PX118: Waves

Written by Emma Towlson
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1 Important Ideas and Properties

We may describe a wave mathematically as a function of position and time, say $u(x,t)$. In general, the shape of the waveform does not vary with time. Consider a one-dimensional sinusoidal wave, with the hopefully familiar forms

$$\begin{aligned} u(x,t) &= A\cos(kx-\omega t) \\ &= A\cos\left(\frac{2\pi}{\lambda}(x-vt)\right) \\ &= A\cos\left(2\pi\left(\frac{x}{\lambda}-\nu t\right)\right) \\ &= A\cos\left(2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right). \end{aligned}$$

What do we know about this wave? It is right-travelling ($+\omega t$ for left-travelling waves) and:

- A is its *amplitude*;
- k is the *wavenumber*;
- ω is the *angular frequency*;
- $\lambda = 2\pi/k$ is the *wavelength*;
- $\nu = \omega/2\pi = 1/T$, where ν is the *frequency* and T the *period* of the wave;
- $v = \omega/k$ is the 'wave speed' or *phase velocity* (the speed of the 'carrier wave');
- $v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$ is the *group velocity*;
- the speed of vertical displacement of the point on the waveform with coordinate x (the *transverse velocity* at x) is given, as a function of time, by $\left(\dfrac{\partial u}{\partial t}\right)_x$.

Now consider the waveform $u(x,t) = A\cos(kx-\omega t+\varphi)$. This wave has the same shape as our first wave, only with a *phase difference* of φ (imagine it is spatially 'shifted' by a constant amount φ with respect to the first wave); φ is variously called the *initial phase*, or the phase *at* $x=0, t=0$.

What if these two waves 'meet' in space?

The Principle of Superposition. The total displacement when two waves superpose is the sum of their individual displacements.

This means that we may 'add waves'. Consider again the sinusoidal waves

$$u_1(x,t) = A\cos(kx-\omega t) \qquad \text{and} \qquad u_2(x,t) = A\cos(kx-\omega t+\varphi).$$

Using the trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

we find that

$$u_1(x,t) + u_2(x,t) = 2A\cos(kx-\omega t+\varphi/2)\cos(\varphi/2).$$

By the Principle of Superposition, this is the total displacement at (x,t) due to the contribution of the two waves u_1 and u_2.

Notice that if $\varphi = 0$, then $u_1 = u_2$ and

$$u_1 + u_2 = 2u_1 = 2u_2;$$

this is *constructive interference*, producing a waveform with double the amplitude of the original waves. On the other hand, if $\varphi = \pi$, we have that

$$u_1 + u_2 = 0;$$

the waves have completely cancelled each other out (this is *destructive interference*).

What if our original waves had slightly different frequencies? Say

$$\begin{aligned} u_1(x,t) &= A\cos(k_1 x - \omega_1 t) \qquad k_1 = k_0 + \Delta k;\ \omega_1 = \omega_0 + \Delta\omega \\ u_2(x,t) &= A\cos(k_2 x - \omega_2 t) \qquad k_2 = k_0 - \Delta k;\ \omega_2 = \omega_0 - \Delta\omega. \end{aligned}$$

Adding as above, we arrive at:

$$u(x,t) := u_1(x,t) + u_2(x,t) = 2\underbrace{A\cos(\Delta k x - \Delta\omega t)}_{\text{beat wave}}\underbrace{\cos(k_0 x - \omega_0 t)}_{\text{carrier wave}}.$$

But what do we 'hear'? For the above wave, the average frequency is:

$$\frac{\omega_1 + \omega_2}{2} = \frac{(\omega_0 + \Delta\omega) + (\omega_0 - \Delta\omega)}{2} = \omega_0,$$

which is the note we hear! The *beat wave* has a speed $\Delta\omega/\Delta k$, and its amplitude is *modulated* by $2\Delta\omega = \omega_1 - \omega_2$.
Now, suppose we have a group of superimposed waves with very similar wavenumbers and angular frequencies; this is equivalent to letting Δk be small and assuming a 'nice' dependence of ω on k. The speed of the resulting beat pattern (the *group velocity*) thus tends to

