambda a_n) \to \infty$, and if $\lambda < 0$ then $(\lambda a_n) \to -\infty$.

Proof. Fix $C > 0$. For $(a_n + b_n)$, take N_1 such that $a_n > C/2$ for $n > N_1$ and N_2 such that $b_n > C/2$ for $n > N_2$; then take $N = \max\{N_1, N_2\}$, so that $a_n + b_n > C$ for $n > N$. For $(a_n b_n)$ replace $C/2$ by $\sqrt{C}$. For (λa_n), take N such that $a_n > C/\lambda$ for $n > N$, then $\lambda a_n > C$ when $\lambda > 0$; similar when $\lambda < 0$. □

2.2.2 Tending to Zero: Null Sequences

We next define what it means for a sequence to "tend to zero".

Definition 2.7. A sequence (a_n) *tends to zero* if, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n > N$.

We write this as $(a_n) \to 0$ (as $n \to \infty$) or $\lim_{n\to\infty} a_n = 0$. We also call (a_n) *a null sequence*.

Example.
1. $\left(\frac{1}{n}\right) \to 0$: given $\varepsilon > 0$, pick $N > \frac{1}{\varepsilon}$: then $n > N \implies 0 < \frac{1}{n} < \varepsilon$, so $\left|\frac{1}{n}\right| < \varepsilon$.
2. $\left(\frac{1}{n^2}\right) \to 0$: given $\varepsilon > 0$, pick $N > \frac{1}{\sqrt{\varepsilon}}$: then $n > N \implies n^2 > \frac{1}{\varepsilon} \implies 0 < \frac{1}{n^2} < \varepsilon$, so $\left|\frac{1}{n^2}\right| < \varepsilon$.
3. The sequence $(6) \not\to 0$, i.e. it does not tend to zero: take $\varepsilon = 1$, then we want to find N such that $|a_n| < 1$ for all $n > N$. But this is impossible, since $|a_n| = 6$ for all n.

We can now relate sequences tending to zero and sequences tending to infinity in a natural way:

Lemma 2.8. If $(a_n) \to \infty$ then $\left(\frac{1}{a_n}\right) \to 0$.

Proof. Given $\varepsilon > 0$, set $C = \frac{1}{\varepsilon}$; take an N s.t. $a_n > C$ for $n > N$; then $0 < \frac{1}{a_n} < \frac{1}{C} = \varepsilon$ when $n > N$. □

Lemma 2.9 (Absolute Value Rule)**.** $(a_n) \to 0$ if and only if $(|a_n|) \to 0$.

Proof. This follows immediately from the fact that $\big||a_n|\big| = |a_n|$. □

It should be noted the converse of the previous lemma is false: if $\left(\frac{1}{a_n}\right) \to 0$, then $(|a_n|) \to \infty$, but it is not necessarily the case that $(a_n) \to \infty$; for example, consider $a_n = (-1)^n n$.

Theorem 2.10 (Sandwich Theorem for null sequences)**.** Let (a_n) and (b_n) be sequences s.t. $0 \le |b_n| \le a_n$ for all n. Then if $(a_n) \to 0$, then $(b_n) \to 0$.

Proof. Given $\varepsilon > 0$, take N such that $|a_n| < \varepsilon$ when $n > N$. As $a_n \ge 0$, $a_n < \varepsilon$; thus for $n > N$, $0 \le |b_n| \le a_n < \varepsilon$, so $(b_n) \to 0$. □

Example. We have $0 \le \left|\frac{(-1)^n}{n^2}\right| \le \frac{1}{n}$ for all n; as $\left(\frac{1}{n}\right) \to 0$, $\left(\frac{(-1)^n}{n^2}\right) \to 0$.

Proposition 2.11 (Sum and Product Rules for null sequences)**.** Let $(a_n) \to 0$ and $(b_n) \to 0$. Then for any $\lambda \in \mathbb{R}$, $(\lambda a_n) \to 0$; $(a_n + b_n) \to 0$; and $(a_n b_n) \to 0$.

We can combine the first two parts of these and say that for any $\lambda, \mu \in \mathbb{R}$, $(\lambda a_n + \mu b_n) \to 0$.

Proof. We first prove $(\lambda a_n) \to 0$. If $\lambda \neq 0$, then given $\varepsilon > 0$, take N such that $|a_n| < \frac{\varepsilon}{|\lambda|}$ for $n > N$; then $|\lambda a_n| < \varepsilon$. (If $\lambda = 0$ it is trivial.) To prove $(a_n + b_n) \to 0$, given $\varepsilon > 0$, pick N_1 such that $|a_n| < \frac{\varepsilon}{2}$ for $n > N_1$, and pick N_2 such that $|b_n| < \frac{\varepsilon}{2}$ for $n > N_2$. Put $N = \max\{N_1, N_2\}$; then by the triangle inequality, $|a_n + b_n| \le |a_n| + |b_n| < \varepsilon$ for $n > N$. The proof of $(a_n b_n) \to 0$ uses $\sqrt{\varepsilon}$ instead of $\frac{\varepsilon}{2}$. □

Example. $\frac{n^2+5n+6}{n^4} = \frac{1}{n^2} + \frac{5}{n^3} + \frac{6}{n^4}$ is the sum of three null sequences and so it is null.

2.2.3 Convergent Sequences

We next define what it means for a sequence to converge to a number other than zero.

Definition 2.12. A sequence (a_n) *converges* (or *tends*) to $a \in \mathbb{R}$ if, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$.

