PX109: Relativity

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March 2009

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First edition written by Emma Towlson and Typeset by Kieran Bhardwaj in March 2009.
Based off notes from the lectures given in 2007.

1 Introduction

> *"It is known that Maxwell's electrodynamics–as usually understood at the present time–when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise–assuming equality of relative motion in the two cases discussed–to electric currents of the same path and intensity as those produced by the electric forces in the former case."*
>
> *- A. Einstein, On the Electrodynamics of Moving Bodies, June 30 1905*

Special Relativity applies the principal of relativity (that all motion is relative rather than there being a well defined state of rest, proposed by Galileo) to frames in uniform relative motion. So we're just going to brush gravity and acceleration under the carpet for now!
This guide will take you through the principles and consequences of Special Relativity and hopefully provide a helpful resource for preparing for your exam (sorry, last time I'll mention it!)

> *"The hardest thing in the world to understand is income tax."*
>
> *- A. Einstein*

2 Before Einstein

2.1 A Quick Revision Of Newtonian Physics

You're probably sick to death of this by now, so this section is just intended as a summary of the main principles of mechanics we know (or assume!) and love from classical physics.

- **Newton's First Law:** A body continues in its state of rest or uniform motion unless it is acted upon by an external force.
- **Newton's Second Law:** The rate of change of a body's momentum is equal to the total force acting on it:
- $\underline{F} = \frac{d}{dt}(\underline{p})$
- **Newton's Third Law:** For every action, there is an equal and opposite reaction.

Definition: Uniform motion is motion in a straight line at constant velocity.
Notice: Newton's Second Law assumes that mass is a constant, which we shall see is not the case.

- Velocity is the rate of change of position in a specified direction: $V_x = \frac{dx}{dt}$
- Acceleration is the rate of change of velocity in a specified direction: $a_x = \frac{dV_x}{dt}$
- Linear momentum $\underline{p} = m\underline{v}$

- Momentum must be conserved.
- Energy cannot be created or destroyed, so must be conserved.
- The work done by a force, F, in moving a body from x=0 to x=x1 (independent of the path chosen):

$$W = \int_{x=0}^{x=x_1} F_x dx = \int_0^v mv\, dv = \Delta E_{KE}$$

Definition: A quantity is INVARIANT if it can be considered the same in any frame of reference. Newton believed that time was invariant, but this is not in fact the case.

2.2 Galilean Transformations

Suppose we have two frames of reference, S and S'. Suppose also that S' is moving at a constant velocity, u, in the positive x-direction, and that at time $t = 0$ their origins coincided ($O = O'$). (See Figure 1) It should be clear (treated classically) that at time $t = t$, the distance between the

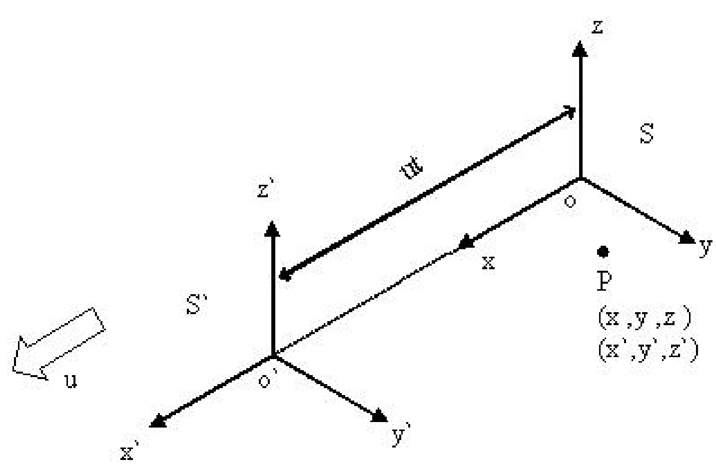

Figure 1: S and S' frames of reference

two frames is ut. So we must have that $x' = x - ut$. Since the motion is entirely in the x-direction, we have that at any time, $y = y$' and $z = z$'.
Now consider a stationary point P observed from both frames (as in Figure 1). We may relate its position relative to frame S to its position relative to S' using the above. These are the intuitive Galilean transformations:

Transform	Inverse Transform
$x' = x - ut$	$x = x' + ut$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = t$	$t = t'$

What if P had a velocity? Suppose it is moving at a constant speed in the x-direction, and its velocity relative to frame S is v (and v' relative to S'). Then simply by differentiating the top terms in transform equations, we see that:

$$v' = \frac{d}{dt}x' = v - u \qquad v = \frac{d}{dt}v' + u$$

Unfortunately, this is no longer accurate at speeds close to that of light.
Definition: An inertial frame is a uniformly moving reference frame.
Remember: Motion must always be referred to a frame of reference.