$$v_g = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}.$$

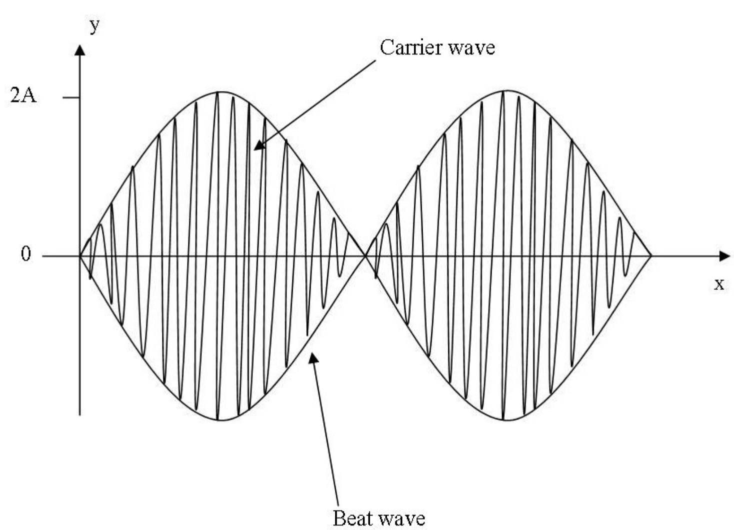

Figure 1: An example of beats

The group velocity is usually thought of as the speed at which energy or information is passed along the wave. It is an important concept for example, electromagnetic waves in the ionosphere appear to have a wave speed greater than the speed of light! However, their group velocity is always less than c.

Definition. Waves are said to *disperse* if the wave speed v, depends on the wavenumber k. Or, equivalently, if ω is NOT proportional to k.

Examples. (a) For sound, we have that

$$\omega = \sqrt{\frac{\beta}{\rho}} \implies v = \sqrt{\frac{\beta}{\rho}}$$

(so NOT dispersive). Notice that in this case

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{\frac{\beta}{\rho}} = v \implies v = v_g.$$

(b) For waves in deep water,

$$\omega = \alpha\sqrt{gk} \implies v = \alpha\sqrt{\frac{g}{k}}$$

(the waves disperse). Here

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\alpha}{2}\sqrt{\frac{g}{k}} = \frac{v}{2} \implies v_g \neq v.$$

2 The Wave Equation

It can be shown that any one-dimensional wave $y(x,t) = f(x - vt)$ will satisfy the following equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}.$$

(Note: this may be extended to higher dimensions; for 3-dimensional waves by replacing $\frac{\partial^2 y}{\partial x^2}$ with the *Laplacian* $\Delta\, y$.)

For electromagnetic waves, we find that

$$\frac{\partial^2 E}{\partial x^2} = \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2},$$

giving us an expression for the speed of light

$$c = \sqrt{\frac{1}{\mu_0\varepsilon_0}}.$$

3 Mechanical Waves

3.1 Waves on a String

By resolving forces, it can be shown that for waves on a taut string, the wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2},$$

where T is the tension in the string and μ the mass per unit length. Thus, the speed of the wave is

$$v = \sqrt{\frac{T}{\mu}},$$

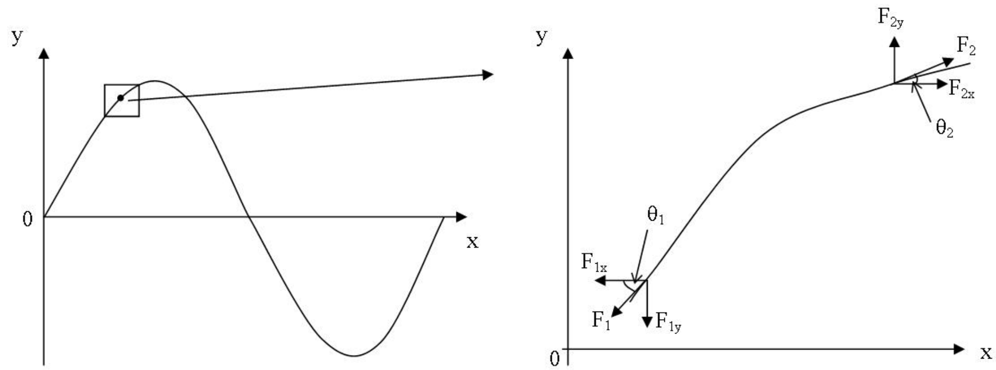

Figure 2: A section of a wave on a string

and the wave is *not* dispersive. How does this change for a 'real' string? The taut string will stretch according to Hooke's Law:

$$F = -Kx$$

(Note: do not confuse K, the spring constant, with k the wavenumber.)
Now we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - \frac{\beta}{\mu}y,$$

yielding the dispersion relation

$$\omega(k) = \sqrt{\frac{T}{\mu}k^2 + \frac{K}{\mu}} \implies v(k) = \sqrt{\frac{T}{\mu} + \frac{K}{\mu k^2}}$$

3.2 Waves on a Rod

Suppose we apply a force F at one end of a rod with cross-sectional area A acting along the length of the rod (see Figure 3), creating an elastic longitudinal wave.