We write this as $(a_n) \to a$ (as $n \to \infty$) or $\lim_{n \to \infty} a_n = a$.

We can think of this (and the other forms of convergence) as a game. You choose a sequence you think converges, and you choose a. Your opponent chooses $\varepsilon > 0$ – possibly extremely small. You try and find an N such that all the terms beyond that point are within ε of a, i.e. $a - \varepsilon < a_n < a + \varepsilon$. If you can always find an N, no matter how small an ε you are given, you win; the sequence tends to a. If there is some ε for which you cannot find an N, you lose; it doesn't tend to a.

Example. Consider $\left(\frac{n+1}{2n}\right)$. To show this tends to $\frac{1}{2}$, given ε, we want an N such that $\left|\frac{n+1}{2n} - \frac{1}{2}\right| < \varepsilon$ for $n > N$. As $\frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$, we need an N such that $\left|\frac{1}{2n}\right| < \varepsilon$, whence we see that $N = \frac{1}{2\varepsilon}$ suffices.

Just by looking at the definition it should be clear that:

Lemma 2.13. $(a_n) \to a$ if and only if $(a_n - a) \to 0$.

Is it possible that a sequence (a_n) could converge to two different numbers? No:

Proposition 2.14. A sequence cannot converge to more than one limit.

Proof. All uniqueness proofs follow the same idea: suppose there are two, and show they must be equal. So suppose $(a_n) \to a$ and $(a_n) \to b$, with $a < b$, and let $\varepsilon = \frac{b-a}{2}$. Then pick N_1 s.t. $|a_n - a| < \frac{b-a}{2}$ for $n > N_1$, and hence that $a_n < \frac{a+b}{2}$. Then pick $n > N_2$ s.t. $|a_n - b| < \frac{b-a}{2}$ for $n > N_2$, and hence that $a_n > \frac{a+b}{2}$. Then when $n > \max\{N_1, N_2\}$, $a_n < \frac{a+b}{2}$ and $a_n > \frac{a+b}{2}$, which is impossible. Thus $a = b$. ☐

Proposition 2.15. A convergent sequence is bounded.

Proof. Let $(a_n) \to a$. Then taking $\varepsilon = 1$, $a - 1 < a_n < a + 1$ for $n > N$, so $U = \max\{a_1, \dots, a_N, a+1\}$ and $L = \min\{a_1, \dots, a_N, a-1\}$ are upper and lower bounds respectively[5]. ☐

Theorem 2.16 (Sum, Product and Quotient Rules)**.** Let $(a_n) \to a$ and $(b_n) \to b$. Then for any $\lambda, \mu \in \mathbb{R}$, $(\lambda a_n + \mu b_n) \to \lambda a + \mu b$; $(a_n b_n) \to ab$; and if $b \neq 0$ then $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.

Proof. Since these results are quite long to prove in full, we instead give the key steps. In each case we apply one or more intermediary steps to form a null sequence, then apply the appropriate rules.
Sum rule: simply note that $(\lambda a_n + \mu b_n - (\lambda a + \mu b))$ is a null sequence, and apply the sum rule for null sequences.
Product Rule: First show that $a_n b_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a)$ (multiply out first term on rhs) then apply sum and product rules for null sequences to show lhs is null.
Quotient rule: First we note that $(bb_n) \to b^2$, and so $bb_n > \frac{b^2}{2}$ for $n > N_0$. So, we can show that eventually, $0 \leq \left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{2}{b^2}|b - b_n|$. Since $b - b_n$ is null, by sandwich rule we have that $\frac{1}{b_n} \to \frac{1}{b}$. Finally, we write $\frac{a_n}{b_n} = a_n \frac{1}{b_n}$ and apply product rule for sequences.

☐

Theorem 2.17 (Sandwich Rule)**.** Let (a_n), (b_n), (c_n) be such that $a_n \leq c_n \leq b_n$ for every n. If $(a_n) \to l$ and $(b_n) \to l$, then $(c_n) \to l$.

This is also known as the *squeeze rule*, for obvious reasons.

Proof. Simply note that $a_n \leq c_n \leq b_n$ if and only if $0 \leq c_n - a_n \leq b_n - a_n$, and $(b_n - a_n) \to 0$, and apply the sandwich rule for null sequences. ☐

[5] The crucial point here is that the sets we are taking the maximum and minimum over are finite.

2.3 Useful Results on Sequences

Having defined various notions of convergence, we can extend them to more sequences in various ways.

Definition 2.18. A sequence (a_n) satisfies a property *eventually* if there exists $N \in \mathbb{N}$ such that (a_{n+N}) satisfies the property.

This may seem like a hindrance; we only show that part of the sequence satisfies a property, and we want it all to have it. Sometimes, we get this for free:

Lemma 2.19. If a sequence (a_n) is eventually bounded, i.e. (a_{n+N}) is bounded for some N, then (a_n) is bounded.

Proof. This is similar to showing a convergent sequence is bounded. We know that $\{a_{n+1}, a_{n+2}, \dots\}$ is bounded, i.e. $L \leq a_{n+N} \leq U$ for all n. Thus $U' = \max\{a_1, \dots, a_N, U\}$ and $L' = \min\{a_1, \dots, a_N, L\}$ are upper and lower bounds for (a_n). □

In fact, the same is true of convergence:

Proposition 2.20 (Shift Rule)**.** For any $n \in \mathbb{N}$, $(a_n) \to a$ if and only if $(a_{n+N}) \to a$.

