# PX3A3 Electrodynamics 

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## Acknowledgements

These lecture notes follow Prof David Leadley's 2022 lectures for PX3A3 Electrodynamics closely. However, they are altered significantly after each lecture ${ }^{1}$, and are by no means an accurate reflection of what is covered in the module (but should hopefully still be of good reference for students taking PX3A3). In particular, I am most certainly responsible for all errors; if you see something weird, I'd say consulting the course Discord/WhatsApp group chats would be a good idea, people there usually have 200 IQ minimum.

Throughout this document, I also reference Prof Sandra Chapman's book "Core Electrodynamics", Prof David Tong's lecture notes $\$^{2}$ on Vector Calculus and Electromagnetism, and good ol' Griffiths "Introduction to Electrodynamics" extensively, though apologies in advance for the lack of proper citations as these notes were originally compiled for personal use.

I shall also thank the Wikipedia authors and all the geniuses on StackExchange for both making this document possible and carrying my whole MathPhys degree, you guys are absolute saints.

## Typos

As with any lecture notes, there certainly are typos hidden somwhere (or everywhere, maybe). If you spot any, please let me know via email at enoch.ko@warwick.ac.uk. Any other suggestions are also very much appreciated!

## Scope and Remark

Note that these notes do not cover all of PX3A3, but rather only up to week 6 of the 2022 schedule in terms of content. This is partly (maybe $80 \%$ ) because I am lazy, but also because I found the remaining content less interesting, and frankly, I did not feel like typing out Prof Leadley's lectures word by word (don't get me wrong, the lectures were not bad by any stretch, I just feel like I will not contribute anything extra or provide any interesting perspective to those topics as compared to what you will get by simply attending the remaining lectures).

Regardless, I hope these notes are more or less helpful in providing a slightly tidier, more comprehensive, and hopefully mathematically coherent introduction to electrodynamics :)

[^0]
## 1 Electromagnetism and Special Relativity

God said, "Let there be light!"
And so, there was Maxwell and his equations.
Also, Tim Gershon was there. Somehow.

### 1.1 Revision: The Maxwell Equations \& EM Waves

Recall from PX263:
Maxwell's Equations (in free space)

$$
\begin{gather*}
\nabla \cdot \mathbf{E}=\frac{\rho}{\epsilon_{0}}  \tag{M1}\\
\nabla \cdot \mathbf{B}=0  \tag{M2}\\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}  \tag{M3}\\
\nabla \times \mathbf{B}=\mu_{0}\left(\mathbf{J}+\epsilon_{0} \frac{\partial \mathbf{E}}{\partial t}\right) \tag{M4}
\end{gather*}
$$

Equivalently, we can rewrite the first and fourth equation as $\nabla \cdot \mathbf{D}=\rho$ and $\nabla \times \mathbf{H}=\mathbf{J}+\partial_{t} \mathbf{D}$ respectively, where $\mathbf{D}=\epsilon_{0} \mathbf{E}$ and $\mathbf{H}=\mathbf{B} / \mu_{0}$ (in free space).

Note: We will not consider dielectrics in PX3A3, thus one can always use the above relations.
Using the vector identity $\nabla \times(\nabla \times \mathbf{A})=\nabla(\nabla \cdot \mathbf{A})-\nabla^{2} \mathbf{A}$, we also obtained that in free space,

$$
\nabla^{2} \mathbf{E}=\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}} \quad \text { and } \quad \nabla^{2} \mathbf{B}=\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{B}}{\partial t^{2}}
$$

where $c=1 / \sqrt{\mu_{0} \epsilon_{0}} \approx 3 \times 10^{8} \mathrm{~ms}^{-1}$ is the speed of light in vacuum, showing that light is in fact electromagnetic waves(!).

### 1.2 Revision: Energy in EM Fields and Waves

Again recalling from PX263, conservation of energy gives

$$
\frac{\partial u}{\partial t}+\nabla \cdot \mathbf{S}=-\mathbf{E} \cdot \mathbf{J}_{f}=\mathbf{E} \cdot\left(\nabla \times \mathbf{H}-\frac{\partial \mathbf{D}}{\partial t}\right)=\nabla \cdot(\mathbf{E} \times \mathbf{H})+\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}+\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}
$$

where we can identify $\mathbf{S}=\mathbf{E} \times \mathbf{H}$ as the Poynting vector, and in linear, isotropic materials $\left(\mathbf{H}=\mathbf{B} / \mu_{0}\right.$ and $\left.\mathbf{D}=\epsilon_{0} \mathbf{E}\right)$, we have

$$
\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\left(\frac{B^{2}}{2 \mu_{0}}+\frac{\epsilon_{0} E^{2}}{2}\right) \Longrightarrow u=\frac{1}{2}\left(\frac{B^{2}}{\mu_{0}}+\epsilon_{0} E^{2}\right)
$$

For waves, we let $\mathbf{E}=\mathbf{E}_{0} e^{i(\omega t-\mathbf{k} \cdot \mathbf{r})}$ and $\mathbf{B}=\mathbf{B}_{0} e^{i(\omega t-\mathbf{k} \cdot \mathbf{r})}$ such that $\nabla \mapsto-i \mathbf{k}$ and $\partial_{t} \mapsto i \omega$. The Maxwell equations (in free space) then becomes

$$
\begin{align*}
\mathbf{k} \cdot \mathbf{E} & =0  \tag{1.1}\\
\mathbf{k} \cdot \mathbf{B} & =0  \tag{1.2}\\
\mathbf{k} \times \mathbf{E} & =\omega \mathbf{B}  \tag{1.3}\\
\mathbf{k} \times \mathbf{B} & =-\frac{\omega}{c^{2}} \mathbf{E} \tag{1.4}
\end{align*}
$$

Thus the power flow (aka energy flux) is given by the time-averaged Poynting vector

$$
\mathbf{S}=\mathbf{E} \times \mathbf{H}=E H \hat{\mathbf{k}}=\frac{E^{2}}{Z} \hat{\mathbf{k}} \Longrightarrow\langle\mathbf{S}\rangle=\frac{1}{2} \frac{E_{0}^{2}}{Z} \hat{\mathbf{k}}
$$

with impedance Z defined as

$$
Z=\frac{|\mathbf{E}|}{|\mathbf{H}|}=\sqrt{\frac{\mu_{0} \mu_{r}}{\epsilon_{0} \epsilon_{r}}}=Z_{0} \sqrt{\frac{\mu_{r}}{\epsilon_{r}}}
$$

where $Z_{0}=\sqrt{\mu_{0} / \epsilon_{0}} \approx 377 \Omega$ is the impedance of free space.

### 1.3 EM in terms of Potentials

Looking at (M2), one might define a magnetic field via

$$
\begin{equation*}
\mathbf{B}=\nabla \times \mathbf{A} \tag{1.5}
\end{equation*}
$$

where $\mathbf{A}$ is some vector field known as the magnetic vector potential. Note that this automatically satisfies M2) since $\nabla \cdot(\nabla \times \mathbf{A})=0$ for all vector fields $\left(\text { in } \mathbb{R}^{3}\right)^{3}$.
Similarly, since

$$
0=\nabla \times \mathbf{E}+\frac{\partial(\nabla \times \mathbf{A})}{\partial t}=\nabla \times\left(\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}\right)
$$

we can define a scalar potentia $4^{4}$ via

$$
\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}=-\nabla \phi
$$

Rearranging, we have

$$
\begin{equation*}
\mathbf{E}=-\nabla \phi-\frac{\partial \mathbf{A}}{\partial t} \tag{1.6}
\end{equation*}
$$

This reduces the 6 components of $\mathbf{E}$ and $\mathbf{B}$ to the 4 components of $\mathbf{A}$ and $\phi$ !

## Example 1.

Consider a uniform magnetic field

$$
\mathbf{B}=\left(0,0, B_{0}\right)=\nabla \times \mathbf{A}=\left(\partial_{y} A_{z}-\partial_{z} A_{y}, \partial_{z} A_{x}-\partial_{x} A_{z}, \partial_{x} A_{y}-\partial_{y} A_{x}\right)
$$

Here, $\mathbf{A}$ can be $B_{0}(0, x, 0), B_{0}(-y, 0,0)$, or $\frac{B_{0}}{2}(-y, x, 0)$. In particular, there is a freedom to choose between different potentials for a given field (though the third one is usually preferred for its symmetry).

Now, note that for any scalar field $\psi, \nabla \times(\nabla \psi)=0$. Therefore, for any magnetic field $\mathbf{B}=\nabla \times \mathbf{A}$,

$$
\mathbf{B}^{\prime}=\nabla \times(\mathbf{A}+\nabla \psi)=\mathbf{B}
$$

i.e. we can arbitrarily choose $\psi$ and have $\mathbf{B}$ expressed in terms of $\mathbf{A}^{\prime} \equiv \mathbf{A}+\nabla \psi$.

