MA134

Geometry & Motion Revision Guide

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Introduction

Geometry & Motion is an extension of differentiation and integration learned at A-level to higher dimensions. The focus of the course is on calculation and applying the concepts taught. As such there are a great many definitions to commit to memory, but not many proofs. It is also very important to learn to correctly parameterise curves, surfaces and solids. Not only does it mean you will end up with the correct answer at the end of the question, but once you have a parameterisation more often than not the rest of the question is just a case of carefully integrating or differentiating.

Dwight also records all of his lectures as screen casts, which are available through the undergraduate handbook and moodle site. This should be the first place to look for course material as it is the most relevant and up to date source for the current course, as well as further examples.

I would recommend trying to make your own summary notes from the lectured course, as this will help you learn the material better than just looking through these, which are in fact a typeset copy of my efforts in first year. But, these will serve as a quick reference for those who do not find summarising notes a useful form of revision.

Note 0.1 (A note on notation). Throughout I have tried to be consistent with notation, but along the way I may have slipped up. Where you see $\boldsymbol{r}'(t)$ this means the derivative of $\boldsymbol{r}(t)$ with respect to t, elsewhere, and more usually this is denoted $\dot{\boldsymbol{r}}(t)$, but I'm sure you will know what I mean.

If you don't like my notation feel free to pick your own, but be sure to explain in your work precisely what you mean with your notation, especially if it is non-standard.

Authors

This revision guide for MA134 Geometry & Motion has been designed as an aid to revision, not a substitute for it. Written by Jack Betteridge.
Based upon lectures given by Prof. Dwight Barkley at the University of Warwick in 2011 and updated for the 2015 course
Any corrections or improvements should be entered into our feedback form at http://tinyurl.com/WMSGuides (alternatively email revision.guides@warwickmaths.org).

1 Curves & Parameterisations

$\boldsymbol{r} : I \to \mathbb{R}^3, I \subset \mathbb{R}$ s.t. $t \mapsto \boldsymbol{r}(t) = (x(t), y(t), z(t))$ is a vector valued function of time

Definition 1.1. $\mathcal{C} \subset \mathbb{R}$ is a curve (or path) if $\exists\ \boldsymbol{r} : I \to \mathbb{R}^n$ continuous s.t. $\mathcal{C} = \{\boldsymbol{r}(t) : t \in I\}$, where I is an interval.

Definition 1.2. The mapping $t \mapsto \boldsymbol{r}(t),\ \boldsymbol{r} : I \to \mathbb{R}^n$ is called a parameterisation of $\mathcal{C}$ if it consists of n continuous functions of one variable t; $x_1(t), \ldots, x_n(t)$, called the components of $\boldsymbol{r}$ (a parameterisation is not unique!)

Example 1.3 (Simple parameterisation)**.** Parameterise the triangle $(0,0), (1,0), (0,1)$ using the unit interval.

Step 1: Draw a picture (I leave space here for you to practise this essential skill)

Step 2: Split up the interval into $[0, 1/3], [1/3, 2/3], [2/3, 1]$ then
$(0,0)$ to $(1,0)$ is just $(3t,\ 0)$
$(1,0)$ to $(0,1)$ is $(1 - 3(t - 1/3),\ 3(t - 1/3)) = (2 - 3t,\ 3t - 1)$
$(0,1)$ to $(0,0)$ is $(0,\ 1 - 3(t - 2/3)) = (0,\ 3 - 3t)$

Step 3: Put it all together:

$$\boldsymbol{r}(t) = \begin{cases} (3t,\ 0) & \text{on } [0, 1/3] \\ (2 - 3t,\ 3t - 1) & \text{on } [1/3, 2/3] \\ (0,\ 3 - 3t) & \text{on } [2/3, 1] \end{cases}$$

Example 1.4 (Parameterisation)**.** Parameterise a helix starting at $(a, 0, 0)$, ending at $(-a, 0, 1)$ with radius a and 1.5 turns anticlockwise around the z-axis.