Proof. Given $\varepsilon > 0$, take N_1 such that $|a_n - a| < \varepsilon$ for $n > N_1$; then for $n > N_1 - N$, $|a_{n+N} - a| < \varepsilon$; thus $(a_n) \to a$ implies $(a_{n+N}) \to a$. The converse is almost identical. □

Corollary 2.21 (Sandwich Rule with Shift Rule)**.** Let (a_n), (b_n), (c_n) be such that $a_n \leq c_n \leq b_n$ eventually. If $(a_n) \to l$ and $(b_n) \to l$, then $(c_n) \to l$.

Often, we are not concerned with exactly what the limit of a sequence is, but rather that it must lie between, say, 0 and 1. If we know that the terms of a convergent sequence are between 0 and 1, must its limit lie between 0 and 1?

Lemma 2.22. Let $(a_n) \to a$. If $a_n \geq 0$ for all n, then $a \geq 0$.

Proof. Suppose $a < 0$; then for $\varepsilon = -a$, when $n > N$, $|a_n - a| < \varepsilon$, i.e. $2a < a_n < 0$ – contradiction. □

Corollary 2.23 (Inequality Rule)**.** Let $(a_n) \to a$ and $(b_n) \to b$, with $a_n \leq b_n$ for all n. Then $a \leq b$.

Corollary 2.24 (Closed Interval Rule)**.** Let $(a_n) \to a$. If $L \leq a_n \leq U$ for every n, then $L \leq a \leq U$.

2.3.1 Subsequences

A subsequence of a sequence is formed by picking out some (or all) terms of a sequence to form a new sequence: for instance we might take $(a_1, a_4, a_9, a_{16}, \dots)$. In general, the index is a strictly increasing sequence of natural numbers:

Definition 2.25. A *subsequence* of $(a_n)_{n=1}^{\infty}$ is a sequence $(a_{n_i})_{i=1}^{\infty}$, where $(n_i)_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers.

It is immediate that:

Lemma 2.26. If $(a_n) \to a$, then every subsequence $(a_{n_i}) \to a$ as well.

Note that the shifted sequence (a_{n+N}) is simply a special kind of subsequence, so this is a generalisation of the shift rule. Furthermore, if a sequence is bounded, then so is every subsequence:

Lemma 2.27. If (a_n) is bounded, then every subsequence (a_{n_i}) is bounded as well.

It is a useful fact that *every* sequence contains a subsequence which is monotonic:

Proposition 2.28. Every sequence has a monotonic subsequence.

Proof. We say that a_f is a *floor term* for (a_n) if $a_n \geq a_f$ for all $n \geq f$. If there are infinitely many floor terms, then the subsequence of floor terms is an increasing sequence. If there are finitely many floor terms, then beyond the last floor term we can always construct a decreasing sequence. □

2.3.2 Powers and the Ratio Lemma

We now demonstrate a standard result on limits which comes in very useful all over analysis.

Proposition 2.29. Consider (x^n). If $x > 1$, $(x^n) \to \infty$; if $x = 1$, $(x^n) \to 1$; if $-1 < x < 1$, $(x^n) \to 0$; and if $x \leq -1$, (x^n) does not converge.

Very often, a sequence is not exactly x^n but it grows or shrinks at least as fast, so we can sandwich them by a geometric sequence x^n. We can formalise this by considering the ratio of two terms, $\frac{a_{n+1}}{a_n}$.

Lemma 2.30 (Ratio Lemma)**.** Let (a_n) be a sequence with $a_n > 0$ for all n. Suppose $0 < l < 1$ and $\frac{a_{n+1}}{a_n} \leq l$ for all n. Then $(a_n) \to 0$.

Proof. By induction, we can show that $0 \leq a_{n+1} \leq l^n a_1$. Then as $0 < l < 1$, we have $l^n a_1 \to 0$ as $n \to \infty$, so by the sandwich rule $(a_n) \to 0$. □

The same result is true if $\frac{a_{n+1}}{a_n} \leq l$ eventually, using the shift rule. In fact, if the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ converges, we get a more useful form of the same thing:

Corollary 2.31 (Ratio Lemma, limit form)**.** Let (a_n) be a sequence with $a_n > 0$ for all n. If $\left(\frac{a_{n+1}}{a_n}\right) \to l$ with $0 \leq l < 1$ then $(a_n) \to 0$; while if $\left(\frac{a_{n+1}}{a_n}\right) \to l$ with $l > 1$ then $(a_n) \to \infty$.

Note that we cannot conclude anything if $\left(\frac{a_{n+1}}{a_n}\right) \to 1$; for instance $(n)_{n=1}^{\infty} \to \infty$, $\left(\frac{1}{n}\right)_{n=1}^{\infty} \to 0$, and $(k)_{n=1}^{\infty}$ for any constant $k > 0$ all have their ratios tending to 1.

Proof. If $\left(\frac{a_{n+1}}{a_n}\right) \to l$, $0 \leq l < 1$, then for $\varepsilon = \frac{1-l}{2}$, $\frac{a_{n+1}}{a_n} \leq \frac{l+1}{2} < 1$ for $n > N$, so $(a_n) \to 0$. If $\left(\frac{a_{n+1}}{a_n}\right) \to l$ and $l > 1$, then $\left(\frac{a_n}{a_{n+1}}\right) \to \frac{1}{l}$, so $\left(\frac{1}{a_n}\right) \to 0$ by the first case; hence $(a_n) \to \infty$ (as $a_n > 0$). □

2.3.3 Standard Results

We present, without proof, some standard limits which crop up all over analysis. You may be asked to prove any one of these, but it is more likely you will need to apply them in finding the limit of a more complicated sequence, so remembering the result is much more important.

