

PX153 Maths for Physicists

Section 2 - Fourier Series

6.1 Fourier Series and Coefficients

A Fourier Series can be used to represent a function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Where $-L \leq x \leq L$ and a_n and b_n are constants

For a function $f(x)$ to be represented by a Fourier Series it must:

- Be periodic $\rightarrow L$ is the period
- Be single valued and continuous, except possibly at a finite number of finite discontinuities
- have a finite number of maxima and minima within one period
- the integral over one period of $|f(x)|$ must converge

The coefficients a_0 , a_n and b_n can be found using

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Orthogonality Relations:

$$(\alpha, \beta \in \mathbb{Z})$$

$$\int_{-\pi}^{\pi} \sin(\alpha x) \cos(\beta x) dx = 0$$

$$\int_{-\pi}^{\pi} \sin(\alpha x) \sin(\beta x) dx = \int_{-\pi}^{\pi} \cos(\alpha x) \cos(\beta x) dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \pi & \text{if } \alpha = \beta \end{cases}$$

6.2 Properties of Fourier Series

- The Fourier Series is periodic, so $f(x+2L) = f(x)$
- If a function is symmetric, $f(x) = -f(-x)$, and $b_n = 0$ for all n
 - e.g. $\cos(nx)$, $|x|$
- If a function is antisymmetric, $f(x) = -f(-x)$, and $a_n = 0$ for all n
 - e.g. $\sin(nx)$, x
- A Fourier Series can be used to prove the result of a series

Example: Show $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} (2n+1)^{-2}$ using the Fourier Series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} \dots \right)$$

Let $x=0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left(1 + \frac{1}{9} + \frac{1}{25} \dots \right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} \dots$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)^2} \right)$$

6.3 Parseval's theorem

A function $f(x)$ has a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$
then Parseval's theorem gives

$$\frac{1}{L} \int_{-L}^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

6.4 Fourier Sine and Cosine Series

If you want to compute a fourier series for a function within a range $0 \leq x \leq L$, a Sine or Cosine Series can be used:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

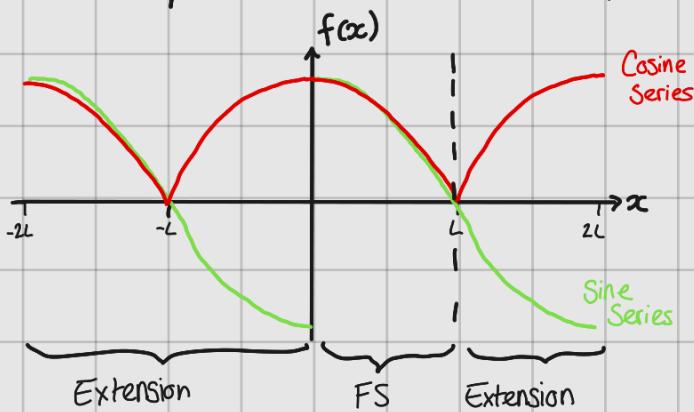
With Coefficients

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

The periodic extension of a Sine and Cosine Series for the same function $f(x)$ will differ as Cosine is symmetric and Sine is antisymmetric



6.5 Gibbs Phenomenon

- When a function is discontinuous at a value, the Fourier Series at that value will be halfway between the upper and lower value
- For discontinuity $f(x_d)$ at x_d , the Fourier Series will converge to

$$\frac{f(x_d^-) + f(x_d^+)}{2}$$

- At discontinuities, the Fourier Series will overshoot before the discontinuity. As the number of terms increases, the overshoot moves closer to the discontinuity

6.6 Convergence

Pointwise Convergence

- If $f(x)$ and $f'(x)$ are continuous over $[-L, L]$, except possibly at a finite number of points, the Fourier Series converges at every point

$$\tilde{f}(x) = \frac{1}{2} [f(x^+) + f(x^-)]$$

Where $f(x^+) = \lim_{\varepsilon \rightarrow 0} f(x+\varepsilon)$ and $f(x^-) = \lim_{\varepsilon \rightarrow 0} f(x-\varepsilon)$

Uniform Convergence

- If $f(x)$ and $f'(x)$ exist and are continuous over $[-L, L]$ and $f(L) = f(-L)$ the Fourier Series converges uniformly to $f(x)$