# PX264: Physics of Fluids Summary

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## Basics

A fluid is a continuous substance which deforms continuously under a shear stress. There are two ways of defining a fluid element:

- Eulerian: elements are fixed in space, the mass may vary over time.
- Lagrangian: elements move with the fluid, their mass is constant but their shape becomes deformed over time.

The viscosity $\mu$ of a fluid is its resistance to deformation. The viscous force on a fluid is described by

$$\mathbf{F}_V = \mu\nabla^2\mathbf{v}$$

The force due to a pressure gradient is described by

$$\mathbf{F}_P = -\boldsymbol{\nabla}P$$

## Equations of Fluid Dynamics

There are three key equations with which we can describe a fluid. The first is the Navier-Stokes equation:

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\boldsymbol{\nabla}P + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

The operator $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the material (or advective) derivative. The Navier-Stokes equation relates forces and momentum, essentially Newton's second law for fluids. The second equation is the continuity equation

$$\frac{\partial\rho}{\partial t} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

This can be derived by considering the mass conservation in a fluid using the divergence theorem. The final equation is the adiabatic pressure equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{P}{\rho^\gamma}\right) = 0$$

This is assuming that any processes considered are fast enough that heat transfer can be ignored. These equations require boundary conditions to be solved; typically, we use a far-field solution of constant $\mathbf{v}_0, P_0, \rho_0$, and $\mathbf{v} = 0$ on a stationary object. To solve these equations analytically, we can make approximations:

- Incompressible: constant $\rho$ so $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$.
- Inviscid: $\mu = 0$ is valid for fluids with a large Reynolds number $\mathrm{Re} = \frac{\rho_0 v_0 L_0}{\mu}$. Re is the ratio of the inertial and viscous terms in the Navier-Stokes equation. So we can instead ignore the inertial term ($\rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$) in the case of a small Re.
- Steady: $\frac{\partial\mathbf{v}}{\partial t} = 0$

# Flow Potentials and Streamfunctions

Bernoulli's principle states that

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant along a streamline}$$

This is valid for incompressible, inviscid, steady flows.

If the flow is irrotational ($\nabla \times \mathbf{v} = 0$), then we can write the velocity as the gradient of some potential function $\mathbf{v} = \nabla\phi$. This is called the flow potential. For a 2D incompressible flow, we can say $v_x = \frac{\partial\psi}{\partial y}$ and $v_y = -\frac{\partial\psi}{\partial x}$ so that $\nabla \cdot \mathbf{v} = 0$ is satisfied. $\psi$ is called the streamfunction and lines of constant $\psi$ are the streamlines of the flow. Some simple 2D potential flows are

- Uniform: $\phi = Vx,\ \psi = Vy$
- Point source: $\phi = \frac{q}{2\pi}\ln(r),\ \psi = \frac{q\theta}{2\pi}$
- Vortex: $\phi = \frac{K\theta}{2\pi},\ \psi = -\frac{k}{2\pi}\ln(r)$
- Dipole: $\phi = \frac{a\cos\theta}{r},\ \psi = -\frac{a\sin\theta}{r}$

We can combine these flows to find other flows, such as flows over objects. These will have stagnation points where $\mathbf{v} = 0$. The value of $\psi$ at this point can be found; this value will be constant along that streamline, which will define the boundary of the object.

# Circulation and the Magnus Effect

The problem with these streamfunctions is that there will be no circulation around these objects: the circulation $K$ around some closed path $\Gamma$ is

$$K = \oint_\Gamma \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S}$$

Where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity. Kelvin's circulation theorem states that the circulation is constant for inviscid, incompressible flow:

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint_\Gamma \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} = 0$$

The material derivative needs to be used because the path $\Gamma$ moves with the fluid over time.

In a potential flow, the force on an object is given by $\mathbf{F} = \rho\mathbf{v} \times \mathbf{K}$. The direction of $\mathbf{K}$ is determined by $\boldsymbol{\omega}$. If the circulation occurs due to an object rotating, this force is called the Magnus effect. Whilst this gives a force perpendicular to the fluid flow, it does not predict a force parallel to the flow (so no drag). This is obviously incomplete.

# Boundary Layers

Whilst potential flows provide an accurate description away from an object, a viscous boundary layer of width $\delta \approx \frac{L_0}{\mathrm{Re}^{\frac{1}{2}}}$ must be considered close to an object. Downstream from an object, there will be an adverse pressure gradient which can sometimes lead to flow reversal. The point where this becomes possible is called the separation point.

Consider an aerofoil. When the viscous boundary layer is considered, flow reversal causes a detached vortex to form which drops off the back of the aerofoil. This process is called vortex shedding and by conservation of circulation (Kelvin's theorem) the moving wing must now have a non-zero vorticity, which generates lift.

Vortex shedding is also the origin of aerodynamic drag. Flow downstream of an object will not be perfectly symmetric; there will be continuous vortex shedding with asymmetric interactions. There will now be a net force on the object in the direction of the fluid flow. Vortex interactions lead to vortices moving to smaller scales over time. This is called turbulence: energy is injected at the large scale and the dissipates to smaller scales.