Proposition 2.32. (i) If $x > 0$ then $(x^{1/n}) \to 1$. (ii) $(n^{1/n}) \to 1$.

The following proposition tells us that while (x^n), (n^k), (n^n) and $(n!)$ all tend to infinity, they don't all do it at the same speed:

Proposition 2.33. (i) $\left(\frac{x^n}{n!}\right) \to 0$ for all values of x.
(ii) $\left(\frac{n!}{n^n}\right) \to 0$ as $n \to \infty$.
(iii) $\left(\frac{x^n}{n^k}\right) \to 0$ if $x \leq 1$, while $\left(\frac{x^n}{n^k}\right) \to \infty$ if $x > 1$.

The following bounds on $n!$ come in useful sometimes:

Proposition 2.34. $n^n e^{-n+1} \leq n! \leq n^{n+1} e^{-n+1}$.

Example. Consider $\left(\frac{n!2^n}{n^n}\right)$. Now, $n^n e^{-n+1} \leq n! \leq n^{n+1} e^{-n+1} \implies 2^n e^{-n+1} \leq \frac{n!2^n}{n^n} \leq n2^n e^{-n+1} \implies e\left(\frac{2^n}{e^n}\right) \leq \frac{n!2^n}{n^n} \leq e\left(\frac{n2^n}{e^n}\right)$. By the ratio lemma, $\left(\frac{2^n}{e^n}\right) \to 0$ and $\left(\frac{n2^n}{e^n}\right) \to 0$, so $\left(\frac{n!2^n}{n^n}\right) \to 0$.

3 Completeness

Completeness is the key property of the real numbers which the rational numbers lack: essentially, the real number line has no "holes". The holes which are present in the rational numbers are the *irrational* numbers such as $\sqrt{2}$, π, e and so on. However, the rationals and irrationals are both "dense", in the sense that between any two real numbers there are always infinitely many rational numbers and infinitely many irrational numbers.

3.1 Rational Numbers

Example. $\frac{1}{2}$, $-\frac{5}{6}$, 0 are all rational numbers.

Definition 3.1. A real number is rational if it can be written in the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. A real number that is not rational is termed irrational.

Theorem 3.2. $\sqrt{2}$ is irrational.

Proof. Suppose $\sqrt{2}$ is rational then it can be written in form $\frac{m}{n}$ where m and n are integers which are coprime (only 1 can divide both m and n) so $\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2$ so m^2 is even but that means that m is even so we can write m as $m = 2p$ where p is an integer. So $m^2 = 4p^2 = 2n^2$, $2p^2 = n^2$ so n^2 is also even and so n is even but that contradicts the claim made above as both m and n are divisible by 2, so $\sqrt{2}$ cannot be rational. □

Theorem 3.3. Between any two distinct real numbers there is a rational number.

Proof. In order to prove this statement it will be useful to define the integer part of x notation, that is if x is a real number then $[x]$ is the integer part of that number (e.g. $[\pi] = 3$)

Let $a, b \in \mathbb{R}$ such that $a < b$,by the definition above we know that $[x] \leq x < [x] + 1$ by setting $x = 2^n a$ we get $\frac{[2^n a]}{2^n} \leq a \leq \frac{[2^n a]}{2^n} + \frac{1}{2^n}$, we know that $\frac{1}{2^n} \to 0$ so $\exists N \in \mathbb{N}$ such that $|\frac{1}{2^n}| < b - a$ for $n > N$, but then $a < \frac{[2^n a]}{2^n} + \frac{1}{2^n} < b$. □

Corollary 3.4. Let $a < b$. There is an infinite number of rational numbers in the open interval (a,b).

Theorem 3.5. Between any two distinct real numbers there is an irrational number.

Proof. Let $p, q, x, y \in \mathbb{Z}$ and $y, q \neq 0$ and $\frac{p}{q} < \frac{x}{y}$ we know that $\frac{\sqrt{2}}{n} \to 0$ so $\exists N \in \mathbb{N}$ such that $|\frac{\sqrt{2}}{n}| < \frac{x}{y} - \frac{p}{q}$ for $n > N$, so $\frac{p}{q} < \frac{p}{q} + \frac{\sqrt{2}}{n} < \frac{x}{y}$ so there is an irrational between two rational numbers. □

Corollary 3.6. Let $a < b$. There is an infinite number of irrational numbers in the open interval (a, b).

So what is it that makes the real numbers different?

3.2 Least Upper Bounds and Greatest Lower Bounds

Much as we defined bounds for sequences, we can define bounds for sets of real numbers as follows:

Definition 3.7. Let $A \subseteq \mathbb{R}$ be non-empty. We say that:
1. U is an *upper bound* for A if, for every $a \in A$, $a \leq U$;
2. A is *bounded above* if it has an upper bound;
3. L is a *lower bound* for A if, for every $a \in A$, $a \geq L$;
4. A is *bounded below* if it has a lower bound.

Again, the bounds do not have to be the best possible – if U is an upper bound, then so is any number greater than U. We rectify this by defining a *least upper bound* and a *greatest lower bound*:

Definition 3.8. Let $A \subseteq \mathbb{R}$ be non-empty. We say that U is a *least upper bound* for A if it is an upper bound, and for any other upper bound U' we have $U \leq U'$. Similarly, we say that L is a *greatest lower bound* for A if it is a lower bound, and for any other lower bound L' we have $L \geq L'$.