2.3 The Michelson-Morley Experiment

In the nineteenth century, physicists believed in a stationary ether - the thought was that light must travel through some medium, like waves may travel through water or along a string etc. So, supposing there is an ether, it follows that an 'ether wind' would be induced in the laboratory by the motion of the Earth through the ether, and that this would hinder the progress of light travelling against it. In 1887, Michelson and Morley carried out an ingenious experiment to detect the hypothesised drag.
Using the configuration of mirrors shown in Figure 2, they split a beam of light into two perpendicular beams (with the half-slivered mirror) and then rejoined these beams. Now, they reasoned that the beam travelling perpendicular to the ether wind would have to travel further than the parallel beam. Thus there would be a slight delay in one of the beams when it recombined through interference with the other beam - this would result in a predicted fringe shift of about one twenty-fifth of a fringe. (NB The apparatus was free to rotate, allowing any direction relative to the ether wind.)

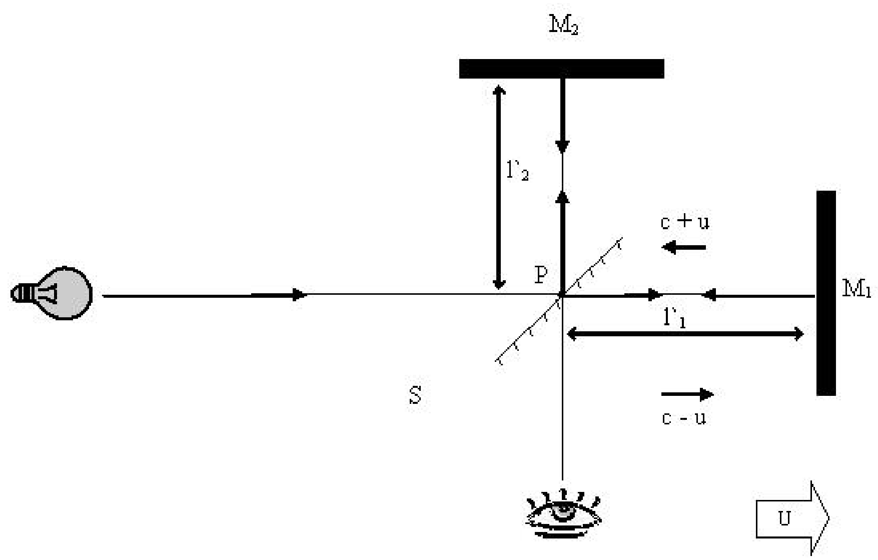

Figure 2: The Michelson-Morley Experiment

but they did not observe any significant fringe shift, providing strong evidence against the idea of an ether.

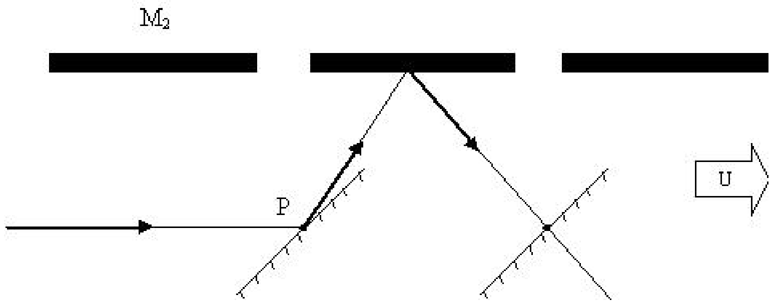

Figure 3: The predicted movement of the lightbeam and mirror in the ether

3 After Einstein

3.1 Einsteins Postulates

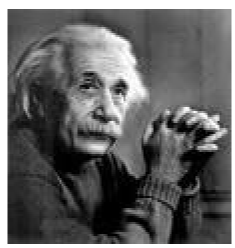

Figure 4: Einstein

1. The laws of physics are the same in all inertial frames.
2. The speed of light in empty space (a vacuum) is the same in all inertial frames and is independent of the motion of its source.

Remember these - you will almost certainly be asked about them!