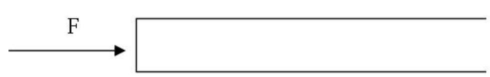

Figure 3: An elastic longditudinal wave

Hooke's Law tells us that, at all times, the stress is proportional to the strain, so if $\sigma(x,t)$ is the coordinate, at time t, of the cross-section of the rod which occupies position x in the rest configuration, then

$$\text{Stress} = \frac{F}{A}, \qquad \text{Strain} = \frac{\partial \sigma}{\partial x}$$

and we find that

$$F = YA\frac{\partial \sigma}{\partial x} \implies \frac{\partial F}{\partial x} = YA\frac{\partial^2 \sigma}{\partial x^2},$$

where Y is *Young's modulus*, the constant ratio of stress to strain.
But we know that

$$F = \text{mass} \times \text{acceleration} = m\frac{\partial^2 \sigma}{\partial t^2},$$

and since the element of mass at x is $\mathrm{d}m = \rho \mathrm{d}V = \rho A \mathrm{d}x$,

$$\mathrm{d}F = \rho A \mathrm{d}x \frac{\partial^2 \sigma}{\partial t^2} \implies \frac{\partial F}{\partial x} = \rho A \frac{\partial^2 \sigma}{\partial t^2}.$$

Equating the two expressions for $\frac{\partial F}{\partial x}$, we are left with

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \sigma}{\partial t^2}$$

. . . the wave equation! So, here

$$v = \sqrt{\frac{Y}{\rho}}.$$

If were to have applied a force normal to the rod (see Figure 4), we would have created a transverse shear wave and with speed

$$v = \sqrt{\frac{G}{\rho}},$$

where G is called the *shear modulus*.

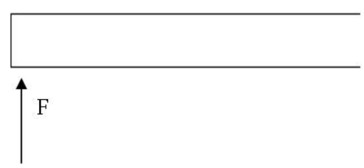

Figure 4: A sheer wave

3.3 Seismic Waves

Earthquakes usually create two types of 3-dimensional wave:

- The *Shear S-waves* are transverse, with speed

$$v_S = \sqrt{\frac{G}{\rho}}$$

- The *Compressional P-waves* are longitudinal and always travel faster than S-waves, with speed

$$v_P = \sqrt{\frac{B + 4G/3}{\rho}},$$

where B is the *bulk modulus*.

3.4 Transfer of Power

Suppose we apply a force to one end of a string, as in section 3.1. Then the rate of energy transfer (the *power*) is

$$\begin{aligned} P = F \cdot v &= -F_{1y} \frac{\partial y}{\partial t} \\ &= -F_{1x} \left(\frac{\partial y}{\partial x} \right) \left(\frac{\partial y}{\partial t} \right). \end{aligned}$$

Remember that this is the instantaneous power at position x and time t, and is valid for any waveform. Let us consider an arbitrary sinusoidal wave, $y(x,t) = A\cos(kx - \omega t)$. We see that

$$\frac{\partial y}{\partial x} = -Ak\sin(kx - \omega t) \qquad \text{and} \qquad \frac{\partial y}{\partial t} = A\omega \sin(kx - \omega t).$$

So using the above equation for power, we find:

$$P(x,t) = FA^2 k\omega \sin^2(kx - \omega t)$$

or, since $v = \omega/k$ and $v^2 = F/\mu$:

$$P(x,t) = \omega^2 A^2 \sqrt{\mu F} \sin^2(kx - \omega t).$$

Thus, averaging over the function, we find P_{av} ,the average power, to be

$$P_{av} = \frac{1}{2}\omega^2 A^2 \sqrt{\mu F}.$$

Note: *the power is proportional to the square of the amplitude and to the square of the frequency.* This is in fact true for all *mechanical* waves. For *electromagnetic* waves, however, the power is still proportional to the square of the amplitude but NOT to the square of the frequency.