The least upper bound of A is called the *supremum* of A and is denoted by $\sup A$. The greatest lower bound of A is called the *infimum* of A and is denoted by $\inf A$.

Lemma 3.9. A non-empty set $A \subseteq \mathbb{R}$ can have at most one least upper bound and at most one greatest lower bound.

Proof. Let U_1, U_2 be least upper bounds for A. As U_1 is a least upper bound, $U_1 \leq U$ for all upper bounds U; thus $U_1 \leq U_2$. Similarly, $U_2 \leq U_1$. Hence $U_1 = U_2$. □

Completeness Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

If you only used rational numbers, this would be false: the set $\{x \in \mathbb{Q} : x < \pi\}$ has rational upper bounds $4, 3.2, 3.15, 3.142, 3.1416, \ldots$, but no *rational* least upper bound. Considered as a subset of the reals, its least upper bound is π. So the Completeness Axiom captures the idea that the reals have no "holes".

The following property of the supremum is used frequently throughout analysis.

Lemma 3.10. Suppose a set A is non-empty and bounded above. Given $\varepsilon > 0$, there is an $a \in A$ such that $\sup A - \varepsilon < a \leq \sup A$.

Proof. If not, then $\sup A - \varepsilon$ would be an upper bound less than $\sup A$, which is a contradiction. □

Proposition 3.11. Suppose A is a non-empty set of real numbers which is bounded below. Then $-A = \{-x : x \in A\}$ is bounded above and $\inf A = -\sup(-A)$.

This leads immediately to another version of the completeness axiom:

Theorem 3.12. Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

3.3 Completeness and Sequences

The mathematician Weierstrass was the first to pin down the ideas of completeness in the 1860s and to point out that all the deeper results of analysis are based upon completeness. The most immediately useful consequence is the following theorem:

Theorem 3.13. Every bounded increasing sequence is convergent.

Proof. Let (a_n) be increasing and bounded; then $A = \{a_n : n \in \mathbb{N}\}$ has a least upper bound, $\sup A$. Thus there exists a_N such that $\sup A - \varepsilon < a_N \leq \sup A$; as (a_n) is increasing, $\sup A - \varepsilon < a_n \leq \sup A$ for all $n > N$ as well. Hence $(a_n) \to \sup A$. □

Theorem 3.14. Every bounded decreasing sequence is convergent.

This gives us a very simple test for convergence: if we can show a sequence is monotonic and bounded, then it must converge. This is particularly useful in the following example:

Example (Recursive sequences)**.** Define $a_1 = 4$ and $a_{n+1} = \sqrt{a_n + 6}$. First, $3 \leq a_1 \leq 4$, and $3 \leq a_n \leq 4 \implies 3 \leq a_{n+1} = \sqrt{a_n + 6} \leq \sqrt{10} \leq 4$, so by induction $a_n \in [3, 4]$ for all n, i.e. (a_n) is bounded. Second, note that $a_n^2 - a_n - 6 = (a_n + 2)(a_n - 3) \geq 0$ for all n, since $3 \leq a_n \leq 4$, and so we have $a_n \geq \sqrt{a_n + 6} = a_{n+1}$, i.e. (a_n) is decreasing. So by completeness, it converges; say $(a_n) \to a$. Hence $a_{n+1} = \sqrt{a_n + 6} \to \sqrt{a + 6}$, so $a = \sqrt{a + 6}$. This gives $a^2 - a - 6 = 0$, with solutions $a = -2, 3$. Since $a_n \geq 3$ for all n, $(a_n) \to 3$.

The sequence $(\sin n)$ is far from convergent, though it is bounded: $-1 \leq \sin n \leq 1$. However, the following consequence of completeness guarantees that it does have a convergent subsequence:

Theorem 3.15 (Bolzano-Weierstrass)**.** Every bounded sequence has a convergent subsequence.

Proof. Every bounded sequence has a monotonic subsequence, which is thus bounded; by completeness, every bounded monotonic sequence converges, hence every bounded sequence has a convergent subsequence. □

3.3.1 Cauchy Sequences

We have seen that any bounded monotonic sequence converges; thus we have the following simple test:

Convergence Test. A monotonic sequence converges if and only if it is bounded.

The big advantage of this is that it works *without* knowing exactly what the limit is. Is there a similar test for non-monotonic sequences? Intuitively, we might hope that if $(a_{n+1} - a_n) \to 0$, this would be enough to guarantee (a_n) converges. But $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$ while $(\sqrt{n}) \to \infty$, so this is not true. What we need is not just that neighbouring terms get close, but that all the terms beyond a certain point are close:

Definition 3.16. A sequence is *Cauchy* if, for all $\varepsilon > 0$, there is a number $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for $n, m > N$.

In words, the Cauchy property means that for any positive ε, no matter how small, we can find a point in the sequence beyond which any two of the terms are at most ε apart. So the terms are getting more and more "clustered" or "crowded".

Proposition 3.17. Every convergent sequence is Cauchy.

Proof. Let $(a_n) \to a$: then given $\varepsilon > 0$, there is N such that $|a_n - a| < \frac{\varepsilon}{2}$ for $n > N$. So for $n, m > N$, we have $|a_n - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon$. □

This result does not depend on completeness; this is true over the rationals as well. But the following result, which is the really useful part, *is* a consequence of completeness:

Theorem 3.18. Every Cauchy sequence converges.