3.2 Minkowski Diagrams And Simultaneous Events

Suppose lightning struck both the front and back of a train (it's a pretty unlucky train), and that both flashes appear simultaneous to an observer on the ground. To an observer on the train, however, lightning struck the rear first how can this be?
Minkowski Diagrams (Space-Time Diagrams) allow us to see what is going on much more easily. Along the horizontal axis, we have x, the distance from a defined origin. Along the vertical axis, we have ct (notice that this has units of distance, but may be thought of as time increments). So a beam from of light emitted from the origin would be represented by the line $x = ct$ (at 45^o to the axes).
Consider two stationary observers at points A and C (see Figure 5(a)). If a beam of light were emitted from point B, it would reach both observers at the same time, $t1$. Now suppose that both observers are moving at a speed u in the positive x-direction. Refer to Figure 5(b) - the light beam will now reach A first (at time $t1$), and C later (at time $t2$). The lesson is that events which are simultaneous in one inertial frame are not necessarily simultaneous in another.

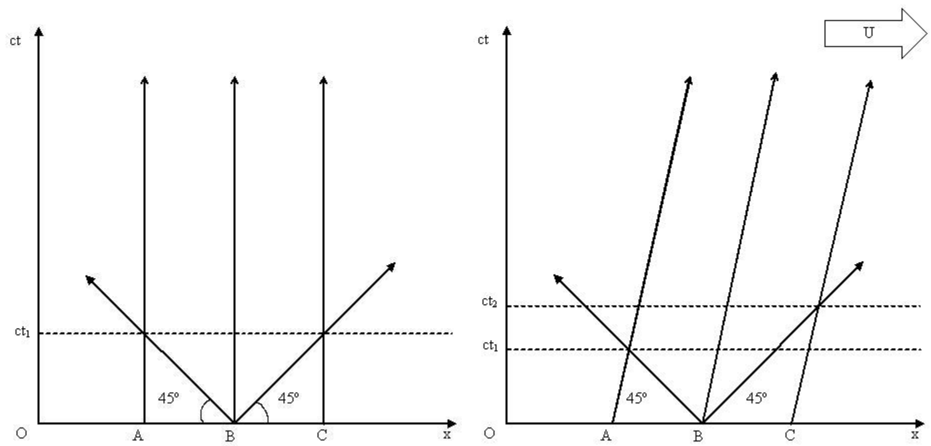

Figure 5: a)Two stationary observers, b) Two moving observers

So can we construct a Minkowski Diagram? Suppose frame S' (co-ordinates (x', ct')) is moving at a speed u relative to frame S (co-ordinates (x, t)). We have the situation in Figure 6. But we need to calibrate the axes. Notice that the ct' axis is the line $x' = 0$.

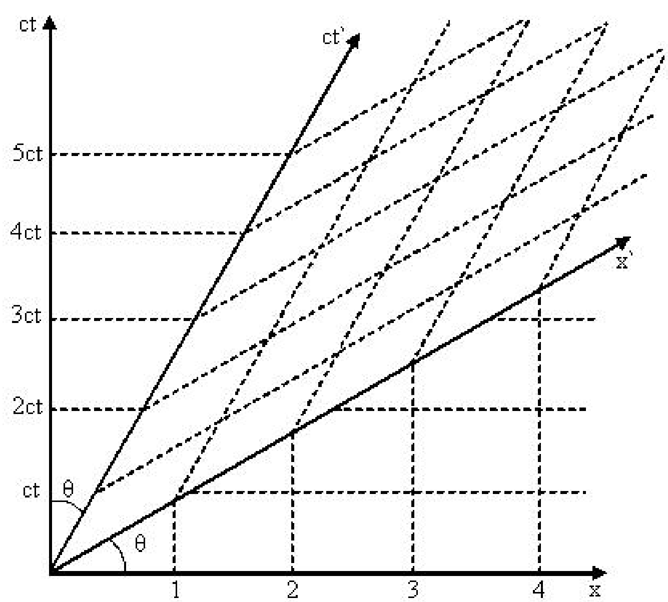

Figure 6: Minkowski diagram calibration

So from the Lorentz Transformations (see section 3.3.1), we see that it must be the line $ct = (c/u)x$. Similarly, the x' axis is the line $ct' = 0$, so must be the line $ct = (u/c)x$.
Fact: $(ct')^2 - (x)'^2 = (ct)^2 - x^2 = s^2$. s^2 is invariant - it is the same for all observers. The intersection of x'-axis with $(x')^2 - (ct)^2 = 1$ defines $x = 1$.

3.3 What Does This Mean Spatially?

3.3.1 Lorentz Transformations

To see how the Galilean Transformations must change to accommodate Einstein's postulates, let us consider the following arrangement. Suppose there are two inertial reference frames, S and S', where S' is travelling at a speed u in the positive x-direction as measured in S (see Figure 7). (From now on, whenever you see S and S', assume this definition.)