Step 1: Draw a picture (I leave space here for you to practise this essential skill)

Step 2: We know a circle with radius a has parameterisation

$$(a\cos t,\ a\sin t), \qquad t \in [0, 2\pi]$$

This is traversed anticlockwise. To go around 1.5 times $t \in [0, 3\pi]$. We also need the curve to climb to a height of 1 using the same t in the z-component i.e:

$$\frac{t}{3\pi}$$

Step 3: Put it all together and we have the parameterisation:

$$\boldsymbol{r}(t) = (a\cos t,\ a\sin t,\ t/3\pi)$$

Alternatively we can have the unit interval parameterisation:

$$\boldsymbol{r}(t) = (a\cos(3\pi t),\ a\sin(3\pi t),\ t)$$

Definition 1.5. If $\mathcal{C}$ is parameterised by $\boldsymbol{r}(t)$ & t is actually time then velocity is the vector:

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

Component by component:

$$\boldsymbol{v}(t) = \left(\frac{dx_1}{dt}(t), \ldots, \frac{dx_n}{dt}(t)\right)$$

Speed is the magnitude of velocity, a scalar, i.e:

$$\textit{speed} = \|\boldsymbol{v}(t)\| = \left\|\frac{d\boldsymbol{r}}{dt}(t)\right\|$$

Acceleration is the rate of change of velocity (a vector):

$$\boldsymbol{a}(t) := \frac{d\boldsymbol{v}}{dt}(t)$$

Definition 1.6. A closed curve or loop is where $\boldsymbol{r} : [a, b] \to \mathbb{R}^n$ & $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. To rule out intersections $\boldsymbol{r}(t_1) \neq \boldsymbol{r}(t_2)$ given $t_1 \neq t_2$ unless $t_1, t_2 \in \{a, b\}$.

Definition 1.7. A curve is regular if there exists a parameterisation s.t. $\frac{d\boldsymbol{r}}{dt}$ is defined and non-zero at all points, so has no corners or cusps.

Definition 1.8. Let $\mathcal{C}$ be a curve parameterised by $\boldsymbol{r} : [a, b] \to \mathbb{R}^n$ the length of $\mathcal{C}$:

$$\ell(\mathcal{C}) := \int_a^b \left\|\frac{d\boldsymbol{r}}{dt}(t)\right\| .dt$$

Note 1.9. The length is independent of parameterisation.

Definition 1.10. Given a parameterisation of $\mathcal{C}$ we know $\boldsymbol{r}'(t)$ is tangent to $\mathcal{C}$ at $\boldsymbol{r}(t)$.
The tangent vector $\boldsymbol{\tau}$ is given by:

$$\boldsymbol{\tau}(t) := \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

Principle normal $\boldsymbol{n}$ is:

$$\boldsymbol{n}(t) := \frac{\boldsymbol{\tau}'(t)}{\|\boldsymbol{\tau}'(t)\|}$$

Curvature:
Define κ as curvature:

$$\kappa(t) := \frac{\|\boldsymbol{\tau}'(t)\|}{\|\boldsymbol{r}'(t)\|}$$

Define ρ as radius of curvature:

$$\rho(t) := \frac{1}{\kappa(t)}$$

The binormal vector $\boldsymbol{b}$ is:

$$\boldsymbol{b}(t) := \boldsymbol{\tau}(t) \times \boldsymbol{n}(t)$$

Note 1.11. κ, ρ are scalars & independent of parameterisation.

Note 1.12. $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{b}$ form the Frenet basis.

Example 1.13 (Properties of curves)**.** Find the speed, acceleration and length of the curve

$$\mathcal{C}: \qquad \boldsymbol{r}(t) = (a\cos(3\pi t),\ a\sin(3\pi t),\ t^3) \qquad t \in [0,1]$$

Furthermore find an expression for the Frenet basis and radius of curvature.