Proof. Fix $\varepsilon = 1$ and fix $m > N$, then $|a_n - a_m| < 1$ for all $n > m$, so a Cauchy sequence is bounded. Thus by completeness it has a convergent subsequence, $(a_{n_i}) \to a$ say. Now, by the triangle inequality, $|a_n - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a|$. The Cauchy property means that $|a_n - a_{n_i}| < \frac{\varepsilon}{2}$ for $n, n_i \geq N_1$, and as $(a_{n_i}) \to a$, $|a_{n_i} - a| < \frac{\varepsilon}{2}$ for $n_i \geq N_2$; thus for $n, n_i > \max\{N_1, N_2\}$, $|a_n - a| < \varepsilon$. □

So we arrive at our convergence test:

Convergence Test. A sequence is convergent if and only if is Cauchy.

4 Series

4.1 Defining Infinite Sums

Series are a very useful construction: they are, in essence, infinite sums - expressions such as $\sum_{k=1}^{\infty} a_k$. In order to define what we mean by this, we first remind readers of the notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

where $m \leq n$ are integers and the a_i are real numbers. It would be a waste to define series with no mention of all our careful work on sequences, so we make use of this and define the sequence of partial sums:

Definition 4.1. Let (a_k) be a sequence, and consider the series $\sum_{k=1}^{\infty} a_k$ with partial sums (s_n), where $s_n := \sum_{k=1}^{n} a_k$. If $(s_n) \to s$, we say the series *converges* to s. If $(s_n) \to \pm\infty$, we say that the series *diverges* to $\pm\infty$. If (s_n) does not converge, we say that the series diverges.

For brevity, we sometimes write $\sum a_n$ for $\sum_{n=1}^{\infty} a_n$. Note that there are two sequences associated with a series $\sum_{k=1}^{\infty} a_k$; the sequence of its terms, (a_k), and the sequence of its partial sums, (s_n). Don't get them mixed up!

Example. Consider the series $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \cdots$. By the geometric progression formula, the sequence of partial sums is

$$s_n = \sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{3} + \cdots + \frac{1}{3^n} = \frac{1}{3} \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} = \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right).$$

As $n \to \infty$, $\left(\frac{1}{3}\right)^{n+1} \to 0$, so $(s_n) \to \frac{1}{2}$. Hence $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

In fact, this example is just one particular case of the general result for geometric series:

Proposition 4.2 (Geometric Series)**.** The series $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$, with sum $\frac{1}{1-x}$, and diverges if $|x| \geq 1$.

Proof. By the geometric progression formula, the sequence of partial sums is

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Now, if $|x| < 1$, then $x^{n+1} \to 0$ as $n \to \infty$, so $\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} s_n = \frac{1}{1-x}$. If $|x| > 1$, then clearly x^{n+1} does not converge, so the series $\sum_{n=0}^{\infty} x^n$ diverges. (Check $x = -1$ carefully.) □

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is called the *harmonic series*:

Proposition 4.3 (Harmonic Series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

Proof. Put $s_n = \sum_{k=1}^{n} \frac{1}{k}$; then show $s_{2^n} \geq 1 + \frac{n}{2}$ by induction; then as s_n is increasing, $(s_n) \to \infty$. □

4.1.1 Properties of Convergent Series

Just as with convergent sequences, there are various properties of convergent series which are rather useful.

Proposition 4.4 (Sum Rule for series)**.** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then for any $\lambda, \mu \in \mathbb{R}$, $\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n)$ is convergent, and $\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$.

Proof. Apply the sum rule for sequences to the sequences of partial sums. □

Proposition 4.5 (Shift Rule for series)**.** For any $N \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_{n+N}$ converges.

Proof. Set $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} a_{k+N}$. Then $s_{n+N} = a_1 + \cdots + a_N + t_n$, so as $a_1 + \cdots + a_N$ is finite, (t_n) converges iff (s_{n+N}) converges, thus (shift rule for sequences) (t_n) converges iff (s_n) converges. □

4.2 Testing for Convergence

Just as with sequences, we often don't care exactly what the limit of a series is, but only that it converges. There are a number of useful tests by which we can easily determine if a series converges.

Theorem 4.6 (Null Sequence Test)**.** If $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$.

Proof. If $s_n = \sum_{k=1}^{n} a_k$, then $a_{n+1} = s_{n+1} - s_n$; so if $(s_n) \to s$, then $(a_n) \to s - s = 0$. □

This is typically useful only to prove divergence: if $(a_n) \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ does not converge. For example, since $(\sqrt{n})_{n=1}^{\infty}$ is not null, $\sum_{n=1}^{\infty} \sqrt{n}$ is divergent. The converse does not hold, however: $\left(\frac{1}{n}\right) \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

4.2.1 Series with Positive Terms

There are a number of very useful results which depend on the series having only positive terms.

Proposition 4.7 (Boundedness Condition)**.** Let $a_n \geq 0$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums $(\sum_{k=1}^{n} a_k)_{n=1}^{\infty}$ is bounded.

Proof. If $s_n = \sum_{k=1}^{n} a_k$ is convergent, then (s_n) is bounded. Conversely, if (s_n) is bounded, then since $a_n \geq 0$, (s_n) is increasing and hence convergent (by completeness). □

Theorem 4.8 (Comparison Test)**.** Suppose $0 \leq a_n \leq b_n$ for every n. If $\sum b_n$ converges then $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$. As $0 \leq a_n \leq b_n$, we have $0 \leq s_n \leq t_n$. If $\sum b_n$ converges, then by the boundedness condition (t_n) is bounded, and so (s_n) is bounded and hence $\sum a_n$ converges. □

Example ($\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges)**.** For every n, we have $0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$. By partial fractions, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, hence $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} \to 1$ as $n \to \infty$. Hence as $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges[6].