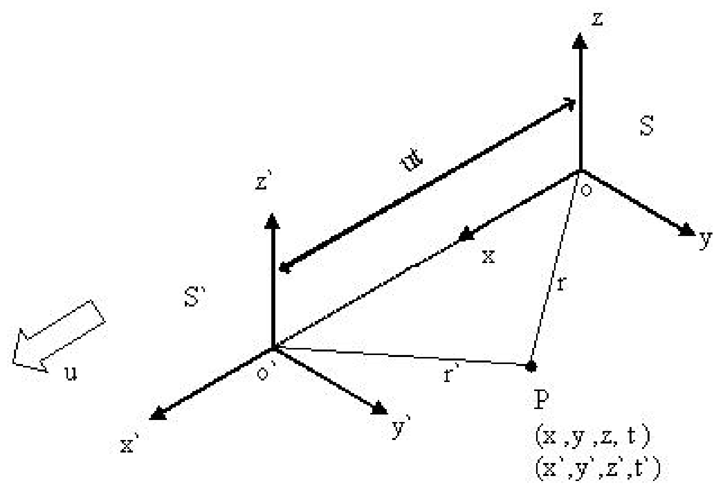

Figure 7: Inertia Frames S and S'

Let us also suppose that at time $t = 0$, the origins of the two frames (O and O') coincided and that a flash of light is emitted from O. After a time t, an observer at O notices that the beam of light has reached point P, which is a distance r from O and a distance r' from O'. If we know the co-ordinates of P in frame S, how can we relate them to the co-ordinates in S'? As before, the y- and z- co-ordinates are easy: frame S' is only travelling in the x-direction, so we may assume that $y = y'$ and $z = z'$. What about the x-co-ordinate? Firstly, note that $r = ct$, where c is the speed of light. But from the Pythagorean Theorem, we also see that:

$$r^2 = x^2 + y^2 + z^2$$

So we must have that:

$$x^2 + y^2 + z^2 = c^2t^2$$

Now, we cannot assume that the same amount of time has passed in frame S' (as S and S' are in relative motion). So let us say that a time t' has passed. Then applying the same principles as above, we have that:

$$(x')^2 + (y')^2 + (z')^2 = (r')^2 = c^2(t')^2$$

How do we proceed? Well, we know that for $u << c$, we must be left with our original Galilean Transformations (these are what we observe at small velocities!). So let us say that:

$$x'(x, t) = \gamma(x - ut)$$

for some γ which increases in significance as u approaches c, but approaches 1 as u is decreased, and that:

$$t'(x, t) = \alpha(t - \beta x)$$

for some α with the same conditions as γ and some β which approaches 0 as u is decreased. Substituting $x'(x,t) = \gamma(x - ut)$ and $t'(x,t) = \alpha(t - \beta x)$ into $(x')^2 + (y')^2 + (z')^2 = (r')^2 = c^2(t')^2$ and trawling through some messy algebra (the reader should feel free to try this an exercise!), we arrive at our destination: the Lorentz Transformations, which are as follows:

$$x' = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{ux}{c^2})$$

and, naturally, the inverse Lorentz Transformations:

$$x = \gamma(x' + ut')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma(1 - \frac{ux'}{c^2})$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Remember: u is the velocity of S' as measured in S.
Does γ behave as we dictated? Plotting it as a function of u (see Figure 8), we see that its value only really starts to grow from 1 at around 0.5c this is 1.510^8 m/s!