4 Reaching a Boundary

4.1 On a String

So what happens when a wave hits a boundary? Some energy is reflected and some is transmitted, and certain conditions will be imposed on the wave at a boundary, as we shall now explore. Let us consider waves on a string again.

- **Fixed end.** Suppose one end of the string is attached to a wall at position x_0, as in Figure 5.

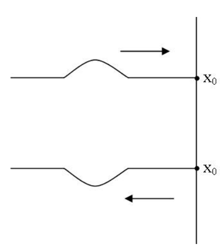

Figure 5: A fixed end boundary

Our boundary condition is then that

$$y(x_0, t) = 0 \quad \text{for all } t$$

(at x_0, the string cannot move). The pulse is reflected and inverted as a consequence.

- **Free end.** Suppose that one end of the string is attached to a frictionless ring of negligible mass which is free to slide along a frictionless rod, as in Figure 6.

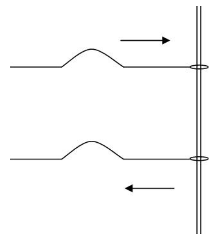

Figure 6: A free end boundary condition

Now the boundary condition is that

$$\left(\frac{\partial y}{\partial x}\right)_{x_0} = 0$$

(at x_0, the rate of change of the vertical displacement must be zero, as the only force is from the rod and acts normal to the rod). The pulse will be reflected but NOT inverted.

- **Two strings, mass per unit length μ_1 and μ_2, joined together.**
  - ⋆ $\mu_1 < \mu_2$: after the wave hits the boundary of the strings, we have the situation in Figure 7.

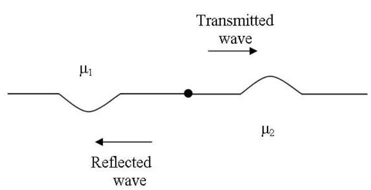

Figure 7: $\mu_1 < \mu_2$

The reflected wave is inverted (c.f. fixed end).

  - ⋆ $\mu_1 > \mu_2$: now, as shown in Figure 8, the reflected wave is NOT inverted (c.f. free end).

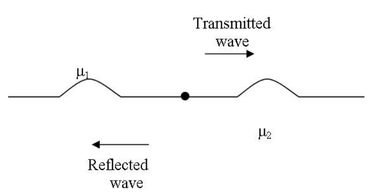

Figure 8: $\mu_1 > \mu_2$

  - ⋆ $\mu_1 = \mu_2$: shown in Figure 9, there is complete transmission of the wave, i.e. all of the energy is transmitted and there is no reflected pulse (the strings are *impedance matching*).

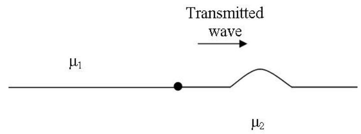

Figure 9: $\mu_1 = \mu_2$

We can also impose two restrictions on the point where the two strings join.
Firstly, if we define this point to be on the line $x = 0$, the continuity of the string demands that

$$y(x \to 0, t)_{\text{string 1}} = y(x \to 0, t)_{\text{string 2}}.$$

If we call the incident wave y_i, the reflected wave y_r and the transmitted wave y_t, this condition tells us that

$$y_i + y_r = y_t.$$

For a typical wave, this also means that:

$$A_i + A_r = A_t \qquad \text{and} \qquad P_i = P_r + P_t,$$

where A is the amplitude of each wave and P the power.
In addition, the tension in each string must be the same at the point where they meet, i.e. the transverse force in each string must be equal at $x = 0$. So:

$$\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x} = \frac{\partial y_t}{\partial x}.$$

In a sinusoidal wave, this means that

$$k_i(A_i - A_r) = k_t A_t \qquad \text{or} \qquad \sqrt{\mu_i}(A_i - A_r) = \sqrt{\mu_t} A_t,$$

where k is the wavenumber.