By taking the contrapositive of the standard Comparison Test, we get the following additional test:

Corollary 4.9 (Comparison Test extension)**.** Suppose $0 \leq a_n \leq b_n$. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Example. Note that $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$. As $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\sqrt{n}}$ diverges as well.

For sequences, by sandwiching by a geometric sequence, we got a very useful Ratio Lemma. We can do the same for series by comparing with a geometric series, to get the Ratio Test:

Theorem 4.10 (Ratio Test)**.** Suppose $a_n > 0$ for all n, and $\left(\frac{a_{n+1}}{a_n}\right) \to l$. Then $\sum a_n$ converges if $0 \leq l < 1$ and diverges if $l > 1$.

Note that, again, the case $l = 1$ is omitted: both $\left(\frac{1}{n}\right)$ and $\left(\frac{1}{n^2}\right)$ have their ratios tending to 1, but $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges.

Example. Consider $\sum \frac{2^n}{n!}$. The ratio is

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to 0.$$

Hence by the ratio test $\sum \frac{2^n}{n!}$ converges.

Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!} = \frac{1!2!}{3!} + \frac{2!4!}{6!} + \frac{3!6!}{9!} + \cdots$. The ratio is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(2(n+1))!}{(3(n+1))!} \cdot \frac{(3n)!}{n!(2n)!} = \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} = \frac{4n^3+10n^2+8n+2}{27n^3+54n^2+33n+6} \to \frac{4}{27}.$$

Since $\frac{4}{27} < 1$, $\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}$ converges by the ratio test.

4.2.2 Integral Test

By using integrals to bound sums, we can see that

$$\int_{m+1}^{n+1} f(x)\,\mathrm{d}x \leq \sum_{k=m+1}^{n} f(k) \leq \int_{m}^{n} f(x)\,\mathrm{d}x.$$

This leads us to the Integral Test for convergence, which is also very useful:

Theorem 4.11 (Integral Test for convergence)**.** Suppose the function $f(x)$ is non-negative and decreasing for $x \geq 1$. Then $\sum_{n=1}^{\infty} f(n)$ converges if the increasing sequence $\left(\int_1^n f(x)\,\mathrm{d}x\right)$ is bounded.

Example. Consider $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$. Observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\log x}\right) = -\frac{1}{x(\log x)^2}$, so

$$\int_2^n \frac{1}{x(\log x)^2}\,\mathrm{d}x = -\frac{1}{\log x}\Big|_2^n = \frac{1}{\log 2} - \frac{1}{\log n}.$$

As $n \to \infty$, $\log n \to \infty$, so $\frac{1}{\log n} \to 0$, hence $\left(\int_2^n \frac{1}{x(\log x)^2}\right) \to \frac{1}{\log 2}$, and $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.

[6] In fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$: this fact is proved in MA250 Introduction to PDEs using Fourier series.

Similarly, we can use integrals to test for divergence:

Theorem 4.12 (Integral Test for divergence)**.** Suppose the function $f(x)$ is non-negative and decreasing for $x \geq 1$. Then $\sum_{n=1}^{\infty} f(n)$ diverges if the increasing sequence $\left(\int_1^n f(x)\,\mathrm{d}x\right)$ is unbounded.

Example. Consider $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. Observe that

$$\int_2^n \frac{1}{x \log x}\,\mathrm{d}x = \int_2^n \frac{\frac{1}{x}}{\log x}\,\mathrm{d}x = \log(\log x)\Big|_2^n = \log\log n - \log\log 2.$$

Now as $n \to \infty$, $\log\left(\frac{\log n}{\log 2}\right) \to \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

4.2.3 What is e?

We now define the base of the natural logarithm e.

Definition 4.13. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

A simple application of the Ratio Test shows that this series converges. We can use some trickery[7] with the Binomial Theorem to show the following two very useful limits:

Theorem 4.14. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$, and $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$.

Example. We apply these limits to show that $\sum \frac{n!}{n^n}$ converges, using the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \to \frac{1}{e} < 1.$$

4.2.4 Error Bounds

If we have established that a series $\sum a_n$ converges then the next question is to calculate the total sum $\sum_{n=1}^{\infty} a_n$. Usually there is no hope of getting an explicit formula for the sum and we must be content with an approximate answer – for example, correct to 10 decimal places.

Example. Take $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Can we estimate the size of the error $\left|\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{6}\right|$? If we can, we will know how many terms we need to add up to get a good estimate of $\frac{\pi^2}{6}$. Using the integral estimates, we have

$$\int_{N+1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2}\,\mathrm{d}x.$$

Hence $\frac{1}{1+N} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{N}$. For the error to be less than 10^{-10} we need $N \geq 10^{10}$, which is a huge number of terms, and thus this formula is not much use for calculating $\frac{\pi^2}{6}$.

4.3 Series with Positive and Negative Terms

Up to now, all our tests for convergence (except for the Null Sequence Test) have required the terms of our sequence to be positive. For general series, this is not always true.

One very special kind of series is an alternating series, i.e. one of the form $\sum(-1)^{n+1} a_n$. The following test tells us we are guaranteed that it converges, as long as (a_n) is decreasing and tends to zero:

Theorem 4.15 (Alternating Series Test)**.** Suppose (a_n) is decreasing and null. Then the alternating series $\sum(-1)^{n+1} a_n$ is convergent.