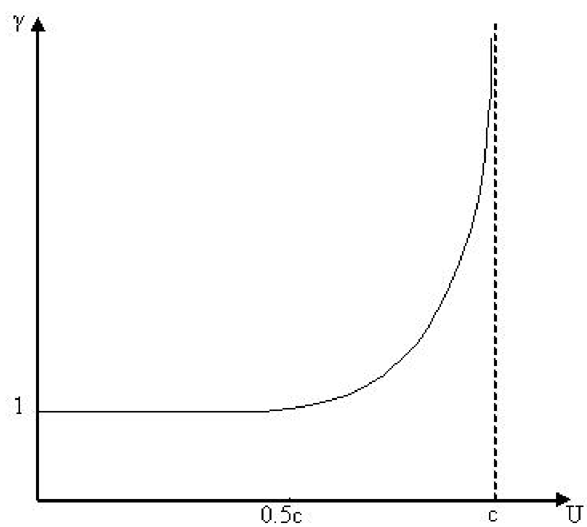

Figure 8: Graph of γ

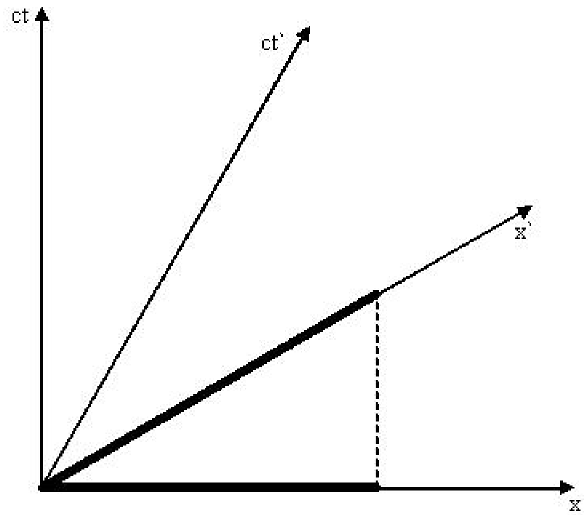

Figure 9: Length Contraction Visually

3.3.2 Length Contraction

The hint is in the title - the length of a body in motion relative to the observer will be measured to be shorter than if it were at rest. To see this, let us simply consider a straight bar observed from frames S and S' (see Figure 9 for a Minkowski diagram illustrating the bar). The bar lies along the x-axis, with its ends at x_1 and x_2. The bar is at rest relative to S, so we may define the proper length, L_o, to be:

$$L_o = x_2 - x_1$$

What would an observer in S' measure? Remembering that time is always changing, we must measure the position of each of its end points (now located at x'_1 and x'_2) in one instantaneous measurement at time $t = t'$. Using the inverse Lorentz Transformations, we see that:

$$x_1 = \gamma(x'_1 + ut')$$

$$x_2 = \gamma(x'_2 + ut')$$

$$=> x_2 - x_1 = L_0 = \gamma(x'_2 - x'_1)$$

But $x'_2 - x'_1 = L'$, where L' is the length of the bar measured in S'. So we have that:

$$L_0 = \gamma L' \qquad \text{or} \qquad L' = \frac{L_0}{\gamma}$$

Remember: If $L' > L_0$ you've done something wrong!

3.4 What Does This Mean For Time?

3.4.1 Time Dilation

This is a very similar calculation to that for length contraction, but with the opposite result: the time difference between two events is smaller in the rest frame of the events than in any other frame in relative motion.

Suppose an observer in frame S at position x_0 measures the time for Event 1 to be t_1 and for Event 2 to be t_2. This is the rest frame, and we define the proper time, Δt, to be:

$$\Delta t = t_2 - t_1$$

This much is obvious, but what is the time difference $\Delta t'$ in S', which is travelling at a velocity u relative to S? Let us say that Event 1 occurs at time $t = t'_1$ and Event 2 at time $t = t'_2$. Then applying the Lorentz Transformations, we see that:

$$t'_1 = \gamma(t_1 - \frac{ux_0}{c^2}) \qquad and \qquad t'_2 = \gamma(t_2 - \frac{ux_0}{c^2})$$

$$=> t'_2 - t'_1 = \gamma(t_2 - t_1)$$

$$=> \Delta t' = \gamma \Delta t \qquad or \qquad \Delta t = \frac{\Delta t'}{\gamma}$$

3.5 What Does This Mean For Velocity?

3.5.1 Lorentz Transformations Of Velocity

Let us return to our S and S' frames, and consider an object moving in the positive x-direction with speed v_x relative to S (and $v'_{x'}$ relative to S') - as in Figure 10. How can we relate v_x and $v'_{x'}$? Firstly, note from the Lorentz Transformations that:

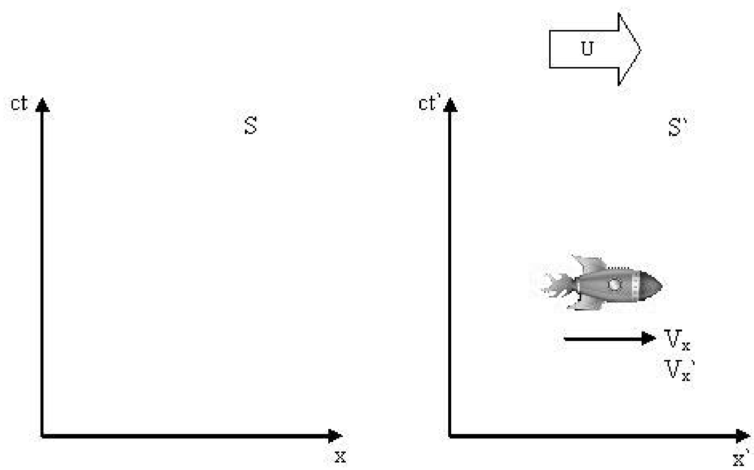

Figure 10: Velocity in S and S' Frames

$$\Delta x' = \gamma(\Delta x - u\Delta t)$$

$$\Delta t' = \gamma(\Delta t - \frac{u\Delta x}{c^2})$$

Also, by definition:

$$v_x = \frac{dx}{dt} \qquad v'_x = \frac{dx'}{dt}$$

and then please grit your teeth through this abuse of mathematics! (Physicists will generally wave their hands and declare authoritatively that "it works") Using $\Delta x' = \gamma(\Delta x - u\Delta t)$:

$$dx' = \gamma(dx - udt)$$

$$=> dx' = \gamma(\frac{dx}{dt} - u\frac{dt}{dt})dt$$

$$dx' = \gamma(v_x - u)dt$$

And using $\Delta t' = \gamma(\Delta t - \frac{u\Delta x}{c^2})$:

$$dt' = \gamma(dt - \frac{udx}{c^2})$$

$$dt' = \gamma(\frac{dt}{dt} - \frac{u}{c^2}\frac{dx}{dt})dt$$

$$dt' = \gamma(1 - \frac{uv_x}{c^2})dt$$

So:

$$v'_{x'} = \frac{dx'}{dt'} = \frac{\gamma(v_x - u)dt}{\gamma(1 - \frac{uv_x}{c^2})dt}$$

$$\therefore v'_{x'} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \qquad or \qquad v_x = \frac{v'_{x'} + u}{1 + \frac{uv'_{x'}}{c^2}}$$

Remember: It's easy to get confused here - u is the speed of S' as measured in S; v_x is the speed of the object relative to S; $v'_{x'}$ is the speed of the object relative to S'.
Notice: Try 'plugging in' v_x=c. What is $v'_{x'}$?

3.5.2 The Relativistic Doppler Effect

You should be familiar with the Doppler Effect, most easily observed in the form of red or blue shifts from stars moving relative to the Earth (see Figure 11). But surely there are relativistic effects here? Let's work it through.

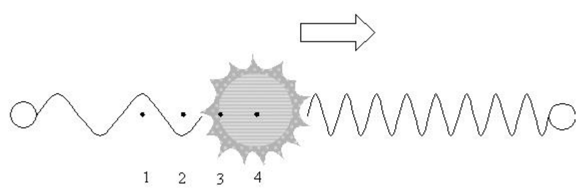

Figure 11: Doppler Shift

Call the rest frame of the star S, and the rest frame of the Earth S' (we shall assume that the star is moving away from Earth, thus S from S', at a relative speed u in the positive x-direction). In frame S, the light emitted is of frequency F_0 and period T_0. This is observed in S' as frequency F' and period T'.
During one period, a time $\Delta t' = \gamma T_0$ passes in frame S' BUT - the star is receding during this time, moving a distance of $\Delta x' = \gamma u T_0$, observed on Earth. This delays the light by:

$$\frac{\Delta x'}{c} = \frac{\gamma u T_0}{c}$$

Thus the period T' observed in S' (Earth) is:

$$T' = \gamma T_0(1 + \frac{u}{c})$$

$$T' = \frac{T_0(1 + \frac{u}{c})}{[(1 + \frac{u}{c})(1 - \frac{u}{c})]^{1/2}}$$

$$T' = T_0\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$$

Then noting that $F' = \frac{1}{T'}$ and $F_0 = \frac{1}{T_0}$, we conclude that:

$$F' = f_0\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}}$$

Remember: This equation is for a receding star. If the star is approaching, we simply have $F' = f_0\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$.