However, from (1.6),

$$
-\mathbf{E}=\nabla \phi^{\prime}+\frac{\partial \mathbf{A}^{\prime}}{\partial t}=\nabla \phi^{\prime}+\frac{\partial \mathbf{A}}{\partial t}+\frac{\partial(\nabla \psi)}{\partial t}
$$

Therefore, for $\mathbf{E}$ to remain unchanged, we require

$$
\begin{equation*}
\nabla \phi^{\prime}=\nabla \phi-\frac{\partial(\nabla \psi)}{\partial t} \Longrightarrow \phi^{\prime}=\phi-\frac{\partial \psi}{\partial t} \tag{1.7}
\end{equation*}
$$

[^1]The selection of such scalar field $\psi$ is called "choosing the gauge", which is usually expressed in terms of $\nabla \cdot \mathbf{A}$. A useful choice for relativity is the Lorenz gaug $\overline{\text { 雨 }}$

## Lorenz gauge

$$
\begin{equation*}
\nabla \cdot \mathbf{A}+\frac{1}{c^{2}} \frac{\partial \phi}{\partial t}=0 \tag{1.8}
\end{equation*}
$$

Substituting (1.6) into (M1), we get

$$
\begin{equation*}
-\nabla^{2} \phi-\frac{\partial(\nabla \cdot \mathbf{A})}{\partial t}=\frac{\rho}{\epsilon_{0}} \tag{1.9}
\end{equation*}
$$

Similarly, substituting (1.5) and (1.6) into (M4), we get

$$
\begin{equation*}
-\nabla^{2} \mathbf{A}+\nabla(\nabla \cdot \mathbf{A})+\frac{1}{c^{2}}\left[\frac{\partial(\nabla \phi)}{\partial t}+\frac{\partial^{2} \mathbf{A}}{\partial t^{2}}\right]=\mu_{0} \mathbf{J} \tag{1.10}
\end{equation*}
$$

Finally, substituting the Lorenz gauge (1.8) into (1.9) and (1.10) gives

$$
\begin{align*}
\nabla^{2} \phi-\frac{1}{c^{2}} \frac{\partial^{2} \phi}{\partial t^{2}} & =-\frac{\rho}{\epsilon_{0}}  \tag{1.11}\\
\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}} & =-\mu_{0} \mathbf{J} \tag{1.12}
\end{align*}
$$

Defining the d'Alembertian by

$$
\begin{equation*}
\square:=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}-\nabla^{2} \tag{1.13}
\end{equation*}
$$

we can rewrite the equations as

## Maxwell's Equations in terms of potentials

$$
\begin{equation*}
\square \phi=\frac{\rho}{\epsilon_{0}} \quad \text { and } \quad \square \mathbf{A}=\mu_{0} \mathbf{J} \tag{1.14}
\end{equation*}
$$

These two equations contain all the information we had in Maxwell's equations. In particular, in free space ( $\rho=0, \mathbf{J}=0$ ), we recover wave equations for $\mathbf{A}$ and $\phi(\square \phi=0$ and $\square \mathbf{A}=0$ ), which gives a description of electromagnetic waves (of speed $c$, as expected).

### 1.4 EM Fields in Materials

Note: This is mainly PX263 revision, and serves as a clarification between $\mathbf{E}$ and $\mathbf{D}$; it is not particularly related to the rest of this module as we will only consider EM fields in free space.

Recall that

$$
\nabla \cdot \mathbf{E}=\frac{\rho}{\epsilon_{0}} \quad \text { and } \quad \nabla \cdot \mathbf{D}=\rho_{\text {free }}
$$

where

$$
\rho=\rho_{\text {free }}+\rho_{p} \Longrightarrow \nabla \cdot\left(\epsilon_{0} \mathbf{E}\right)=\nabla \cdot \mathbf{D}-\nabla \cdot \mathbf{P} \Longrightarrow \mathbf{D}=\epsilon_{0} \mathbf{E}+\mathbf{P}
$$

Usually (for linear, isotropic materials), we have

$$
\mathbf{P}=\epsilon_{0} \chi \mathbf{E} \Longrightarrow \mathbf{D}=\epsilon_{0} \epsilon_{r} \mathbf{E}
$$

[^2]where $\epsilon_{r}=1+\chi$. For non-linear and/or anisotropic materials, this generalizes to the non-linear tensor relation (Einstein notation, summing over repeated indices)
$$
P_{i}=\epsilon_{0} \chi_{i j} E_{j}+\epsilon_{0}^{2} \chi_{i j k}^{(2)} E_{j} E_{k}+\cdots
$$

Again, we (fortunately) will not consider such complications in this module as we will only be looking at EM fields in free space.

### 1.5 Prelude to Special Relativity (PX148 Revision)

In 1887, Michelson-Morley disproved the existence of the aether (medium for propagation of light waves) by improving upon the Michelson interferometer (see PX148 notes for more details). In 1892, Lorentz first derived the Lorentz Force Law

$$
\begin{equation*}
\mathbf{F}=\frac{d \mathbf{p}}{d t}=q(\mathbf{E}+\mathbf{v} \times \mathbf{B}) \tag{1.15}
\end{equation*}
$$

bridging Maxwell's EM fields to Newtonian mechanics. Later in the same year, as an attempt to reconcile the aether with Michelson-Morley, he proposed a coordinate transformation for a frame of reference moving at velocity $v$ in the $+x$ direction:

$$
\begin{aligned}
t^{\prime} & =\gamma\left(t-\frac{v}{c^{2}} x\right) \\
x^{\prime} & =\gamma(x-v t) \\
y^{\prime} & =y \\
z^{\prime} & =z
\end{aligned}
$$

where he derived much later in 1903 that

$$
\begin{equation*}
\gamma=\frac{1}{\sqrt{1-v^{2} / c^{2}}}=\left(1-\beta^{2}\right)^{-1 / 2} \tag{1.16}
\end{equation*}
$$

where $\beta:=v / c$. However, as we know, it wasn't until later in 1905 that Einstein provided the correct physical interpretation for this transformation.

### 1.6 Special Relativity

In his paper The Electrodynamics of Moving Bodies (1905), Einstein reconciled Lorentz Transformation with mechanics by introducing a new understanding of time, under which Gallilean transformation and Newtonian mechanics work as approximations for the case $v / c \ll 1$.
Later in 1907, Minkowsk $7^{7}$ coined the term spacetime (more formally the Minkowski spac $母^{8}$ ), a 4 -dimensional object on which events are points with coordinates

$$
X^{\mu}=\left(X^{0}, X^{1}, X^{2}, X^{3}\right)=(c t, x, y, z)
$$

with indices deliberately written as superscripts to denote contravariant rank-1 tensors (vectors); we shall introduce its counterpart (covariant vectors) later, where indices are written as subscripts (e.g. $X_{\mu}$ ).
Note that we are free to choose the origin of each axis and the orientation of the spatial axes via a rotation matrix, typically (rotating by $\theta$ around the $z$-axis),

$$
R=\left(\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right)
$$

[^3]We can also change our frame of reference via a Lorentz Boost

$$
\left(\begin{array}{c}
c t^{\prime} \\
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right)=\left(\begin{array}{cccc}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
c t \\
x \\
y \\
z
\end{array}\right)
$$

or equivalently,

$$
X^{\prime \mu}=\Lambda_{\nu}^{\mu} X^{\nu}
$$

where

$$
\Lambda:=\left(\begin{array}{cccc}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)=\left(\begin{array}{cccc}
\cosh \alpha & -\sinh \alpha & 0 & 0 \\
-\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

with $\alpha:=\tanh ^{-1} \beta(\operatorname{such}$ that $\Lambda$ looks like a rotation matrix) 910 ,
We also define the metric distance (Minkowski norm squared)

$$
s^{2}:=(c t)^{2}-x^{2}-y^{2}-z^{2}
$$

which is invariant under Lorentz transformations (LT). We often consider the difference between two events (Minkowski inner product ${ }^{11}$, aka the relativistic dot product) given by

$$
d s^{2}:=(c d t)^{2}-d x^{2}-d y^{2}-d z^{2}
$$

This is usually referred to as the spacetime interval, which is again Lorentz invariant. Here, the opposite signature of time and spatial coordinates gives three cases (inside, on, and outside light cone):

$$
\begin{array}{r}
d s^{2}>0: \text { time-like interval } \\
d s^{2}=0: \text { light-like interval } \\
d s^{2}<0: \text { space-like interval }
\end{array}
$$

### 1.7 Introduction to 4 -vectors

4 -vectors represent physical quantities and respect the symmetry of spacetime, i.e. transform according to the Lorentz Boosts. They are usually of the form

$$
X^{\mu}=(c t, \mathbf{r})=(c t, x, y, z)
$$

Often, we denote 4 -vector components with Greek indices $X^{\mu}$, where $\mu=0,1,2,3$, and reserve Italic indices $X^{i}, i=1,2,3$ for ONLY the spatial components.
In particular, the Minkowski distance is

$$
s^{2}=\left(X^{0}\right)^{2}-\left(X^{1}\right)^{2}-\left(X^{2}\right)^{2}-\left(X^{3}\right)^{2}
$$

As hinted earlier, we can also define the covariant vector $X_{\mu}$, where the lower indices are defined as $X_{0}=X^{0}, X_{i}=-X^{i}$, i.e.

$$
X_{\mu}=(c t,-x,-y,-z)
$$

[^4]Therefore, we have
Spacetime interval

$$
\begin{equation*}
s^{2}:=X_{\mu} X^{\mu}=X_{0} X^{0}+X_{1} X^{1}+X_{2} X^{2}+X_{3} X^{3} \tag{1.17}
\end{equation*}
$$

where repeated indices are again summed over with the Einstein summation convention.
More generally, for any 4 -vector $v$ and $w$, we have the (Lorentz invariant) inner product $v \cdot w \equiv$ $v_{\mu} w^{\mu}=v^{\mu} w_{\mu}$, and again all 4-vectors $a$ transform according to LT via

$$
a^{\prime \mu}=\Lambda_{\nu}^{\mu} a^{\nu}
$$

### 1.7.1 4-vector Examples

There are various useful 4 -vectors one can define. For relativistic mechanics, we have

- 4-position:

$$
X^{\mu}=(c t, \mathbf{r})=(c t, x, y, z)
$$

- 4-velocity:

$$
U^{\mu}=\gamma(c, \mathbf{v})
$$

where a factor of $\gamma$ is added such that $U_{\mu} U^{\mu}=c^{2}$ is Lorentz invariant.