4.2 Reflection and Transmission Coefficients

We can use these relations to define two ratios, the reflection (R) and transmission (T) coefficients, as follows:

$$T = \frac{A_t}{A_i} = \frac{2\sqrt{\mu_i}}{\sqrt{\mu_i} + \sqrt{\mu_t}} = \frac{2Z_i}{Z_i + Z_t},$$
$$R = \frac{A_r}{A_i} = \frac{\sqrt{\mu_i} - \sqrt{\mu_t}}{\sqrt{\mu_i} + \sqrt{\mu_t}} = \frac{Z_i - Z_t}{Z_i + Z_t},$$

where Z is the *impedance*, $Z = \sqrt{\mu F}$.

Notice that if $\mu_i < \mu_t$, then $Z_i < Z_t$ and R is negative — this indicates that the reflected wave is inverted. The maximum power transfer occurs when $Z_i = Z_t$ (no reflected wave).

5 The Doppler Effect

This relates to an observed change in frequency of emitted waves if the source is moving relative to the observer, for example the well-known phenomenon of red-shift of radiation from galaxies.

5.1 Sound Waves

Remark. Unlike with electromagnetic waves (c.f. PX109 – Relativity), it makes sense for mechanical waves to talk about 'moving' sources (resp. observers) with 'fixed' observers (resp. sources); the frame of reference of the medium is the obvious choice.

5.1.1 Moving Source, Fixed Observer

Suppose a source moving at speed u emits a sound wave of frequency ν and wavelength λ. Note that the period of the wave is $T = \nu^{-1}$. Figure 10 illustrates the difference in what we would see from a stationary and moving source.

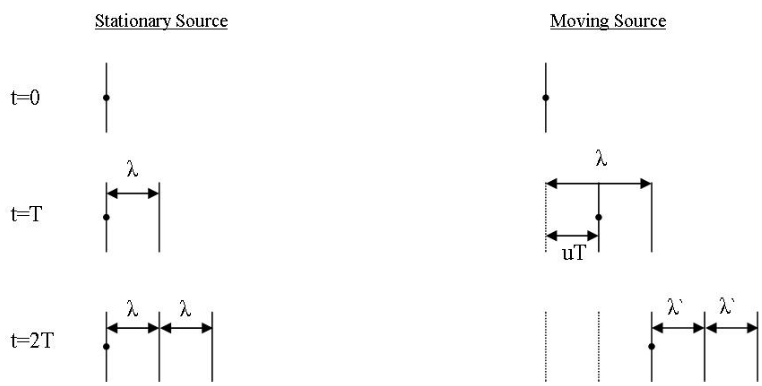

Figure 10: Stationary and Moving Source

Note that

$$\lambda' = \lambda - uT = \lambda - \frac{u}{\nu};$$

recalling that $\lambda = c/\nu$, we see that then

$$\frac{c}{\nu'} = \frac{c}{\nu} - \frac{u}{\nu} \implies \nu' = \frac{c\nu}{c-u} = \frac{\nu}{1-\frac{u}{c}}.$$

Remember that ν' is the observed frequency, ν the emitted frequency and c here is the speed of sound, not light! (Simply change the '-' to a '+' if you are considering a receding source.) The expression is often approximated to

$$\nu' \approx \nu\left(1 + \frac{u}{c}\right).$$

5.1.2 Moving Observer, Fixed Source

Suppose now that the source is stationary, but the observer is moving towards it at a speed u. The apparent speed of sound is now $u + c$, so

$$\nu' = \frac{c+u}{\lambda} = \nu\left(1 + \frac{1}{c}\right).$$

5.1.3 Moving Source and Observer

Now we have that

$$\nu' = \nu\frac{c+u_o}{c+u_s},$$

where u_0 is the speed of the observer, and u_s the speed of the source (watch out for negative signs!). If the source and observer are moving towards each other, you will find that $\nu' > \nu$ (and that $\nu' < \nu$ when moving apart).

5.2 Electromagnetic Waves

Electromagnetic waves travel at the speed of light, which is a universal constant (same in *any* reference frame!); we thus expect a slightly different result. If v is the velocity of the observer relative to the source and c now the speed of light, we have that:

$$\nu' = \nu\sqrt{\frac{c-v}{c+v}}.$$

When $v \ll c$, we may approximate this to

$$\nu' = \nu\left(\frac{c-v}{c+v}\right).$$

(NB: v is *positive* if the observer is moving *away* from the source!)