Step 1: Draw a picture (I leave space here for you to practise this essential skill)

Velocity:

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = (-3\pi a\sin(3\pi t),\ 3\pi a\cos(3\pi t),\ 3t^2)$$

Speed:

$$\begin{aligned}\|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| &= \sqrt{(-3\pi a\sin(3\pi t))^2 + (3\pi a\cos(3\pi t))^2 + (3t^2)^2} \\ &= \sqrt{9\pi^2a^2\sin^2(3\pi t) + 9\pi^2a^2\cos^2(3\pi t) + 9t^4} \\ &= 3\sqrt{\pi^2a^2 + t^4}\end{aligned}$$

Acceleration:

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t) = (-9\pi^2a\cos(3\pi t),\ -9\pi^2a\sin(3\pi t),\ 6t)$$

Magnitude of acceleration:

$$\begin{aligned}\|\boldsymbol{a}(t)\| = \|\boldsymbol{v}'(t)\| = \|\boldsymbol{r}''(t)\| &= \sqrt{(-9\pi^2 a\cos(3\pi t))^2 + (-9\pi^2 a\sin(3\pi t))^2 + (6t)^2} \\ &= \sqrt{81\pi^4 a^2\cos^2(3\pi t) + 81\pi^4 a^2\sin^2(3\pi t) + 36t^2} \\ &= 3\sqrt{9\pi^4 a^2 + 4t^2}\end{aligned}$$

Hence:

$$\begin{aligned}\boldsymbol{\tau}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} &= \frac{1}{\sqrt{\pi^2a^2+t^4}}(-\pi a\sin(3\pi t),\ \pi a\cos(3\pi t),\ t^2) \\ \boldsymbol{n}(t) = \frac{\boldsymbol{\tau}'(t)}{\|\boldsymbol{\tau}'(t)\|} &= \frac{1}{\sqrt{9\pi^4a^2+4t^2}}(-3\pi^2 a\cos(3\pi t),\ -3\pi^2 a\sin(3\pi t),\ 2t) \\ \boldsymbol{b}(t) = \boldsymbol{\tau}(t)\times\boldsymbol{n}(t) &= \frac{1}{\sqrt{\pi^2a^2+t^4}\sqrt{9\pi^4a^2+4t^2}}(-\pi a\sin(3\pi t),\ \pi a\cos(3\pi t),\ t^2) \\ &\qquad \times(-3\pi^2 a\cos(3\pi t),\ -3\pi^2 a\sin(3\pi t),\ 2t) \\ &= \frac{1}{\sqrt{\pi^2a^2+t^4}\sqrt{9\pi^4a^2+4t^2}}(2t\pi a\cos(3\pi t) + 3t^2\pi a\sin(3\pi t), \\ &\qquad -3t^2\pi a\cos(3\pi t) - 2t\pi a sin(3\pi t),\ 3\pi^3a^2\cos(6\pi t)) \\ \rho(t) = \frac{1}{\kappa(t)} = \frac{\|\boldsymbol{r}'(t)\|}{\|\boldsymbol{\tau}'(t)\|} &= \frac{3\sqrt{\pi^2a^2+t^4}}{3\sqrt{9\pi^4a^2+4t^2}} = \sqrt{\frac{\pi^2a^2+t^4}{9\pi^4a^2+4t^2}}\end{aligned}$$

Finally:

$$\begin{aligned}\ell(\mathcal{C}) &= \int_a^b \|\boldsymbol{r}'(t)\|.dt \\ &= \int_0^1 3\sqrt{\pi^2a^2+t^4}.dt \\ &= \int_0^{\operatorname{arsinh}(1/\pi a)} 3\pi^2a^2\cosh^2(\theta).d\theta \qquad \text{Using } t = \pi a\sinh(\theta) \\ &= \frac{3}{2}\pi a\left(\pi a \operatorname{arsinh}\left(\frac{1}{\pi a}\right) + \sqrt{1+\frac{1}{\pi a}}\right)\end{aligned}$$

2 Derivatives

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$, then the <u>partial derivative</u> of f with respect to x_k is:

$$\frac{\partial f}{\partial x_k}(x_1,\ldots,x_n) := \lim_{h\to 0}\left[\frac{f(x_1,\ldots,x_k+h,\ldots,x_n) - f(x_1,\ldots,x_k,\ldots,x_n)}{h}\right]$$

Definition 2.2. The <u>directional derivative</u> of $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\boldsymbol{x}$ in a direction $\boldsymbol{v}$, $(\boldsymbol{x},\boldsymbol{v}\in\mathbb{R}^n)$ is:

$$(D_{\boldsymbol{v}}f)(x) := \lim_{h\to 0}\frac{f(\boldsymbol{x}+h\boldsymbol{v}) - f(\boldsymbol{x})}{h}$$