- 4-momentum:

$$
P^{\mu}=(E / c, \mathbf{p})
$$

which is defined such that $P_{\mu} P^{\mu}=E^{2} / c^{2}-p^{2}$, and $E^{2}=m^{2} c^{4}+p^{2} c^{2} \Longrightarrow P_{\mu} P^{\mu}=m^{2} c^{2}$, i.e. Lorentz invariant ( $m=$ rest mass). Note that this is consistent with the 4 -velocity via

$$
P^{\mu}=m U^{\mu}=\left(\gamma m c^{2} / c, \gamma m \mathbf{v}\right)=(E / c, \mathbf{p})
$$

We shall derive the expressions for the 4 -velocity and 4 -momentum more carefully later.
Furthermore, for electromagnetism, we have

- 4-current (density):

$$
j^{\mu}=(\rho c, \mathbf{J})
$$

- 4-potential:

$$
A^{\mu}=(\phi / c, \mathbf{A})
$$

We shall also derive these expressions properly later. However, observe that since $\square \phi=\frac{\rho}{\epsilon_{0}}$ and $\square \mathbf{A}=\mu_{0} \mathbf{J}$, we have

## Electrodynamics in one equation

$$
\begin{equation*}
\square A^{\mu}=\mu_{0} j^{\mu} \tag{1.18}
\end{equation*}
$$

### 1.8 More on Tensors

Note: If you are also a MathPhys student and find this section too handwavy, I highly recommend David Tong's brilliant lecture notes $\int^{12}$ for a far more comprehensive introduction to tensors.

Tensors are usually classified in terms of their rank, which is roughly defined by the number of spacetime indices ${ }^{13}$. Some useful examples include

- Rank 0 (scalar): $d s^{2}=d X_{\mu} d X^{\mu}, c^{2}=U_{\mu} U^{\mu}$
- Rank 1: 4-vectors, e.g. $X^{\mu}, P^{\mu}, j^{\mu}, A^{\mu}, \ldots$
- Rank 2: metric tensor $g^{\mu \nu}$, Faraday tensor $F^{\mu \nu}$
- Rank 3: $\partial_{\lambda} F_{\mu \nu}$
- Rank 4: antisymmetric tensor $\epsilon_{\alpha \beta \gamma \delta}$, Riemann curvature tensor $R_{\beta \gamma \delta}^{\alpha}$

All tensors must transform according to LT, e.g. $F^{\prime \mu \nu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} F^{\alpha \beta}$.
We shall now define the metric tensor, which is given by ${ }^{14}$

## Metric Tensor (Special Relativity)

$$
g^{\mu \nu}=g_{\mu \nu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{1.19}\\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

In particular, this allows us to transform between covariant and contravariant vectors, i.e.

$$
X^{\mu}=g^{\mu \nu} X_{\nu} \quad \text { and } \quad X_{\mu}=g_{\mu \nu} X^{\nu}
$$

which lets us rewrite the spcaetime interval as

$$
d s^{2}=g_{\mu \nu} X^{\mu} X^{\nu}
$$

Note: In general, the metric tensor $g_{\mu \nu}$ is more complicated than this (more to come in GR!). To distinguish this "special" relativity case from the "general" case (pun intended), we often replace $g_{\mu \nu}$ by

$$
\eta_{\mu \nu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)=\eta^{\mu \nu}
$$

[^5]
## 2 Introduction to Relativistic Electrodynamics

I went to a wedding in a Faraday cage...
There was no reception ${ }^{15}$

### 2.1 Relativistic Equations of Motion

A consistent set of relativistic eqautions of motion must

1. approach the Newtonian results in the classical limit $v / c \ll 1$; and
2. transform as 4 -vectors via Lorentz transformations.

To formulate such equations, we shall first look at the proper time $\tau$, which is the time in the co-moving frame of our particle in question, i.e. the time measured in a frame of reference where the particle is stationary. In this co-moving frame, the spacetime interval between two events is

$$
d s^{2}=\eta_{\mu \nu} d X^{\mu} d X^{\nu}=c^{2} d \tau^{2}-d x_{\tau}^{2}-d y_{\tau}^{2}-d z_{\tau}^{2}
$$

In particular, we have $d x_{\tau}=d y_{\tau}=d z_{\tau}=0$ for the particle in this frame, so, assuming the interval is timelike ${ }^{16}$, i.e. $d s^{2}>0$, we have

$$
d s^{2}=c^{2} d \tau^{2} \Longrightarrow d s=c d \tau
$$

or equivalently,

$$
\begin{equation*}
d \tau=\frac{d s}{c} \tag{2.20}
\end{equation*}
$$

By definition, the proper time is a scalar agreeable by all observers (Lorentz invariant).
Note that in a different frame of reference, we have

$$
\begin{aligned}
c^{2} d \tau^{2} & =d s^{2}=c^{2} d t^{2}-d x^{2}-d y^{2}-d z^{2} \\
d \tau^{2} & =d t^{2}\left(1-\frac{1}{c^{2}} \frac{d x^{2}}{d t^{2}}-\frac{1}{c^{2}} \frac{d y^{2}}{d t^{2}}-\frac{1}{c^{2}} \frac{d z^{2}}{d t^{2}}\right) \\
& =d t^{2}\left(1-\beta^{2}\right)
\end{aligned}
$$

Therefore,

## Proper time

$$
\begin{equation*}
d \tau=\frac{d t}{\gamma} \tag{2.21}
\end{equation*}
$$

as expected from PX148.
Now, we are ready to properly formulate the 4-velocity by

$$
\begin{aligned}
U^{\mu} & :=\frac{d X^{\mu}}{d \tau} \\
& =\left(\frac{d X^{0}}{d \tau}, \frac{d X^{1}}{d \tau}, \frac{d X^{2}}{d \tau}, \frac{d X^{3}}{d \tau}\right) \\
& =\gamma\left(\frac{d X^{0}}{d t}, \frac{d X^{1}}{d t}, \frac{d X^{2}}{d t}, \frac{d X^{3}}{d t}\right)
\end{aligned}
$$

[^6]i.e.

4-Velocity

$$
\begin{equation*}
U^{\mu}=\gamma(c, \mathbf{v}) \tag{2.22}
\end{equation*}
$$

where we differentiated position against $\tau$ to ensure $U_{\mu} U^{\mu}=c^{2}$ is observer-independent ${ }^{17}$,
Similarly, we define the 4 -momentum ${ }^{18}$ by

$$
P^{\mu}:=m U^{\mu}=\gamma m(c, \mathbf{v})=\left(\gamma m c^{2} / c, \gamma m \mathbf{v}\right)
$$

i.e.

## 4-Momentum

$$
\begin{equation*}
P^{\mu}=(E / c, \mathbf{p}) \tag{2.23}
\end{equation*}
$$

To define the 4 -force

$$
f^{\mu}:=\frac{d P^{\mu}}{d \tau}=\gamma\left(\frac{d P^{0}}{d t}, \frac{d \mathbf{p}}{d t}\right)
$$

Note that

$$
U_{\mu} f^{\mu}=U_{\mu} \frac{d}{d \tau}\left(m U^{\mu}\right)=m U_{\mu} \frac{d U^{\mu}}{d \tau}=\frac{m}{2} \frac{d}{d \tau}\left(U_{\mu} U^{\mu}\right)=0
$$

In particular,

$$
0=U^{0} f^{0}-\left(U^{1} f^{1}+U^{2} f^{2}+U^{3} f^{3}\right)=\gamma c f^{0}-\gamma \mathbf{v} \cdot \gamma \mathbf{F} \Longrightarrow f^{0}=\frac{\gamma}{c}(\mathbf{v} \cdot \mathbf{F})
$$

Therefore,

## 4-Force

$$
\begin{equation*}
f^{\mu}=\gamma\left(\frac{\mathbf{v} \cdot \mathbf{F}}{c}, \mathbf{F}\right)=\gamma\left(\frac{1}{c} \frac{d E}{d t}, \mathbf{F}\right) \tag{2.24}
\end{equation*}
$$

Looking at the Newtonian work done relationship $d E / d t=\mathbf{F} \cdot \mathbf{v}$, the 4 -force we obtained from the 4 -momentum does somehow resemble what's expected classically.

[^7]
### 2.2 Relativistic EM Forces and the Faraday Tensor

To use our new 4 -vectors, we will need a relativistic form of the Lorentz Force Law. Substituting 1.15 into 2.24 , and using 2.22 , we get ${ }^{19}$

$$
\begin{aligned}
& f^{0}=\frac{\gamma}{c}(\mathbf{v} \cdot \mathbf{F})=\frac{q \gamma}{c}(\mathbf{v} \cdot \mathbf{E}+\mathbf{v} \cdot(\mathbf{v} \times \mathbf{B}))=\frac{q}{c}\left(U^{1} E_{x}+U^{2} E_{y}+U^{3} E_{z}\right) \\
& f^{1}=\gamma F_{x}=\gamma q\left(E_{x}+v_{y} B_{z}-v_{z} B_{y}\right)=q\left(U^{0} \frac{E_{x}}{c}+U^{2} B_{z}-U^{3} B_{y}\right) \\
& f^{2}=\gamma F_{x}=\gamma q\left(E_{y}+v_{z} B_{x}-v_{x} B_{z}\right)=q\left(U^{0} \frac{E_{y}}{c}+U^{3} B_{x}-U^{1} B_{z}\right) \\
& f^{3}=\gamma F_{x}=\gamma q\left(E_{z}+v_{x} B_{y}-v_{y} B_{x}\right)=q\left(U^{0} \frac{E_{z}}{c}+U^{1} B_{y}-U^{2} B_{x}\right)
\end{aligned}
$$

Putting all together, we have

$$
\begin{equation*}
f^{\mu}=q F_{\nu}^{\mu} U^{\nu} \tag{2.25}
\end{equation*}
$$

where $F_{\nu}^{\mu}$ is the Faraday Tensor (aka Electromagnetic Tensor), which is a rank-2 tensor

$$
F_{\nu}^{\mu}:=\left(\begin{array}{cccc}
0 & E_{x} / c & E_{y} / c & E_{z} / c \\
E_{x} / c & 0 & B_{z} & -B_{y} \\
E_{y} / c & -B_{z} & 0 & B_{x} \\
E_{z} / c & B_{y} & -B_{x} & 0
\end{array}\right)
$$

with $\mu$ and $\nu$ indexing the rows and columns respectively, e.g. $F_{2}^{1}=B_{z}$.
Usually, one uses either the (fully) contravariant form or the (fully) covariant form, i.e.