6 Light

6.1 Overview

Maxwell's Equations, which describe electric and magnetic fields, solve in free space (and in 1-D) to the wave equations

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0\mu_0\frac{\partial^2 E}{\partial t^2},$$
$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_0\mu_0\frac{\partial^2 B}{\partial t^2}.$$

Light waves are oscillating electric (E) and magnetic (B) fields. These fields are always both perpendicular to each other and to the direction of motion of the wave (light is a transverse wave, see Figure 11).

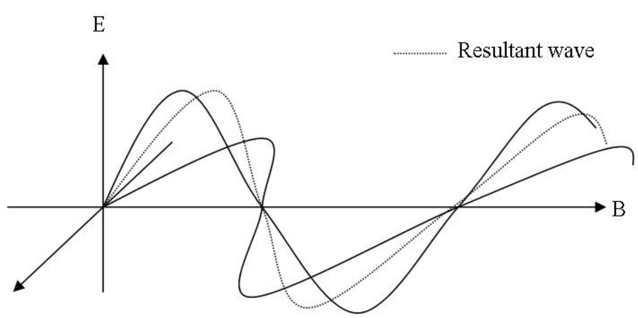

Figure 11: A wave diagram of light

Usually, the magnetic field is very small in magnitude, so we will only consider the electric field.

6.2 Polarisation

Due to the orthogonality restraint, the component of the electric field $\mathbf{E}$ along the direction of motion of an electromagnetic wave (the x-direction, say) is always zero. Normally, the $\mathbf{E}$-field a fairly random mix of y- and z-components — unpolarised. For instance,

$$\mathbf{E}_y(x,t) = E_y(x,t)\hat{\mathbf{j}} = E_{0y}\cos(kx-\omega t)\hat{\mathbf{j}},$$
$$\mathbf{E}_z(x,t) = E_z(x,t)\hat{\mathbf{k}} = E_{0z}\cos(kx-\omega t)\hat{\mathbf{k}}.$$

We can polarise light using a filter (though this can also be done through reflection, scattering, ...), i.e. we can 'filter out' some of the components of the electric field, leaving only those along a chosen axis. If we were to put an identical second filter next to the first, we would find that no light is transmitted when they are at 90° (all of the components are 'filtered out') and that the transmitted light is most intense when they align (so the second filter does not remove any further components).

If we let I_0 be the intensity of the original beam of light, I the intensity of the polarised light and θ the angle between the light's original polarisation direction and the axis of the filter, *Malus' Law* tells us that:

$$I = \frac{I_0}{2}\cos^2\theta.$$

We may have that E_y and E_z are out of phase with each other, say by φ.

If $\varphi = 0, \pm n\pi$, then we have *linear polarisation*.

If $\varphi = \pm\pi/2$, then we have *circular polarisation*. The electric field has a constant magnitude, but now varies in direction — it 'rotates' about the direction of travel. We say we have left-hand polarisation if $\varphi = +\pi/2$ and right-hand polarisation if $\varphi = -\pi/2$. Furthermore, if $E_y \neq E_z$ we will have *elliptical polarisation*.

6.3 Energy and Pressure

Defining the intensity, I, to be the power per unit area, we have that:

$$I = \frac{1}{2}\varepsilon_0 c E_0^2,$$

where E_0 is the amplitude of the electric field, c the speed of light and ε_0 the *permittivity of free space* (note that this is independent of the frequency, unlike in the particle view of light, c.f. section 3.4).

There is a rate of flow of momentum, and consequently radiation exerts a pressure. So, if P_{rad} is the pressure and p the momentum, and using the relativistic expression $E = pc$ for the photon energy, a totally absorbing surface will experience a pressure

$$P_{rad} = \frac{\text{Force}}{\text{Area}} = \frac{1}{\text{Area}}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{\text{Area}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{c}\right) = \frac{1}{c}\frac{\text{Power}}{\text{Area}} = \frac{I}{c}.$$

If the surface is totally reflecting then $P_{rad} = \dfrac{2I}{c}$.

7 Reflection and Refraction

7.1 Snell's Law, Fermat's Principle and Total Internal Reflection

Suppose we have a beam of light travelling in air which passes into a pane of glass, as in Figure 12.

We know that $\theta_i = \theta_r$, but what about the angle of the transmitted ray?