Definition 2.3.

$$\nabla f(\boldsymbol{x}) := \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}),\ldots,\frac{\partial f}{\partial x_n}(\boldsymbol{x})\right)$$

A useful approximation is: $f(\boldsymbol{x}+\boldsymbol{h}) \approx f(\boldsymbol{x}) + \boldsymbol{h}\cdot\nabla f(\boldsymbol{x})$
So $(D_{\boldsymbol{v}}f)(\boldsymbol{x}) = \boldsymbol{v}\cdot\nabla f(\boldsymbol{x}) = \|\boldsymbol{v}\|\|\nabla f(\boldsymbol{x})\|\cos\theta$

$\theta = 0$ gives maximum gradient, $\theta = \pi$ gives minimum gradient, $\theta = \pi \pm \frac{\pi}{2}$ gives the direction for level curve.

Definition 2.4. Tangent plane:

$$\pi(\boldsymbol{x}) : (\boldsymbol{r} - \boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) = 0$$

Definition 2.5. The chain rule:
One dimension:

$$\frac{d}{dt} f(h(t)) = f'(h(t)) \cdot h'(t)$$

Multivariable: If $r : \mathbb{R} \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, let $g = f \circ \boldsymbol{r}$, then for $n = 3$

$$\frac{d}{dt} g(t) = \frac{d}{dt} f(\boldsymbol{r}(t)) = \nabla f(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

or more generally:

$$\frac{dg}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}$$

3 Area & Volume

Definition 3.1. Area under curve f:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x).dx$$

Definition 3.2. Volume under surface f:

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} f(\boldsymbol{x}_i)\Delta A_i = \iint_\Omega f.dA = \int_c^d \left(\int_a^b f(x,y).dx \right).dy$$

Integrating f over a 2D area with limits:

$$V = \int_a^b \int_{g(x)}^{h(x)} f(x,y).dy.dx$$

where: $\Omega = \left\{ (x,y) : \begin{array}{c} a \le x \le b, \\ g(x) \le y \le h(x) \end{array} \right\}$

$$V = \int_c^d \int_{\xi(y)}^{\eta(y)} f(x,y).dx.dy$$

where: $\Omega = \left\{ (x,y) : \begin{array}{c} \xi(y) \le x \le \eta(y), \\ c \le y \le d \end{array} \right\}$

Note 3.3. Area $= \iint_\Omega .dA$

We also have:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\boldsymbol{x}_i)\Delta V_i = \iiint_\Omega f.dV = \int_a^b \int_c^d \int_e^f f(x,y,z).dz.dy.dx$$

Integrating f over a 3D volume with limits:

$$\iiint_\Omega f.dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\xi(x,y)}^{\eta(x,y)} f(x,y,z).dz.dy.dx$$

where: $\Omega = \left\{ (x,y,z) : \begin{array}{c} a \leq x \leq b, \\ g(x) \leq y \leq h(x) \\ \xi(x,y) \leq z \leq \eta(x,y) \end{array} \right\}$

Note 3.4. Volume $= \iiint_\Omega .dV$

Definition 3.5. Centre of Mass: For a body with density $\rho(x,y,z)$, let

$$\bar{x} = \frac{1}{M} \iiint_\Omega x\rho.dV$$

$$\bar{y} = \frac{1}{M} \iiint_\Omega y\rho.dV$$

$$\bar{z} = \frac{1}{M} \iiint_\Omega z\rho.dV$$

where

$$M = \iiint_\Omega \rho.dV$$

Then the centre of mass is

$$\boldsymbol{r}_{COM} := (\bar{x}, \bar{y}, \bar{z})$$

4 Polar & Spherical Coordinates

Definition 4.1. Polar Coordinates (r, θ):

$$\Delta A = \Delta r \Delta\theta\, r$$

so

$$V = \int_\theta \int_r f(r,\theta) r.dr.d\theta$$

Definition 4.2. Cylindrical Coordinates(r, θ, z):

$$\Delta V = \Delta r \Delta\theta \Delta z\, r$$

so

$$V = \int_\theta \int_r \int_z f(r,\theta,z) r.dz.dr.d\theta$$

Definition 4.3. Spherical Coordinates (ρ, θ, ϕ):