## Faraday/Electromagnetic Tensor

$$
\begin{align*}
& F^{\mu \nu}=\eta^{\nu \beta} F_{\beta}^{\mu}=\left(\begin{array}{cccc}
0 & -E_{x} / c & -E_{y} / c & -E_{z} / c \\
E_{x} / c & 0 & -B_{z} & B_{y} \\
E_{y} / c & B_{z} & 0 & -B_{x} \\
E_{z} / c & -B_{y} & B_{x} & 0
\end{array}\right)  \tag{2.26}\\
& F_{\mu \nu}=\eta_{\mu \alpha} \eta_{\nu \beta} F^{\alpha \beta}=\left(\begin{array}{cccc}
0 & E_{x} / c & E_{y} / c & E_{z} / c \\
-E_{x} / c & 0 & -B_{z} & B_{y} \\
-E_{y} / c & B_{z} & 0 & -B_{x} \\
-E_{z} / c & -B_{y} & B_{x} & 0
\end{array}\right) \tag{2.27}
\end{align*}
$$

In particular, note that $F^{\mu \nu}$ and $F_{\mu \nu}$ are fully antisymmetric, i.e. $F^{\mu \nu}=-F^{\nu \mu}$ and $F_{\mu \nu}=-F_{\nu \mu}$. Note also that the matrices all consist of a time-like block (0th row and column) of electric field entries, and a space-like block (bottom-right $3 \times 3$ matrix) of magnetic field entries.

With these new forms, one usually writes the relativistic Lorentz Force Law as
Relativistic Lorentz Force Law

$$
\begin{equation*}
f^{\mu}=q F^{\mu \nu} U_{\nu} \quad \text { or } \quad f_{\mu}=q F_{\mu \nu} U^{\nu} \tag{2.28}
\end{equation*}
$$

[^8]
### 2.3 Motion in Simple EM Fields

Now, using 2.25, we have

$$
\begin{equation*}
f^{\mu}=\frac{d P^{\mu}}{d \tau}=m \frac{d U^{\mu}}{d \tau} \Longrightarrow \frac{d U^{\mu}}{d \tau}=\frac{q}{m} F_{\nu}^{\mu} U^{\nu} \tag{2.29}
\end{equation*}
$$

This is simply an eigenvalue problem of the form

$$
\frac{d \mathbf{u}}{d \tau}=M \mathbf{u}
$$

which has general solution

$$
\mathbf{u}=\sum_{\lambda} A_{\lambda} e^{\lambda \tau} \mathbf{e}_{\lambda}
$$

where $\lambda$ and $\mathbf{e}_{\lambda}$ are the eigenvalues and corresponding eigenvectors of $M$ respectively. We shall now try and solve 2.29 under different (boundary) conditions.

Case 1: Uniform E-field; $\mathbf{E}=(E, 0,0), \mathbf{B}=\mathbf{0}$
The Faraday Tensor is

$$
F_{\nu}^{\mu}=\eta_{\nu \beta} F^{\mu \beta}=\left(\begin{array}{cccc}
0 & E / c & 0 & 0 \\
E / c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

which has eigenvalues $\lambda=0$ (repeated) and $\pm E / c$, with eigenvectors $(0,0,0,1),(0,0,1,0)$, $(1,1,0,0)$, and $(1,-1,0,0)$ respectively. In particular, our general solution is

$$
U^{\mu}=A e^{a \tau}\left(\begin{array}{l}
1 \\
1 \\
0 \\
0
\end{array}\right)+B e^{-a \tau}\left(\begin{array}{c}
1 \\
-1 \\
0 \\
0
\end{array}\right)+C\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right)+D\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right)
$$

where $a=q E / m c$.
Assuming ${ }^{20} v_{x}(\tau=0)=0$, we have $A=B$. Furthermore, from $U^{0}(\tau=0)=\gamma_{0} c=A+B$, we have $A=B=\gamma_{0} c / 2$. Therefore,

$$
\binom{\gamma c}{\gamma v_{x}}=\binom{U^{0}}{U^{1}}=\gamma_{0} c\binom{\cosh \left(\frac{q E}{m c} \tau\right.}{\sinh \left(\frac{q E}{m c} \tau\right)}
$$

In particular, $\gamma$ changes with time via

$$
\gamma(\tau)=\gamma_{0} \cosh \left(\frac{q E}{m c} \tau\right)
$$

and so

$$
v_{x}(\tau)=\frac{U^{1}}{\gamma(\tau)}=c \tanh \left(\frac{q E}{m c} \tau\right)
$$

This should make sense since

$$
\lim _{\tau \rightarrow \infty}\left|v_{x}(\tau)\right|=c \cdot \lim _{x \rightarrow \pm \infty}|\tanh (x)|=c
$$

as is expected for a relativistic accelerator.

[^9]Using this, one can also find the coordinate time $t$ via

$$
d t=\gamma(\tau) d \tau \Longrightarrow t=\int_{0}^{t} \gamma(\tau) d \tau \propto \sinh \left(\frac{q E}{m c} \tau\right)
$$

Now, for the transverse velocities $v_{y}$ and $v_{z}$, note that

$$
U^{2}=\gamma(\tau) v_{y}(\tau)=C \Longrightarrow v_{y}(\tau)=\frac{C}{\gamma(\tau)}
$$

and similarly,

$$
U^{3}=\gamma(\tau) v_{z}(\tau)=D \Longrightarrow v_{z}(\tau)=\frac{D}{\gamma(\tau)}
$$

Therefore, perhaps surprisingly, the transverse velocities decrease as the particle accelerates! But of course, this is expected as

$$
P^{2}=\gamma m v_{y}=\frac{\gamma m C}{\gamma}=m C \quad \text { and } \quad P^{3}=\gamma m v_{z}=\frac{\gamma m D}{\gamma}=m D
$$

i.e. the transverse momtenta are conserved ( $C$ and $D$ are just constants).

Case 2: Uniform B-field; $\mathbf{E}=\mathbf{0}, \mathbf{B}=(B, 0,0)$
The Faraday Tensor is

$$
F_{\nu}^{\mu}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & -B & 0
\end{array}\right)
$$

So, $U^{0}$ and $U^{1}$ are constant. Also,

- $U^{0}=\gamma c \Longrightarrow \gamma=$ constant $\Longrightarrow|\mathbf{v}|=$ constant $\Longrightarrow$ Energy is conserved; and
- $U^{1}=\gamma v_{x} \Longrightarrow$ velocity along the direction of $\mathbf{B}, v_{x}$, remains constant.

This time, $F_{\nu}^{\mu}$ has eigenvalues 0 (repeated) and $\pm i B$, with eigenvectors $(1,0,0,0),(0,1,0,0)$, $(0,0,1, i)$ and $(0,0,1,-i)$ respectively. This has general solution

$$
U^{\mu}=A^{\prime}\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right)+B^{\prime}\left(\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right)+C e^{i \omega \tau}\left(\begin{array}{l}
0 \\
0 \\
1 \\
i
\end{array}\right)+D e^{-i \omega \tau}\left(\begin{array}{c}
0 \\
0 \\
1 \\
-i
\end{array}\right)
$$

where $\omega=q B / m$ is the Larmor frequency. In particular, assuming ${ }^{21} v_{z}(\tau=0)=0$, we have $C=D$, and we can write

$$
\binom{U^{2}}{U^{3}}=A\binom{\cos (\omega \tau)}{-\sin (\omega \tau)}=A\binom{\sin (\omega \tau+\phi)}{\cos (\omega \tau+\phi)}
$$

where $A=C+D$, and a phase $\phi$ is added for generality.
In particular, this results in helical motion around the $x$-axis, with $\omega(t)=\omega(\tau) / \gamma$.
Case 3: Crossed $\mathbf{E}$ and $\mathbf{B}$ fields; $\mathbf{E}=\left(0, E_{y}, 0\right), \mathbf{B}=\left(0,0, B_{z}\right)$
The Faraday Tensor is

$$
F_{\nu}^{\mu}=\left(\begin{array}{cccc}
0 & 0 & E_{y} / c & 0 \\
0 & 0 & B_{z} & 0 \\
E_{y} / c & -B_{z} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

[^10]Clearly, $U^{\mu}=\left(c, E_{y} / B_{z}, 0,0\right)$ is an eigenvector with eigenvalue 0 , and is therefore a steady-state solution for 2.29. In particular, we have

$$
v_{x}=\frac{E_{y}}{B_{z}}
$$

with overall motion perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. This is known as the Hall drift. Aside: More generally, Hall drift occurs whenever $\mathbf{E} \cdot \mathbf{B}=0$, with $\mathbf{v}_{d}=\mathbf{E} \cdot \mathbf{B} / B^{2}$.