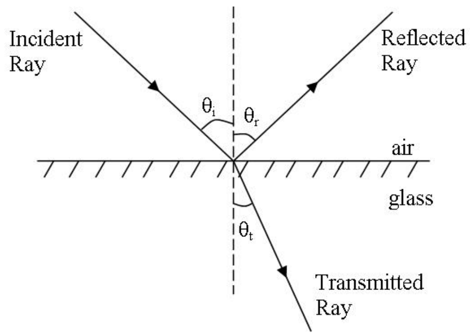

Figure 12: Snell's Law

Snell's Law. $n_i \sin\theta_i = n_t \sin\theta_t$.
Here, n is the *refractive index* of the medium through which the wave is travelling, and defined as $n = c/v$ (where c is the speed of light in vacuum and v the speed of light in the medium). Note that n can be wavelength dependent!
Snell's Law can be shown to be a result of *Fermat's Principle* or the *principle of least time*, which tells us that out of all the hypothetical routes a beam of light could take from a point A to a point B, it will always choose the quickest.

Notice that if the incident ray approaches at an angle greater than a critical angle, θ_c, there will be no transmitted ray (see Figure 13) — we call this phenomenon *total internal reflection.*

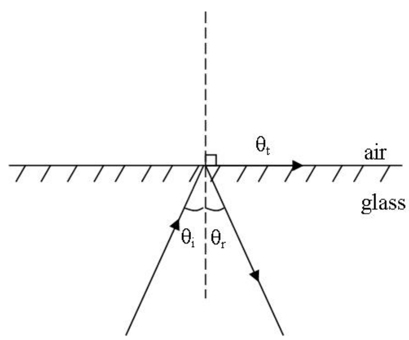

Figure 13: Total internal reflection

It should be clear that θ_t must be at least 90° for total internal reflection to occur, and from this we can find the critical angle:

$$n_i \sin\theta_c = n_t \sin(90°) \implies \sin\theta_c = \frac{n_t}{n_i}.$$

If $\theta_i > \theta_c$, there will be no transmitted wave.

7.2 Brewster's Law

Reflection can polarise light. Let us consider the situation in Figure 14.

If the ray of light is incident at an angle θ_p, the *polarisation angle*, such that the resulting transmitted and reflected waves are perpendicular to one another, we find that the reflected way is linearly polarised and the transmitted ray *partially* polarised. Using this orthogonality, we find:
Brewster's Law. $\tan\theta_p = \dfrac{n_t}{n_i}$.

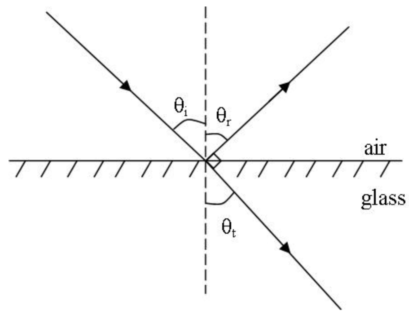

Figure 14: Brewster's Law

7.3 Wavefronts and Huygen's Principle

Huygen's Principle. *Every point on a wave acts as a source of new spherical secondary waves, such that the wavefront at some later time is the envelope of these secondary waves.*

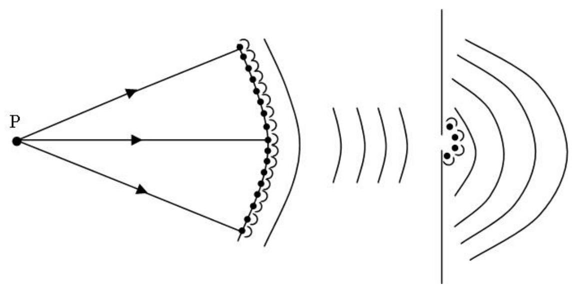

Figure 15: Huygen's Principle

Figure 15 illustrates this idea (note that a solid line is a 'wavefront' — a locus of points at which the wave has a given phase).
Remark. This is only a simplistic model. It is helpful in describing scattering when the wavelength is much smaller than the object's dimensions. The principle's biggest drawbacks are that it does not account for interference (maybe leading to diffraction) and that we would expect a backwards travelling wave to form as a result, which is not observed!

8 Interference

8.1 Overview

Suppose we have two point sources, P_1 and P_2, as shown in Figure 16. What do we observe when they superpose? Note that if we are to see interference effects, we must have *coherent* sources: they must be emitting waves of the same frequency and have a constant phase relationship over time (we shall assume this throughout).