3.6 Conservation Laws

In this section, we consider linear momentum and mass (see section 3.8 for more on mass-energy). Can we find an expression for relativistic mass and momentum? Let us make two sensible assumptions: a) Relativistic mass is conserved in all inertial reference frames; and b) Linear momentum is conserved in all reference frames.
Definition: The rest mass of a body, m_0, is simply that - its mass when it is not in motion. The closer to the speed of light a body is travelling, the heavier it appears to be. For clarity, in this section we shall say that a body has mass $m(u)$ if it is travelling at a speed u.
Now, suppose we have two identical bodies, each of rest mass m_0, each travelling towards the other at a speed u. They collide inelastically and stick together. We need to consider this collision in two frames - frame S will observe the situation described, and frame S' will be the rest frame of body 1 (see Figure 12 for full description). First thing's first: what is v? (NB We need to be

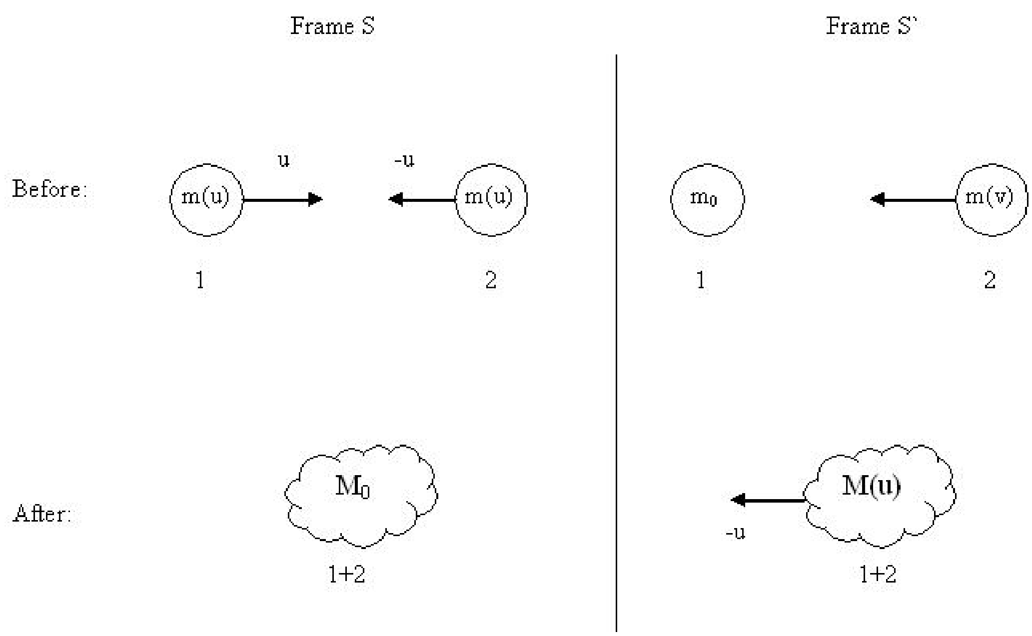

Figure 12

very careful with signs!) We know that in frame S, the speed of body 2 is $-u$. So let's apply the Lorentz Transformation for velocity to find it in frame S:

$$v'_{x'} = \frac{x_v - u}{(1 - \frac{uv_x}{c^2})}$$

But here, $v'_{x'} = -v$ and $v_x = -u$ (warned you about the signs!). So:

$$-v = \frac{-u - u}{1 - \frac{u(-u)}{c^2}}$$

$$=> v = \frac{2u}{1 + \frac{u^2}{c^2}}$$

Now we may continue. Applying conservation of relativistic mass, we see that:

$$\text{Frame } S: \ M_0 = 2m(u)$$

$$\text{Frame } S': \ M(u) = m_0 + m(v)$$

Then applying conservation of linear momentum to frame S', we see that:

$$-M(u)u = -m(v)v$$

This is enough information. Try the algebra yourself (remember that $\frac{M(u)}{M_0} = \frac{m(u)}{m_0}$ must hold if we are to find a universal relationship) - you should find that:

$$m(u) = \gamma(u)m_0$$

$$=> m = \gamma m_0$$

where m is the relativistic mass of the body. Remembering the definition of momentum, it should be easy to see that relativistic momentum, $\underline{p}_{rel}$, is defined as:

$$\underline{p}_{rel} = \gamma m_0 \underline{v}$$

3.7 Force

We have two major problems when considering relativistic forces:

- Acceleration is not invariant (it is in Newtonian mechanics).
- Force is not usually in the same direction as the resultant acceleration (it is parallel in Newtonian mechanics).

Thus there is not much we can do at this stage! We still have the relationship: $\underline{F} = \frac{dp}{dt}$ (Newton's 2nd Law) but what is this now?

$$\underline{F}_{rel} = \frac{d}{dt}(\underline{p}_{rel})$$

$$\underline{F}_{rel} = \frac{d}{dt}(\gamma m_0 \underline{v})$$

$$=> \underline{F}_{rel} = \gamma m_0 \frac{d\underline{v}}{dt} + m_0 \underline{v} \frac{d\gamma}{dt}$$

... as both γ and $\underline{v}$ depend on time.