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

so

$$\Delta V = \Delta\rho \Delta\phi \Delta\theta \rho^2 \sin\phi$$

so

$$\iiint_\Omega f.dV = \int_\phi \int_\theta \int_\rho f(\rho,\theta,\phi)\rho^2 \sin\phi.d\rho.d\theta.d\phi$$

Example 4.4 (Triple integrals and spherical coordinates)**.** Find the mass of the following region

$$\Omega = \{(x, y, z) \in \mathbb{R} : x^2 + y^2 + z^2 \leq 5,\ x^2 + y^2 \leq z^2,\ z \geq 0\}$$

given that its density is given by

$$f(x, y, z) = x^2 + y^2 + z^2 + 1$$

Step 1: Draw a picture (I leave space here for you to practise this essential skill)

Step 2: Notice from your picture that this is just a portion of a sphere and hence it may be best to use spherical coordinates.

Step 3: Reparameterise the domain using spherical coordinates

$$\Omega = \left\{(\theta, \varphi, \rho) \in \mathbb{R} : 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \frac{\pi}{4},\ 0 \leq \rho \leq 5\right\}$$

Step 4: Reparameterise the density function

$$f(\theta, \varphi, \rho) = f(\rho) = \rho^2 + 1$$

Step 5: Integrate

$$\begin{aligned}
\iiint_\Omega f.dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^5 (\rho^2 + 1)\rho^2 \sin(\varphi).d\rho.d\varphi.d\theta \\
&\quad \text{Notice we can split the integral} \\
&= \int_0^{2\pi} .d\theta \int_0^{\pi/4} \sin(\varphi).d\varphi \int_0^5 \rho^4 + \rho^2.d\rho \\
&= 2\pi \Big[-\cos\varphi \Big]_0^{\pi/4} \left[\frac{\rho^5}{5} + \frac{\rho^3}{3}\right]_0^5 \\
&= \frac{2000\left(2 - \sqrt{2}\right)\pi}{3}
\end{aligned}$$

5 Transformations & Generalised Coordinates

2D Case: $\boldsymbol{\psi} : \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{\varphi} : \Gamma \to \Omega,\ (u, v) \mapsto (x(u, v), y(u, v))$ now

$$\Delta A = \left|\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| \Delta u \Delta v$$

so:

$$\iint_{\Omega} f.dA = \underbrace{\int_u \int_v}_{\Gamma} f(\boldsymbol{\psi}(u,v)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| .dv.du$$

Denote

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \text{ as } \frac{\partial(x,y)}{\partial(u,v)}$$

which is called the Jacobian Matrix

3D Case: $\boldsymbol{\psi} : \mathbb{R}^3 \to \mathbb{R}^3$, $\boldsymbol{\varphi} : \Gamma \to \Omega$, $(u,v,w) \mapsto (x(u,v,w), y(u,v,w), z(u,v,w))$ now

$$\Delta V = \left| \det \left(\frac{\partial(x,y,z)}{\partial(u,v,w)} \right) \right| \Delta u \Delta v \Delta w$$

so:

$$\iiint_{\Omega} f.dV = \underbrace{\int_u \int_v \int_w}_{\Gamma} f(\boldsymbol{\psi}(u,v,w)) \left| \det \left(\frac{\partial(x,y,z)}{\partial(u,v,w)} \right) \right| .dw.dv.du$$