### 2.4 Lorentz Transformation of Tensors

Rank-0 tensors (scalars):
Scalars are trivially invariant, i.e. they are the same in all reference frames.
Rank-1 tensors (vectors):
In this module, we consider 4 -vectors which transform according to LT via

$$
X^{\mu}=\Lambda_{\nu}^{\mu} X^{\nu}
$$

with the Lorentz boost in the $+x$-direction given by

$$
\Lambda_{\nu}^{\mu}=\left(\begin{array}{cccc}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

$\underline{\text { Rank-2 tensors (matrices) })^{22}}$
The Lorentz Transformation generalizes easily to higher-rank tensors. For example, we looked at the Faraday Tensor, which transforms according to LT via

$$
F^{\prime \mu \nu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} F^{\alpha \beta}
$$

More explicitly, suppose we want to find $E_{x}^{\prime}$ for a particle travelling along the $x$-direction. Then

$$
\begin{aligned}
\frac{E_{x}^{\prime}}{c} & =F^{\prime 10}=\Lambda_{\mu}^{1} \Lambda_{\nu}^{0} F^{\mu \nu}=\Lambda_{0}^{1}\left(\Lambda_{0}^{0} F^{00}+\Lambda_{1}^{0} F^{01}\right)+\Lambda_{1}^{1}\left(\Lambda_{0}^{0} F^{10}+\Lambda_{1}^{0} F^{11}\right) \\
& =-\beta \gamma\left(0+\beta \gamma \frac{E_{x}}{c}\right)+\gamma\left(\gamma \frac{E_{x}}{c}+0\right)=\left(1-\beta^{2}\right) \gamma^{2} \cdot \frac{E_{x}}{c}=\frac{E_{x}}{c}
\end{aligned}
$$

So, $E_{x}^{\prime}=E_{x}$, i.e. the parallel component of $\mathbf{E}$ remains unchanged.
Similarly, we have

$$
\frac{E_{y}^{\prime}}{c}=F^{\prime 20}=\Lambda_{\mu}^{2} \Lambda_{\nu}^{0} F^{\mu \nu}=\Lambda_{2}^{2}\left(\Lambda_{0}^{0} F^{20}+\Lambda_{1}^{0} F^{21}\right)=\frac{\gamma E_{y}}{c}-\beta \gamma B_{z}
$$

and

$$
\frac{E_{z}^{\prime}}{c}=F^{\prime 30}=\Lambda_{\mu}^{3} \Lambda_{\nu}^{0} F^{\mu \nu}=\Lambda_{3}^{3}\left(\Lambda_{0}^{0} F^{30}+\Lambda_{1}^{0} F^{31}\right)=\frac{\gamma E_{z}}{c}+\beta \gamma B_{y}
$$

In particular, observe that

$$
\begin{aligned}
& E_{y}^{\prime}=\gamma\left(E_{y}-v_{x} B_{z}\right)=\gamma[\mathbf{E}+\mathbf{v} \times \mathbf{B}]_{y} \\
& E_{z}^{\prime}=\gamma\left(E_{z}+v_{x} B_{y}\right)=\gamma[\mathbf{E}+\mathbf{v} \times \mathbf{B}]_{z}
\end{aligned}
$$

Now, for magnetic fields, we have

$$
B_{x}^{\prime}=F^{\prime 32}=\Lambda_{\mu}^{3} \Lambda_{\nu}^{2} F^{\mu \nu}=\Lambda_{3}^{3} \Lambda_{2}^{2} F^{32}=B_{x}
$$

[^11]So, the parallel component of $\mathbf{B}$ remains unchanged.
Similarly,

$$
B_{y}^{\prime}=\Lambda_{\mu}^{1} \Lambda_{\nu}^{3} F^{\mu \nu}=\Lambda_{0}^{1} F^{03}+\Lambda_{1}^{1} F^{13}=\frac{\beta \gamma E_{z}}{c}+\gamma B_{y}
$$

and

$$
B_{z}^{\prime}=\Lambda_{\mu}^{2} \Lambda_{\nu}^{1} F^{\mu \nu}=\Lambda_{0}^{1} F^{20}+\Lambda_{1}^{1} F^{21}=-\frac{\beta \gamma E_{y}}{c}+\gamma B_{z}
$$

Again, we can rewrite these expressions as

$$
B_{y}^{\prime}=\gamma\left[\mathbf{B}-\frac{\mathbf{v} \times \mathbf{E}}{c^{2}}\right]_{y} \quad \text { and } \quad B_{z}^{\prime}=\gamma\left[\mathbf{B}-\frac{\mathbf{v} \times \mathbf{E}}{c^{2}}\right]_{z}
$$

To summarize, we have

## Lorentz Transformation of E and B

$$
\begin{gather*}
\mathbf{E}_{\|}^{\prime}=\mathbf{E}_{\|}  \tag{2.30}\\
\mathbf{B}_{\|}^{\prime}=\mathbf{B}_{\|}  \tag{2.31}\\
\mathbf{E}_{\perp}^{\prime}=\gamma\left(\mathbf{E}_{\perp}+\mathbf{v} \times \mathbf{B}\right)  \tag{2.32}\\
\mathbf{B}_{\perp}^{\prime}=\gamma\left(\mathbf{B}_{\perp}-\frac{\mathbf{v} \times \mathbf{E}}{c^{2}}\right) \tag{2.33}
\end{gather*}
$$

An important consequence of this is quantities that are frame invariant (aka Lorentz invariants).
Theorem 1. The dot product $\boldsymbol{E} \cdot \boldsymbol{B}$ is frame invariant.
Proof. WLOG, assume $\mathbf{v}=(v, 0,0)$, which we are allowed to as spatial rotations always leave dot products unchanged (this has nothing to do with Lorentz Transformations).

$$
\begin{aligned}
\mathbf{E}^{\prime} \cdot \mathbf{B}^{\prime} & =E_{x}^{\prime} B_{x}^{\prime}+E_{y}^{\prime} B_{y}^{\prime}+E_{z}^{\prime} B_{z}^{\prime} \\
& =E_{x} B_{x}+\gamma^{2}\left(E_{y}-v B_{z}\right)\left(B_{y}+\frac{v E_{z}}{c^{2}}\right)+\gamma^{2}\left(E_{z}+v B_{y}\right)\left(B_{z}-\frac{v E_{y}}{c^{2}}\right) \\
& =E_{x} B_{x}+E_{y} B_{y} \gamma^{2}\left(1-\beta^{2}\right)+E_{z} B_{z} \gamma^{2}\left(1-\beta^{2}\right)=\mathbf{E} \cdot \mathbf{B}
\end{aligned}
$$

Alternatively, we can also consider the determinant

$$
\left|F^{\mu^{\prime} \nu^{\prime}}\right|=\left|\Lambda_{\mu}^{\mu^{\prime}} \Lambda_{\nu}^{\nu^{\prime}} F^{\mu \nu}\right|=\left|\Lambda_{\mu}^{\mu^{\prime}}\right|\left|\Lambda_{\nu}^{\nu^{\prime}}\right|\left|F^{\mu \nu}\right|
$$

Since $\left|\Lambda_{\mu}^{\mu^{\prime}}\right|=\left|\Lambda_{\nu}^{\nu^{\prime}}\right|=\left(1-\beta^{2}\right) \gamma^{2}=1$, we have $\left|F^{\mu^{\prime} \nu^{\prime}}\right|=\left|F^{\mu \nu}\right|$. Therefore, the determinant of $F^{\mu \nu}$ is Lorentz invariant. One can then show that

$$
\operatorname{det}\left(F^{\mu \nu}\right)=\frac{1}{c^{2}}(\mathbf{E} \cdot \mathbf{B})^{2}
$$

i.e. $\mathbf{E} \cdot \mathbf{B}$ is Lorentz invariant.

As we can see, this is tedious work. We had to somehow know to calculate $\mathbf{E} \cdot \mathbf{B}$ at the first place, then hope that things work out as above, or grind through the determinant of a $4 \times 4$ matrix. A more useful way would be to find invariants (scalars) directly from the electromagnetic tensor, which we shall demonstrate below.

Theorem 2. $B^{2}-E^{2} / c^{2}$ is frame invariant.

Proof. We simply use the fact that tensor products with all indices contracted (i.e. scalars) are frame invariant. In particular,

$$
F^{\mu \nu} F_{\mu \nu}=2\left(B_{x}^{2}+B_{y}^{2}+B_{z}^{2}\right)-2\left(\frac{E_{x}^{2}}{c^{2}}+\frac{E_{y}^{2}}{c^{2}}+\frac{E_{z}^{2}}{c^{2}}\right)=2\left(B^{2}-\frac{E^{2}}{c^{2}}\right)
$$

### 2.4.1 Aside: The Dual Electromagnetic Tensor

The same idea can be used to prove the invariance of $\mathbf{E} \cdot \mathbf{B}$, but requires the introduction of the dual electromagnetic tensor ${ }^{233}$, which is defined as $\boxed{2}^{24}$

## Aside: The dual electromagnetic tensor $\tilde{F}^{\mu \nu}$

$$
\tilde{F}^{\mu \nu}:=\frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}=\left(\begin{array}{cccc}
0 & -B_{x} & -B_{y} & -B_{z}  \tag{2.34}\\
B_{x} & 0 & E_{z} / c & -E_{y} / c \\
B_{y} & -E_{z} / c & 0 & E_{x} / c \\
B_{z} & E_{y} / c & -E_{x} / c & 0
\end{array}\right)
$$

Note that $\tilde{F}^{\mu \nu}$ arises from $F^{\mu \nu}$ by the substitution $\mathbf{E} \mapsto c \mathbf{B}$ and $\mathbf{B} \mapsto-\mathbf{E} / c$. Furthermore, the fact that $\tilde{F}^{\mu \nu}$ is a tensor (and not just a matrix) means that it also transforms nicely under Lorentz Transformations, with

$$
\tilde{F}^{\mu \nu}=\Lambda_{\rho}^{\mu} \Lambda^{\nu}{ }_{\sigma} \tilde{F}^{\rho \sigma}
$$

Taking the obvious square of $\tilde{F}$ give $2^{25}$

$$
\begin{aligned}
\tilde{F}^{\mu \nu} \tilde{F}_{\mu \nu} & =\frac{1}{4} \epsilon^{\mu \nu \rho \sigma} \epsilon_{\mu \nu \alpha \beta} F^{\alpha \beta} F_{\rho \sigma} \\
& =\frac{1}{4}\left(-2!\delta^{\rho \sigma}{ }_{\alpha \beta}\right) F^{\alpha \beta} F_{\rho \sigma} \\
& =-\frac{1}{2}\left(\delta_{\alpha}^{\rho} \delta_{\beta}^{\sigma}-\delta_{\beta}^{\rho} \delta_{\alpha}^{\sigma}\right) F^{\alpha \beta} F_{\rho \sigma} \\
& =-\frac{1}{2}\left(F^{\alpha \beta} F_{\alpha \beta}-F^{\alpha \beta} F_{\beta \alpha}\right) \\
& =-F^{\alpha \beta} F_{\alpha \beta}
\end{aligned}
$$

So, we get nothing new. Of course, the next most natural thing to $\mathrm{d}^{26}{ }^{26}$ is to contract $\tilde{F}$ with our original $F$, giving

$$
\tilde{F}^{\mu \nu} F_{\mu \nu}=-\frac{4}{c}(\mathbf{E} \cdot \mathbf{B})
$$

and voilá! We have shown for the third time that $\mathbf{E} \cdot \mathbf{B}$ is Lorentz invariant, this time with a much more elegant and powerful tool, i.e. by contracting tensors into scalars.