3.8 Energy And Mass

Now here's an idea to make Newton turn in his grave!
The Principle of Mass-Energy: Mass may be created or destroyed, but at the cost of an equivalent amount of energy either vanishing or appearing respectively.
This must mean that we can find a relationship between mass and energy. Let's consider a body (initially at rest) which is being acted on by a force in the positive x-direction. We know from Newtonian mechanics that *work done = change in kinetic energy* (E_{KE}), ie:

$$E_{KE} = \int_0^v \frac{d\underline{p}}{dt} dx = \int_0^v \frac{dx}{dt} d\underline{p} = \int_0^v v d\underline{p}$$

We need to integrate this by parts, as follows:

$$E_{KE} = pv - \int_0^v p\frac{dv}{dp} dp = pv - \int_0^v pdv$$

But $\underline{p}_{rel} = \gamma m_0 \underline{v}$. So:

$$E_{KE} = \gamma m_0 v^2 - \int_0^v \gamma m_0 v dv = \gamma m_0 v^2 - [-\frac{m_0 c^2}{\gamma}]_0^v$$

$$E_{KE} = \gamma m_0 v^2 + \frac{m_0 c^2}{\gamma} - m_0$$

$$E_{KE} = \gamma m_0 [v^2 + c^2(1 - \frac{v^2}{c^2})] - m_0 c^2$$

$$E_{KE} = \gamma m_0 (v^2 + c^2 - v^2) - m_0 c^2$$

$$E_{KE} = \gamma m_0 c^2 - m_0 c^2$$

$$=> E_{KE} = (\gamma - 1) m_0 c^2$$

In fact, $m_0 c^2$ is the rest mass energy, E_0. So we have that the total energy, E_{tot}, is:

$$E_{tot} = E_{KE} + m_0 c^2$$

$$E_{tot} = \gamma m_0 c^2$$

Note: You may be wondering about potential energy - if you want this answering, take the 4th Year course in General Relativity! You do not need to worry about it for this course.
Let us have another look at relativistic mass. We have that:

$$m = \gamma m_0$$

Note: E_0^2 is invariant. So:

$$m_0^2 = m^2(1 - \frac{v^2}{c^2})$$

$$=> m_0^2 c^2 = m^2 c^2 - m^2 v^2$$

$$=> \frac{E_0^2}{c^2} = \frac{E^2}{c^2} - p^2$$

$$\therefore E^2 = E_0^2 + p^2 c^2 = m_0^2 c^4 + p^2 c^2$$

4 Cheat Sheet

This is by no means everything you need to know, but hopefully a helpful summary of equations you'll need. Don't panic if you're asked about a change in, say, the y direction - just swap x's and y's. Good luck with the exam! (OK, I mentioned it once more)

Newtonian	**Relativistic**
Galilean Transformations:	Lorentz Transformations:
$x' = x - ut$	$x' = \gamma(x - ut)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$t' = t$	$t' = \gamma(t - \frac{ux}{c^2})$
	$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
$v' = v - u$	$v'_{x'} = \frac{x_v - u}{1-\frac{uv_x}{c^2}}$
Mass invariant	$m(u) = \gamma(u)m_0$
Length $= x_2 - x_1$	$L_0 = x_2 - x_1$ but
	$L' = \frac{L_0}{\gamma}$
Time invariant, $\Delta t = t_2 - t_1$	$\Delta t' = \gamma \Delta t$
Momentum: $\underline{p} = m\underline{v}$	$\underline{p}_{rel} = \gamma m_0 \underline{v}$
Force $\underline{F} = \frac{d}{dt}(m\underline{v})$	$\underline{F}_{rel} = \frac{d}{dt}(\underline{p}_{rel})$
$= m\underline{a}$	$= \gamma m_0 \frac{d\underline{v}}{dt} + m_0 \underline{v} \frac{d\gamma}{dt}$
$E_{KE} = \frac{1}{2}mv^2$	$E_{KE} = (\gamma - 1)m_0c^2$
	Rest Mass Energy $= m_0c^2$
	$E_{tot} = \gamma m_0 c^2$
Doppler effect (receeding source):	
$F' = F_0(1 - \frac{u}{c}$	$F' = F_0\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}}$
	$F' = \frac{1}{T'}$; $F_0 = \frac{1}{T_0}$
	$E^2 = E_0^2 + p^2c^2$
	$= m_0^2c^4 + p^2c^2$
	E_0^2 is invariant
	$c^2t^2 - x^2 = l^2$
	l^2 is invariant