6 Line Integrals

Definition 6.1. The arclength parameterisation (or natural parameterisation) is a parameterisation $\boldsymbol{r}(t)$ of a curve $\mathcal{C}$ s.t. $\forall s \in [0,T]$ where $\ell(\mathcal{C}) = T$ we have

$$\frac{d}{ds} \int_0^s \|\boldsymbol{r}(t)\|.dt = \frac{d}{ds} s = 1$$

$\implies \|\boldsymbol{r}'(s)\| = 1$

Note 6.2. When integrating, we often denote integrating with respect to an arc length parameterisation by $ds = \|\tilde{\boldsymbol{r}}'(t)\|.dt$, where $\tilde{\boldsymbol{r}} : [a,b] \to \mathbb{R}^n$ is any parameterisation of a curve $\mathcal{C}$ and $\boldsymbol{r} : [0,T] \to \mathbb{R}^n$ is the arclength parameterisation of the curve $\mathcal{C}$ i.e:

$$\int_{\mathcal{C}} f.ds = \int_0^T f(\boldsymbol{r}(s)).ds = \int_a^b f(\boldsymbol{r}(t)) \|\boldsymbol{r}'(t)\|.dt$$

Example 6.3. Recall for a curve $\mathcal{C}$ with parameterisation $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$:

$$\ell(\mathcal{C}) = \int_a^b \|\boldsymbol{r}'(t)\|.dt$$

So now we are free to denote

$$\ell(\mathcal{C}) = \int_{\mathcal{C}} .ds$$

where

$$ds = \|\boldsymbol{r}'(t)\|.dt$$

denotes integrating with respect to the arc length parameterisation.

Definition 6.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ & $\mathcal{C} \subset \mathbb{R}^n$ be a curve with a parameterisation $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$, the line integral of f over $\mathcal{C}$ is

$$\int_{\mathcal{C}} f.ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|.dt$$

Example 6.5 (Scalar line integral)**.** Evaluate the line integral for a function

$$f(x,y) = x^2 + y^2$$

along the path

$$\mathcal{C}: \qquad \boldsymbol{r}(t) = (t,t) \qquad t \in [0,1]$$

Step 1: Calculate $\boldsymbol{r}'(t)$ and hence $\|\boldsymbol{r}'(t)\|$

$$\boldsymbol{r}'(t) = (1,1)$$

$$\|\boldsymbol{r}'(t)\| = \sqrt{2}$$

Step 2: Use this to calculate the line integral

$$\begin{aligned}
\int_{\mathcal{C}} f.ds &= \int_0^1 f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|.dt \\
&= \int_0^1 f(t,t)\sqrt{2}.dt \\
&= \sqrt{2}\int_0^1 2t^2.dt \\
&= 2\sqrt{2}\left[\frac{t^3}{3}\right]_0^1 \\
&= \frac{2\sqrt{2}}{3}
\end{aligned}$$

Definition 6.6. Let $\boldsymbol{v} : \mathbb{R}^n \to \mathbb{R}^m$ & $\mathcal{C} \subset \mathbb{R}^n$ be a curve with a parameterisation $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$. The line integral of the vector field $\boldsymbol{v}$ over $\mathcal{C}$ is

$$\int_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} = \int_a^b \boldsymbol{v}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t).dt$$

The line integral differs only by the direction $\boldsymbol{r}$ traverses $\mathcal{C}$ i.e: by a factor of -1.

Definition 6.7. Gradient Fields are vector fields with the property that $\boldsymbol{v} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$

Theorem 6.8. Fundamental Theorem for Line Integrals of Vector Fields: Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$, $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$ with endpoints $\boldsymbol{a} = \boldsymbol{r}(a)$, $\boldsymbol{b} = \boldsymbol{r}(b)$ & let $\boldsymbol{v}$ be a gradient field i.e: $\boldsymbol{v} = \nabla f$. Then

$$\int_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

i.e: The line integral of a gradient field depends only upon the end points and not the path taken.

Cor$^{\underline{\text{ly}}}$ 6.9.

$$\oint_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} = 0 \iff \boldsymbol{v} \textit{ is a gradient field.}$$

Where $\oint$ *denotes integrating over a closed curve.*

Example 6.10 (Vector line integral)**.** Evaluate the line integral for a vector field

$$v(x,y) = (2x, 2y)$$

along the path

$$\mathcal{C}: \qquad \boldsymbol{r}(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$$