Example. Since $\left(\frac{1}{n}\right)$ is decreasing and null, the Alternating Series test tells us that $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is convergent. With some delicate work, we can in fact show that $\sum \frac{(-1)^{n+1}}{n} = \log 2$.

[7] Some similar trickery can in fact be used to show that e is irrational.

We could try and exploit the Cauchy property for series, but this becomes much more useful if we first define absolute convergence:

Definition 4.16. The series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Example. 1. $\sum \frac{1}{n^2}$ is absolutely convergent.

2. $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is convergent.

3. $\sum \frac{(-1)^n}{n}$ is not absolutely convergent, since $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ and $\sum \frac{1}{n}$ is divergent.

Theorem 4.17. If $\sum a_n$ converges absolutely, it is convergent.

Proof. Let $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n |a_k|$; we show that (s_n) is Cauchy. As (t_n) is convergent and hence Cauchy, fix $\varepsilon > 0$ and take N such that $|t_n - t_m| < \varepsilon$ whenever $n > m > N$. Then by the triangle inequality,

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| = |t_n - t_m| < \varepsilon. \qquad \square$$

This breathes new life into all our tests for series with positive terms; we can now use them to test for absolute convergence. The Ratio Test can be modified to cope directly with series of mixed terms:

Theorem 4.18 (Ratio Test)**.** Suppose $a_n \neq 0$ and $\left|\frac{a_{n+1}}{a_n}\right| \to l$. Then $\sum a_n$ converges absolutely (and hence converges) if $0 \leq l < 1$ and diverges if $l > 1$.

Theorem 4.19 (Ratio Test Extension)**.** Suppose $a_n \neq 0$ and $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, then $\sum a_n$ diverges.

Example. Consider $\sum \frac{x^n}{n^2}$. When $x = 0$, every term is zero so the series converges. When $x \neq 0$, we can use the new Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = \left(\frac{n}{n+1}\right)^2 |x| \to |x|.$$

Thus if $|x| < 1$, the series converges and if $|x| > 1$ the series diverges. When $|x| = 1$, $|a_n| = \frac{1}{n^2}$, so $\sum |a_n|$ converges, and hence the series converges for $-1 \leq x \leq 1$.

4.3.1 Rearrangements of Series

If you take a *finite* set of numbers and rearrange their order, their sum remains the same. But one truly unsettling fact about *infinite* sums is that, in some cases, you can rearrange the terms to get a totally different sum.

Example. We know that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$. Let us rearrange it as $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$. Then we have $1 - \frac{1}{2} = \frac{1}{2}$, $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, and so on, so

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

which is clearly half our original series, so converges to $\frac{\log 2}{2}$.

Definition 4.20. We say that the sequence (b_n) is a *rearrangement* of (a_n) if there exists a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ (i.e. a permutation on $\mathbb{N}$) such that $b_n = a_{\sigma(n)}$ for all n.

For series with all positive terms it does not matter in what order you add the terms; nor does it matter what order you add the terms of an absolutely convergent series.

Lemma 4.21. Suppose $\sum a_n$ is convergent with $a_n \geq 0$ for all n. If (b_n) is a rearrangement of (a_n), then $\sum b_n$ is convergent and $\sum b_n = \sum a_n$.

Theorem 4.22. Suppose $\sum a_n$ is absolutely convergent. If (b_n) is a rearrangement of (a_n), then $\sum b_n$ is convergent and $\sum b_n = \sum a_n$.

Those series which converge, but do not converge absolutely, are thus the only series which we can conceivably rearrange and get a different sum. We call this *conditional convergence*:

Definition 4.23. The series $\sum a_n$ is *conditionally convergent* if $\sum a_n$ is convergent but $\sum |a_n|$ is not.

Example. $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent since it converges but $\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$ is divergent.

In fact, it turns out that *any* conditionally convergent series can be rearranged to give *any* sum we want! The key to this is the following observation:

Proposition 4.24. Let $\sum a_n$ be conditionally convergent. Then the series formed from just the positive terms of a_n diverges, and the series formed from just the negative terms of a_n diverges.

Given a conditionally convergent series, we can choose any real number we like, say x. We add up enough of the positive terms to get us above x, add enough negative terms until we end up less than x again, and keep repeating; we use up all the terms eventually and so we find a rearrangement that goes to x. This is encapsulated in the following result:

Theorem 4.25 (Riemann's Rearrangement Theorem)**.** Suppose $\sum a_n$ is conditionally convergent. Then for every real number x there is a rearrangement (b_n) of (a_n) such that $\sum b_n = x$.

Closing Remarks

That's all there is to it – it's really not as bad as it looks at first sight. The most important thing about analysis is *make sure you know your definitions*! If you don't know what the definitions of convergent/Cauchy/bounded/monotonic sequences are (etc.), firstly you're throwing away easy marks in the exam, and secondly you're hampering yourself when it comes to proving things since you won't be able to work with a definition you can't remember.

While there will undoubtedly be some proofs on the exam – which is why we have included a selection in this guide – the main focus is on knowing and applying key results. It is thus much more important that you can accurately *state* something like the ratio test and that you can apply it to test convergence, rather than being able to prove it.

Finally, we hope this revision guide has been useful. But it's no use just reading it – practise, practise, PRACTISE! The best source is past exam papers, which can be bought from the Maths General Office, or are available to download from:

`http://www2.warwick.ac.uk/services/exampapers?q=MA131&department=MA&year=Any`

and the solutions at

`http://www2.warwick.ac.uk/fac/sci/maths/undergrad/ughandbook/archives/`

And with that, good luck on the exam!

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