[^12]${ }^{26}$ If we look closer, the fact that $\tilde{F}^{\mu \nu}$ has the word dual in its name does suggest us to perform this operation. For the mathematicians among us physics peasants, see this Wikipedia article on how tensor contraction actually arises from the natural pairing of a vector space and its dual.

### 2.5 Four-gradients

For a general scalar field $\psi(c t, x, y, z)$, its differential is given by

$$
d \psi=\frac{\partial \psi}{\partial(c t)} d(c t)+\frac{\partial \psi}{\partial x} d x+\frac{\partial \psi}{\partial y} d y+\frac{\partial \psi}{\partial z} d z=\sum_{\mu=0}^{4} \frac{\partial \psi}{\partial X^{\mu}} d X^{\mu}
$$

With $d \psi$ on the LHS being a scalar (since it has no indices), we must have $\frac{\partial \psi}{\partial X^{\mu}}$ as a covariant 4 -vector such that all indices are contracted with $d X^{\mu}$. In particular, we define the covariant 4 -derivative by

## Covariant 4-gradient

$$
\begin{equation*}
\partial_{\mu}:=\left(\frac{\partial}{\partial(c t)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)=\left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla\right) \tag{2.35}
\end{equation*}
$$

As usual, we can then raise the indices to get the contravariant 4 -derivative
Contravariant 4-gradient

$$
\begin{equation*}
\partial^{\mu}=\eta^{\mu \nu} \partial_{\nu}=\left(\frac{1}{c} \frac{\partial}{\partial t},-\nabla\right) \tag{2.36}
\end{equation*}
$$

Notice here that the contravariant 4 -gradient has negative spatial components, as opposed to its covariant counterpart (as is the case for other 4 -vectors, e.g. $X_{\mu}=(c t,-\mathbf{r})$ ).
Furthermore, the d'Alembertian operator is now automatically frame invariant, since

$$
\partial_{\mu} \partial^{\mu}=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}-\nabla^{2} \equiv \square
$$

With these 4 -gradients, we can simplify much of the equations we have seen earlier using 4vectors. For example, we have

## Euler's (charge continuity) equation

$$
\begin{equation*}
\partial_{\mu} j^{\mu}=\frac{\partial \rho}{\partial t}+\nabla \cdot \mathbf{J}=0 \tag{2.37}
\end{equation*}
$$

Consider now the inhomogeneous Maxwell's equations. We can rewrite (M1) as

$$
\begin{array}{r}
\nabla \cdot\left(\frac{\mathbf{E}}{c}\right)=\frac{\rho / \epsilon_{0}}{c} \\
\frac{\partial F^{00}}{\partial X^{0}}+\frac{\partial F^{10}}{\partial X^{1}}+\frac{\partial F^{20}}{\partial X^{2}}+\frac{\partial F^{30}}{\partial X^{3}}=\mu_{0}(\rho c) \\
\partial_{\mu} F^{\mu 0}=\mu_{0} j^{0}
\end{array}
$$

and looking at M4):

$$
\nabla \times \mathbf{B}-\frac{1}{c^{2}} \frac{\partial E}{\partial t}=\mu_{0} \mathbf{J}
$$

we can rewrite the $x$-, $y$-, and $z$-components of the equation as

$$
\begin{aligned}
& \frac{\partial B_{z}}{\partial y}-\frac{\partial B_{y}}{\partial z}+\frac{\partial\left(E_{x} / c\right)}{\partial t}=\frac{\partial F^{21}}{\partial X^{2}}+\frac{\partial F^{31}}{\partial X^{3}}+\frac{\partial F^{01}}{\partial X^{0}}=\mu_{0} j^{1} \\
& \frac{\partial B_{x}}{\partial z}-\frac{\partial B_{z}}{\partial x}+\frac{\partial\left(E_{y} / c\right)}{\partial t}=\frac{\partial F^{32}}{\partial X^{3}}+\frac{\partial F^{12}}{\partial X^{1}}+\frac{\partial F^{02}}{\partial X^{0}}=\mu_{0} j^{2} \\
& \frac{\partial B_{y}}{\partial x}-\frac{\partial B_{x}}{\partial y}+\frac{\partial\left(E_{x} / c\right)}{\partial t}=\frac{\partial F^{13}}{\partial X^{1}}-\frac{\partial F^{23}}{\partial X^{2}}+\frac{\partial F^{03}}{\partial X^{0}}=\mu_{0} j^{3}
\end{aligned}
$$

Combining everyting, we get
Inhomogeneous Maxwell's equation

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}=\mu_{0} j^{\nu} \tag{2.38}
\end{equation*}
$$

Similarly for the homogeneous Maxwell equations, from (M2), we have

$$
0=-\nabla \cdot \mathbf{B}=\frac{\partial\left(-B_{x}\right)}{\partial x}+\frac{\partial\left(-B_{y}\right)}{\partial y}+\frac{\partial\left(-B_{z}\right)}{\partial z}=\partial_{1} F_{23}+\partial_{2} F_{31}+\partial_{3} F_{12}
$$

and from (M3):

$$
\frac{1}{c}\left(\nabla \times \mathbf{E}+\frac{\partial \mathbf{B}}{\partial t}\right)=0
$$

we can again rewrite the $x$-, $y$-, and $z$-components separately as

$$
\begin{aligned}
& \frac{\partial\left(E_{z} / c\right)}{\partial y}-\frac{\partial\left(E_{y} / c\right)}{\partial z}+\frac{\partial B_{x}}{\partial(c t)}=\partial_{2} F_{03}+\partial_{3} F_{20}+\partial_{0} F_{32}=0 \\
& \frac{\partial\left(E_{x} / c\right)}{\partial z}-\frac{\partial\left(E_{z} / c\right)}{\partial x}+\frac{\partial B_{y}}{\partial(c t)}=\partial_{3} F_{01}+\partial_{1} F_{30}+\partial_{0} F_{31}=0 \\
& \frac{\partial\left(E_{y} / c\right)}{\partial x}-\frac{\partial\left(E_{x} / c\right)}{\partial y}+\frac{\partial B_{z}}{\partial(c t)}=\partial_{1} F_{02}+\partial_{2} F_{10}+\partial_{0} F_{21}=0
\end{aligned}
$$

Summarizing these, we have the so-called Bianchi Identity

## Bianchi Identity

$$
\begin{equation*}
\partial_{\alpha} F_{\beta \gamma}+\partial_{\beta} F_{\gamma \alpha}+\partial_{\gamma} F_{\alpha \beta}=0 \tag{2.39}
\end{equation*}
$$

or equivalently as

$$
\partial_{[\alpha} F_{\beta \gamma]}=0
$$

where [...] stands for anti-symmetrization of any groups of indicies, i.e. for a rank- $n$ tensor $T$,

$$
T_{[12 \cdots n]}:=\frac{1}{n!} \epsilon^{i_{1} i_{2} \cdots i_{n}} T_{i_{1} i_{2} \cdots i_{n}}
$$

In our case of three indices, it reads

$$
\partial_{[\alpha} F_{\beta \gamma]} \equiv \frac{1}{3!} \epsilon^{\alpha \beta \gamma} \partial_{\alpha} F_{\beta \gamma}=\frac{1}{6}\left(\partial_{\alpha} F_{\beta \gamma}+\partial_{\beta} F_{\gamma \alpha}+\partial_{\gamma} F_{\alpha \beta}-\partial_{\alpha} F_{\gamma \beta}-\partial_{\beta} F_{\alpha \gamma}-\partial_{\gamma} F_{\beta \alpha}\right)
$$

and we recover 2.39) using the fact that $F$ is antisymmetric, i.e. $F_{\mu \nu}=-F_{\nu \mu}$.
In fact, we can define the Faraday tensor alternatively as follow $\overbrace{}^{[27}$
Faraday tensor in terms of the 4-potential

$$
\begin{equation*}
F_{\mu \nu}:=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} \tag{2.40}
\end{equation*}
$$

where $A_{\mu}$ is of course the usual (covariant) 4-potential $A_{\mu}=(\phi / c,-\mathbf{A})$. Notice that $F_{\mu \nu}$ (and therefore $F^{\mu \nu}$ ) is then antisymmetric by definition.