Step 1: Calculate $\boldsymbol{r}'(t)$, remember we don't have to calculate $\|\boldsymbol{r}'(t)\|$ here

$$\boldsymbol{r}'(t) = (-\sin t,\ \cos t)$$

Step 2: Use this to calculate the line integral

$$\begin{aligned}
\int_{\mathcal{C}} \boldsymbol{v}.d\boldsymbol{\ell} &= \int_0^{2\pi} f(\boldsymbol{v}(t)) \cdot \boldsymbol{r}'(t).dt \\
&= \int_0^{2\pi} \boldsymbol{v}(\cos t,\ \sin t) \cdot (-\sin t,\ \cos t).dt \\
&= \int_0^{2\pi} (2\cos t,\ 2\sin t) \cdot (-\sin t,\ \cos t).dt \\
&= \int_0^{2\pi} -2\cos t \sin t + 2\sin t \cos t.dt \\
&= 0
\end{aligned}$$

Note 6.11. $v = \nabla f$ where $f(x,y) = x^2 + y^2$ and $\mathcal{C}$ is a closed curve, so we could have arrived at this result far quicker if we had used the fundamental theorem of line integrals. This is very useful to notice in an exam!

7 Surface Integrals

Note 7.1. For curves we have: $\boldsymbol{r} : I \to \mathbb{R}^n, \quad I = [a,b]$
For surfaces: $\boldsymbol{r} : \Omega \to \mathbb{R}^n \quad \Omega = [a,b] \times [c,d]$

Definition 7.2. The <u>tangent plane</u> to a surface $\boldsymbol{r}$ at the point (u_0, v_0) is given by: $\boldsymbol{p} : \mathbb{R}^2 \to \mathbb{R}^3$, $(h,k) \mapsto \mathbb{R}^3$, where

$$\boldsymbol{p}(h,k) = \boldsymbol{r}(u_0,v_0) + h\frac{\partial \boldsymbol{r}}{\partial u}(u_0,v_0) + k\frac{\partial \boldsymbol{r}}{\partial v}(u_0,v_0)$$

Definition 7.3. The <u>normal vector</u> to the tangent plane is given by

$$\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \qquad \& \qquad \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}$$

Definition 7.4. For $f : \mathbb{R}^3 \to \mathbb{R}$, the <u>surface integral</u> for a surface $\mathcal{S}$, parameterised by $\boldsymbol{r}$, is

$$\iint_{\mathcal{S}} f.dS = \iint_{\Omega} f(\boldsymbol{r}(u,v)) \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| .du.dv$$

Example 7.5 (Surface integral)**.** Find the surface integral of

$$f(x,y,z) = z$$

over the upper unit hemisphere.

Step 1: Draw a picture (I leave space here for you to practise this essential skill)

Step 2: Notice from your picture that this is just a portion of a sphere and hence it may be best to parameterise using trigonometric functions.

Step 3: Parameterise the surface

$$\mathcal{S}: \qquad \boldsymbol{r}(\theta, \varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi) \qquad \theta \in [0, 2\pi],\ \varphi \in [0, \pi/2]$$

Step 4: Evaluate the partial derivatives with respect to each of your parameters

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = (-\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ 0)$$

$$\frac{\partial \boldsymbol{r}}{\partial \varphi} = (\cos\theta, \cos\varphi,\ \sin\theta\cos\varphi,\ -\sin\varphi)$$

Step 5: Calculate $\left\|\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \varphi}\right\|$

$$\begin{aligned}
\left\|\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \varphi}\right\| &= \|(-\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ 0) \times (\cos\theta, \cos\varphi,\ \sin\theta\cos\varphi,\ -\sin\varphi)\| \\
&= \left\|(-\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi)\right\| \\
&= \left\|(-\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin\varphi\cos\varphi)\right\| \\
&= \sqrt{(-\cos\theta\sin^2\varphi)^2 + (-\sin\theta\sin^2\varphi)^2 + (-\sin\varphi\cos\varphi)^2} \\
&= \sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\
&= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\
&= \sin\varphi \qquad \text{since sin is positive for } \varphi \in [0, \pi/2]
\end{aligned}$$