So, at point P, two waves approach from the sources, which we shall describe by:

$$E_1(t) = E_0 \cos(kx_1 - \omega t)$$
$$E_2(t) = E_0 \cos(kx_2 - \omega t)$$

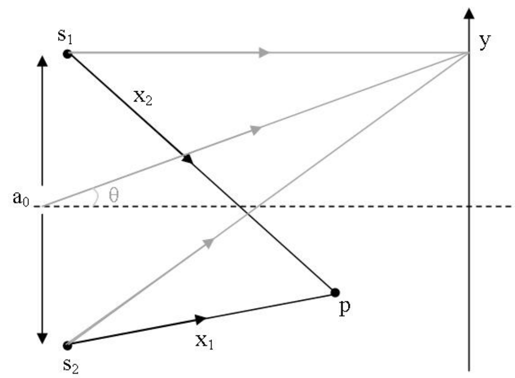

Figure 16: Interference effects

Then the Principle of Superposition (see section 1) tells us that the resulting wave is described by

$$E(t) = 2E_0 \cos\left(\frac{(x_1 - x_2)k}{2}\right) \cos\left(\frac{(x_1 + x_2)k}{2} - \omega t\right).$$

Notice that the amplitude is modulated by the time-independent term, and the time-dependent term describes the envelope wave.
If $x_1 = x_2$ we have constructive interference and $E = 2E_1 = 2E_2$.
If $x_1 - x_2 = \lambda/2$, we have destructive interference and $E = 0$ (λ is the wavelength).

At any position P, the intensity I is given by

$$I = I_0 \cos^2\left(\frac{k\Delta x}{2}\right),$$

where $\Delta x = |x_2 - x_1|$ and $I_0 = 2\varepsilon_0 c^2 E_0^2$. The intensity at the centre of the interference pattern is 4 times that from a single source!

Referring back to Figure 16, if $D \gg a$, we may approximate

$$\Delta x \approx a \sin\theta \implies I \approx I_0 \cos^2\left(\frac{ka}{2}\sin\theta\right).$$

Now, this has maxima when

$$\frac{ka}{2}\sin\theta = n\pi \iff \frac{a}{\lambda}\sin\theta = n$$

and minima when the waves are π out of phase, that is if

$$a\sin\theta = \left(n + \frac{1}{2}\right)\lambda.$$

8.2 Interference in Films; Transmittance and Reflectance

At an interface between two different media, the *transmittance*, T, is given by

$$T = \frac{4n_i n_t}{(n_i + n_t)^2}.$$

The *reflectance*, R, is given by

$$R = \frac{(n_i - n_t)^2}{(n_i + n_t)^2}.$$

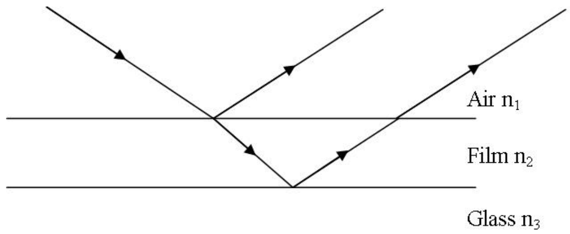

Figure 17: Thin film interference

Let us consider light waves approaching a *thin* film, as in Figure 17. We now have two reflected waves, which will interfere.
The path difference between the two reflected waves is approximately $2t$, and we find (a) constructive interference when $2t = m\lambda_{\text{film}},\ m \in \mathbb{Z}$ and (b) destructive interference when $2t = \left(m + \frac{1}{2}\right)\lambda_{\text{film}},\ m \in \mathbb{Z}$. (NB: If $n_1 < n_2$, the wave is inverted!)

What if we had taken a *thick* film? Suppose our film was a 1 cm thick sheet of glass. As above, we would see, at visible wavelengths, constructive interference at about 10000λ and destructive at 10000.5λ — the effects are 'smeared out'. To exacerbate this, if the beam of light does not hit at precisely $90°$, the difference in path length is to add to the 'smearing'.
In other words, we need to use thin films to see interference effects!

...time to wave goodbye!

That's everything you need to know about waves in a nutshell. Remember to ensure you have learnt all the formulae and definitions before the exam, as there are often some nice simple marks for stating these. Past papers are very useful, as the questions tend to follow a similar format. Also, ensure you are confident working with partial derivatives, as these crop up a lot throughout this module. Finally, good luck for the exam!