[^13]One can easily check that this definition of $F_{\mu \nu}$ is indeed consistent with our previous matrix definitions via direct calculation. For example,

$$
F_{12}=\frac{\partial A_{2}}{\partial X^{1}}-\frac{\partial A_{1}}{\partial X^{2}}=\frac{\partial\left(-A_{y}\right)}{\partial x}-\frac{\partial\left(-A_{x}\right)}{\partial y}=-[\nabla \times \mathbf{A}]_{z}=-B_{z}
$$

Bianchi's identity (and hence the homogeneous Maxwell Equations) now follows automatically:

$$
\begin{aligned}
& \partial_{\alpha} F_{\beta \gamma}+\partial_{\beta} F_{\gamma \alpha}+\partial_{\gamma} F_{\alpha \beta} \\
& =\left(\partial_{\alpha} \partial_{\beta} A_{\gamma}-\partial_{\alpha} \partial_{\gamma} A_{\beta}\right)+\left(\partial_{\beta} \partial_{\gamma} A_{\alpha}-\partial_{\beta} \partial_{\alpha} A_{\gamma}\right)+\left(\partial_{\gamma} \partial_{\alpha} A_{\beta}-\partial_{\gamma} \partial_{\beta} A_{\alpha}\right) \\
& =\partial_{\alpha} \partial_{\beta} A_{\gamma}-\partial_{\alpha} \partial_{\gamma} A_{\beta}+\partial_{\beta} \partial_{\gamma} A_{\alpha}-\partial_{\alpha} \partial_{\beta} A_{\gamma}+\partial_{\alpha} \partial_{\gamma} A_{\beta}-\partial_{\beta} \partial_{\gamma} A_{\alpha} \\
& =0
\end{aligned}
$$

So far, we have been rewriting Maxwell's equations using the 4 -gradient and the Faraday tensor, which are all gauge-independent. In particular, substituting our new definition (2.40) into the inhomogeneous Maxwell's equation (2.38), we get

## Gauge independent Maxwell's equations

$$
\begin{equation*}
\mu_{0} j^{\nu}=\partial_{\mu}\left(\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right)=\square A^{\nu}-\partial^{\nu}\left(\partial_{\mu} A^{\mu}\right) \tag{2.41}
\end{equation*}
$$

However, notice that if we choose the Lorenz gauge, i.e.

$$
\partial_{\mu} A^{\mu} \equiv \frac{1}{c} \frac{\partial \phi}{\partial t}+\nabla \cdot \mathbf{A}=0
$$

Maxwell's equations can be elegantly summarized as

## Maxwell's equations in the Lorenz gauge

$$
\begin{equation*}
\square A^{\mu}=\mu_{0} j^{\mu} \tag{2.42}
\end{equation*}
$$

Noice!

## 3 Narnia $^{\text {TM }}$ : The Dipole, the Antenna \& the Retarded Potential

## "All shall be done, but it may be harder than you think."

- C.S. Lewis, Narnia: The Lion, the Witch and the Wardrobe;
- also me on How to Pass the PX3A3 Exam.


### 3.1 Moving Charges and the Retarded Potential

We first consider the electrostatic setting of a point charge $q$ located at $\mathbf{r}_{0}$. Recall from PX120 that the charge generates an electric field given by

$$
\mathbf{E}(\mathbf{r})=\frac{q}{4 \pi \epsilon_{0}\left|\mathbf{r}-\mathbf{r}_{0}\right|^{2}} \frac{\mathbf{r}-\mathbf{r}_{0}}{\left|\mathbf{r}-\mathbf{r}_{0}\right|}
$$

or equivalently, a scalar electrostatic potential

$$
V(\mathbf{r})=\frac{q}{4 \pi \epsilon_{0}\left|\mathbf{r}-\mathbf{r}_{0}\right|}
$$

For multiple charges, we simply sum over all constituents, giving

$$
V(\mathbf{r})=\sum_{i} \frac{q_{i}}{4 \pi \epsilon_{0}\left|\mathbf{r}-\mathbf{r}_{i}\right|}
$$

We can also integrate over a continuous region of charges with

$$
V(\mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \int d \mathbf{r}^{\prime} \frac{\rho\left(\mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}
$$

where $\rho\left(\mathbf{r}^{\prime}\right)$ is the charge density at $\mathbf{r}^{\prime}$.
Now, this is all well and good, but what if the charges are moving instead? We might reasonably expect $V$ to have the form

$$
V(t, \mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \int d \mathbf{r}^{\prime} \frac{\rho\left(t, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}
$$

i.e. the potential now depends on time due to the time-dependence of $\rho$.

But what should the time $t$ here be in this case? Naturally, we want $t$ to be the time at which we, the observer makes a measurement in our own frame of reference. However, since radiation ${ }^{28}$ from the electric field travels at a finite speed $c$, the speed of light (in vacuum), the charges we measure were actually generated by the source at time $t^{\prime}=t-\left|\mathbf{r}-\mathbf{r}^{\prime}\right| / c$.
We define this as the retarted time

## Retarted time

$$
\begin{equation*}
t_{R}:=t-\frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{c} \tag{3.43}
\end{equation*}
$$

In particular, $\rho$ depends on $t_{R}$, the retarded time, NOT $t$, the time of measurement. That is, for moving charges, the generated electric potential is given by

$$
\begin{equation*}
V(t, \mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \int d \mathbf{r}^{\prime} \frac{\rho\left(t_{R}, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \tag{3.44}
\end{equation*}
$$

[^14]In fact, solving Maxwell's equations $\sqrt{29}$ in the Lorenz gauge $\square A^{\mu}=\mu_{0} j^{\mu}$, or separately as

$$
\square \phi=\frac{\rho}{\epsilon_{0}} \quad \text { and } \quad \square \mathbf{A}=\mu_{0} \mathbf{J}
$$

we get the very similar looking retarded potentials

$$
\begin{align*}
\phi(t, \mathbf{r}) & =\frac{1}{4 \pi \epsilon_{0}} \int d \mathbf{r}^{\prime} \frac{\rho\left(t_{R}, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}  \tag{3.45}\\
\mathbf{A}(t, \mathbf{r}) & =\frac{\mu_{0}}{4 \pi} \int d \mathbf{r}^{\prime} \frac{\mathbf{J}\left(t_{R}, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \tag{3.46}
\end{align*}
$$

which combines into a 4 -potential as
Retarted potential

$$
\begin{equation*}
A^{\mu}=\frac{\mu_{0}}{4 \pi} \int d \mathbf{r}^{\prime} \frac{\left[j^{\mu}\right]}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \tag{3.47}
\end{equation*}
$$

where the square brackets mean that we evaluate $j^{\mu}$ using $\mathbf{r}^{\prime}$ and the retarded time $t_{R}$, i.e.

$$
\left[j^{\mu}\right]=j^{\mu}\left(c t_{R}, \mathbf{r}^{\prime}\right)
$$

instead of the regular 4-position $X^{\mu}=(c t, \mathbf{r})$ (as is the case for $A^{\mu}$ ). Intuitively, this potential should make sense if we compare it to the form of $V(t, \mathbf{r})$ as derived in (3.44).

### 3.2 4-potential from a Hertzian Dipole

As hinted above, one important consequence of the 4 -potential retardation is the emission of radiation from time-dependent charges, as we shall see below.

Consider a pair of opposite oscillating point charges $\pm q(t)$ situated with distance $b$ apart, or more precisely, with a displacement vector $\mathbf{b}=b \hat{\mathbf{z}}$, i.e. we align the dipole along the $+z$-axis. This is commonly set up in, say, radio antennae using some AC current $I(t)$.

The dipole moment $\mathbf{p}$ (with origin set at the midpoint of the two charges) is given by

$$
\mathbf{p}(t)=q(t) \mathbf{b}
$$

which varies in time as

$$
\dot{\mathbf{p}}=\dot{q} \mathbf{b}=I(t) \mathbf{b}
$$

Now, if we assume $|\mathbf{r}| \ggg>b$, where $\lambda$ is the wavelength of radiation, we then have $|\mathbf{r}| \gg$ $\left|\mathbf{r}^{\prime}\right|$, and can therefore approximate $\mathbf{r}^{\prime}$ and hence, $t_{R}$ to be uniform across the dipole. Dipoles satisfying this assumption are known as Hertzian dipoles (sometimes also short dipoles).

Therefore, the vector potential can be approximated as

$$
\mathbf{A}(t, \mathbf{r})=\frac{\mu_{0}}{4 \pi} \int d \mathbf{r}^{\prime} \frac{\mathbf{J}\left(t_{R}, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \approx \frac{\mu_{0}}{4 \pi r} \int d \mathbf{r}^{\prime}[\mathbf{J}]=\frac{\mu_{0} \hat{\mathbf{z}}}{4 \pi r} \int_{-b / 2}^{b / 2} d z[I]=\frac{\mu_{0}[I] b}{4 \pi r} \hat{\mathbf{z}}=\frac{\mu_{0}[\dot{p}]}{4 \pi r} \hat{\mathbf{z}}
$$

where $r:=|\mathbf{r}|$. This is called the electric dipole approximation.
As for the scalar potential, we have

$$
\phi(t, \mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \int d \mathbf{r}^{\prime} \frac{\rho\left(t_{R}, \mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}=\frac{1}{4 \pi \epsilon_{0}}\left[\frac{q\left(t_{R}, \mathbf{r}_{1}\right)}{\left|\mathbf{r}-\mathbf{r}_{1}\right|}+\frac{q\left(t_{R}, \mathbf{r}_{2}\right)}{\left|\mathbf{r}-\mathbf{r}_{2}\right|}\right]
$$

[^15]Labelling $R_{+}:=\left|\mathbf{r}-\mathbf{r}_{1}\right|$ and $R_{-}:=\left|\mathbf{r}-\mathbf{r}_{2}\right|$ as the distances from the respective charges to point $\mathbf{r}$, this becomes

$$
\phi(t, \mathbf{r})=\frac{1}{4 \pi \epsilon_{0}}\left[\frac{q\left(t-\frac{R_{+}}{c}\right)}{R_{+}}+\frac{-q\left(t-\frac{R_{-}}{c}\right)}{R_{-}}\right]
$$