Step 6: Use this to find the surface integral

$$\begin{aligned}
\iint_{\mathcal{S}} f.dS &= \int_0^{2\pi}\int_0^{\pi/2} f(\boldsymbol{r}(\theta,\varphi)) \left\|\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \varphi}\right\| .d\varphi.d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} f(\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)\sin\varphi.d\varphi.d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \cos\varphi\sin\varphi.d\varphi.d\theta \\
&\quad \text{Notice we can split the integral} \\
&= \frac{1}{2}\int_0^{2\pi} .d\theta \int_0^{\pi/2} \sin 2\varphi.d\varphi \\
&= \frac{1}{2}\ 2\pi \left[-\frac{1}{2}\cos 2\varphi\right]_0^{\pi/2} \\
&= \pi
\end{aligned}$$

Note 7.6. Area of S is given by

$$\iint_{\mathcal{S}} .dS = \iint_{\Omega} \left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\| .du.dv$$

Definition 7.7. Let $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ & $\mathcal{S} \subset \mathbb{R}^3$ be a curve with a parameterisation $\boldsymbol{r} : \Omega \to \mathbb{R}^3$. The flux integral is given by:

$$\iint_{\mathcal{S}} \boldsymbol{v} \cdot \hat{\boldsymbol{n}}.dS$$

where

$$\hat{\boldsymbol{n}} = \frac{\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}}{\left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\|}, \qquad dS = \left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\| .du.dv$$

hence

$$\iint_{\mathcal{S}} \boldsymbol{v} \cdot \hat{\boldsymbol{n}}.dS = \iint_{\Omega} \boldsymbol{v} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) .du.dv$$

Example 7.8 (Flux integral)**.** Find the flux of the vector field

$$\boldsymbol{v}(x, y) = (-4y,\ x,\ 0)$$

through the upper unit hemisphere.

Step 1: Notice that steps 1–5 will be identical to the previous example, but we don't need to find the magnitude of the cross product. Again, it will save time in an exam if you spot things like this.

Step 2 (or 6): Use previous work to calculate the flux integral

$$\begin{aligned}
\iint_S \boldsymbol{v}\cdot\hat{\boldsymbol{n}}.dS &= \int_0^{2\pi}\int_0^{\pi/2} \boldsymbol{v}(\boldsymbol{r}(\theta,\varphi))\cdot\left(\frac{\partial \boldsymbol{r}}{\partial\theta}\times\frac{\partial \boldsymbol{r}}{\partial\varphi}\right).d\varphi.d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \boldsymbol{v}(\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi) \\
&\qquad \cdot(-\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin\varphi\cos\varphi).d\varphi.d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} (-4\sin\theta\sin^2\varphi,\ \cos\theta\sin^2\varphi,\ 0) \\
&\qquad \cdot(-\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin\varphi\cos\varphi).d\varphi.d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} 4\sin\theta\cos\theta\sin^3\varphi - \sin\theta\cos\theta\sin^3\varphi.d\varphi.d\theta \\
&\quad \text{Notice we can split the integral} \\
&= \int_0^{2\pi} 3\sin\theta\cos\theta.d\theta \int_0^{\pi/2} \sin^3\varphi.d\varphi \\
&= \frac{3}{2}\int_0^{2\pi} \sin 2\theta.d\theta \int_0^{\pi/2} \sin\varphi - \cos^2\varphi\sin\varphi.d\varphi \\
&= \frac{3}{2}\underbrace{\Big[\cos 2\theta\Big]_0^{2\pi}}_{=0}\left[\cos\varphi + \frac{1}{3}\cos^3\varphi\right]_0^{\pi/2} \\
&= 0
\end{aligned}$$

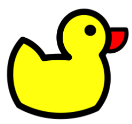
8 Critical Points

When $\nabla f = \mathbf{0}$, let

$$A = \frac{\partial^2 f}{\partial x^2}(\boldsymbol{x}_0), \quad B = \frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{x}_0), \quad C = \frac{\partial^2 f}{\partial y^2}(\boldsymbol{x}_0) \ \& \quad D = AC - B^2$$

f is a maximum at $\boldsymbol{x}_0$ if $D > 0$ & $A < 0$
f is a minimum at $\boldsymbol{x}_0$ if $D > 0$ & $A > 0$
f is a saddle point at $\boldsymbol{x}_0$ if $D < 0$
If $D = 0$ at $\boldsymbol{x}_0$ no conclusion can be made about $f(\boldsymbol{x}_0)$

Here is a duck:



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