If we denote the angle between $\mathbf{r}$ and $\mathbf{b}$ as $\theta$ (such that $\mathbf{r} \cdot \mathbf{b}=\cos \theta$ ), then

$$
R_{ \pm}=\left|\mathbf{r} \mp \frac{\mathbf{b}}{2}\right|=\left(r^{2} \mp \mathbf{r} \cdot \mathbf{b}+\frac{b^{2}}{4}\right)^{1 / 2}=r \pm \frac{\mathbf{r} \cdot \mathbf{b}}{2}+\mathcal{O}\left(b^{2}\right) \approx r \mp \frac{b}{2} \cos \theta
$$

where we assumed $b \ll r$ as before. Therefore,

$$
q\left(t-\frac{R_{ \pm}}{c}\right)=q\left(t-\frac{r}{c} \pm \frac{b \cos \theta}{2 c}\right) \approx q\left(t-\frac{r}{c}\right) \pm \frac{b \cos \theta}{2 c} \dot{q}\left(t-\frac{r}{c}\right)
$$

Hence, we can write

$$
\begin{aligned}
\phi(t, \mathbf{r}) & =\frac{1}{4 \pi \epsilon_{0}}\left(\frac{q+\frac{b \cos \theta}{2 c} \dot{q}}{r-\frac{b}{2} \cos \theta}-\frac{q-\frac{b \cos \theta}{2 c} \dot{q}}{r+\frac{b}{2} \cos \theta}\right)=\frac{1}{4 \pi \epsilon_{0}}\left(\frac{\frac{b \cos \theta}{c} q+\frac{r b \cos \theta}{c} \dot{q}}{r^{2}-\frac{b^{2}}{4} \cos ^{2} \theta}\right) \\
& \approx \frac{1}{4 \pi \epsilon_{0}}\left(\frac{\frac{b \cos \theta}{c} q+\frac{r b \cos \theta}{c} \dot{q}}{r^{2}}\right)=\frac{1}{4 \pi \epsilon_{0} r}\left(\frac{q b}{r}+\frac{\dot{q} b}{c}\right) \cos \theta=\frac{1}{4 \pi \epsilon_{0} r}\left(\frac{p}{r}+\frac{\dot{p}}{c}\right) \cos \theta
\end{aligned}
$$

Now, with the dipole oscillating with some frequency $\omega$, such that

$$
p=p_{0} e^{i \omega t_{R}}
$$

We have (only considering the real parts of $p$ and $\dot{p}$ )

$$
\dot{p}=\omega p \Longrightarrow \frac{\dot{p}}{c}=\frac{\omega}{c} p=\frac{2 \pi}{\lambda} p=2 \pi\left(\frac{r}{\lambda}\right) \cdot \frac{p}{r}
$$

With the far field approximation, i.e. $r \gg \lambda$, this implies

$$
\frac{\dot{p}}{r} \gg \frac{p}{r}
$$

Therefore,

$$
\phi(t, \mathbf{r})=\frac{[\dot{p}] \cos \theta}{4 \pi \epsilon_{0} r c} \Longrightarrow \frac{\phi(t, \mathbf{r})}{c}=\frac{\mu_{0}[\dot{p}]}{4 \pi r} \cos \theta
$$

Finally, combining everything into a 4-potential, we have

## 4-potential from a Hertzian dipole

$$
\begin{equation*}
A^{\mu}=\frac{\mu_{0}[\dot{p}]}{4 \pi r}(\cos \theta, \hat{\mathbf{z}}) \tag{3.48}
\end{equation*}
$$

where as before, $[\dot{p}]:=\dot{p}(t-r / c)$.
Remark 1. Note that since $[\dot{p}]=\omega[p] \propto e^{i \omega t_{R}}$, we have $A^{\mu} \propto e^{i \omega(t-r / c)} / r$, which geometrically are just outgoing spherical waves with a decaying amplitude $\propto 1 / r$.

Remark 2. As shown in the derivation, this 4-potential ONLY holds for Hertzian dipoles in the far field approximation, i.e. we require the assumption $r \gg \lambda \gg b$.

### 3.3 Radiation from a Hertzian Dipole

Now that we have the 4-potential, the corresponding EM fields can be found easily using (2.40).
In spherical coordinates, defining $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}=\hat{\mathbf{r}} \cdot \hat{\mathbf{b}}=\cos \theta$ as before, we have $\hat{\mathbf{z}}=(\cos \theta,-\sin \theta, 0)$, and the 4 -gradient becomes

$$
\partial^{\mu}=\left(\frac{1}{c} \frac{\partial}{\partial t},-\nabla\right)=\left(\frac{1}{c} \frac{\partial}{\partial t},-\frac{\partial}{\partial r},-\frac{1}{r} \frac{\partial}{\partial \theta},-\frac{1}{\sin \theta} \frac{\partial}{\partial \phi}\right)
$$

Therefore, noting that $[\ddot{p}]=\omega[\dot{p}]$ and $\partial[\dot{p}] / \partial r=-\omega[\dot{p}] / c$, we have

$$
\begin{aligned}
& \partial^{0} A^{\nu}=\frac{1}{c} \frac{\mu_{0} \omega[\dot{p}]}{4 \pi r}(\cos \theta, \cos \theta,-\sin \theta, 0) \\
& \partial^{1} A^{\nu}=\frac{\mu_{0}}{4 \pi}\left(\frac{[\dot{p}]}{r^{2}}+\frac{\omega[\dot{p}]}{c r}\right)(\cos \theta, \cos \theta,-\sin \theta, 0) \\
& \partial^{2} A^{\mu}=\frac{\mu_{0}[\dot{p}]}{4 \pi r^{2}}(\sin \theta, \sin \theta, \cos \theta, 0) \\
& \partial^{3} A^{\mu}=0
\end{aligned}
$$

or equivalently,

$$
\partial^{\mu} A^{\nu}=\frac{\mu_{0}[\dot{p}]}{4 \pi r}\left[\frac{\omega}{c}\left(\begin{array}{cccc}
\cos \theta & \cos \theta & -\sin \theta & 0 \\
\cos \theta & \cos \theta & -\sin \theta & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)+\frac{1}{r}\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\cos \theta & \cos \theta & -\sin \theta & 0 \\
\sin \theta & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 0
\end{array}\right)\right]
$$

Now, since $\lambda \ll r$, we have

$$
\frac{\omega}{c}=\frac{2 \pi}{\lambda} \gg \frac{1}{r}
$$

so the first matrix dominates in the far field approximation. Hence,

$$
\begin{aligned}
F^{\mu \nu} & =\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu} \\
& =\frac{\mu_{0} \omega[\dot{p}]}{4 \pi r c}\left[\left(\begin{array}{cccc}
\cos \theta & \cos \theta & -\sin \theta & 0 \\
\cos \theta & \cos \theta & -\sin \theta & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)-\left(\begin{array}{cccc}
\cos \theta & \cos \theta & 0 & 0 \\
\cos \theta & \cos \theta & 0 & 0 \\
-\sin \theta & -\sin \theta & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)\right] \\
& =\frac{\mu_{0}[\ddot{p}]}{4 \pi r c}\left(\begin{array}{cccc}
0 & 0 & -\sin \theta & 0 \\
0 & 0 & -\sin \theta & 0 \\
\sin \theta & \sin \theta & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
\end{aligned}
$$

Indentifying this matrix with

$$
\left(\begin{array}{cccc}
0 & -E_{r} / c & -E_{\theta} / c & -E_{\phi} / c \\
E_{r} / c & 0 & B_{\phi} & B_{\theta} \\
E_{\theta} / c & B_{\phi} & 0 & -B_{r} \\
E_{\phi} / c & -B_{\theta} & B_{r} & 0
\end{array}\right) \equiv F^{\mu \nu}=\frac{\mu_{0}[\ddot{p}]}{4 \pi r c}\left(\begin{array}{cccc}
0 & 0 & -\sin \theta & 0 \\
0 & 0 & -\sin \theta & 0 \\
\sin \theta & \sin \theta & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

we see that in spherical coordinates, a Hertzian dipole generates the following EM field $\$^{30}$.

[^16]
(a) 3D distribution of dipole radiation

(b) Radiation pattern in the $y-z$ plane

Figure 1: Radiation pattern of a Hertzian dipole ${ }^{31}$

## Radiation from a Hertzian dipole

$$
\begin{gather*}
E_{\theta}=\frac{\mu_{0}[\ddot{p}]}{4 \pi r} \sin \theta  \tag{3.49}\\
B_{\phi}=\frac{\mu_{0}[\ddot{p}]}{4 \pi r c} \sin \theta  \tag{3.50}\\
E_{r}=E_{\phi}=0  \tag{3.51}\\
B_{r}=B_{\theta}=0 \tag{3.52}
\end{gather*}
$$

Notice that the fields are mutually orthogonal with $E / B=c$, so we indeed have radiation as expected. Furthermore, recall that the Poynting vector for electromagnetic radiation is defined as

$$
\mathbf{S}:=\mathbf{E} \times \mathbf{H}=\mathbf{E} \times \frac{\mathbf{B}}{\mu_{0}}
$$

Therefore, we have that for radiation from a Hertzian dipole,

$$
\mathbf{S}=\frac{1}{\mu_{0}} \frac{\mu_{0}^{2}[\ddot{p}]^{2}}{(4 \pi r)^{2} c} \sin ^{2} \theta(\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}})=\frac{\mu_{0}[\ddot{p}]^{2}}{(4 \pi r)^{2} c} \sin ^{2} \theta \hat{\mathbf{r}}
$$

Observe the angle dependence of $\mathbf{S}$ : It has maximum magnitude when $\theta=\pi / 2$, and minimum magnitude when $\theta=0$ or $\pi$ (with $P_{\max } / 2$ at $\theta=\pi / 4$ ). In particular, since $\mathbf{S}$ represents the directional energy flux (aka power flow) of an electromagnetic field, the radiation is concentrated around the $(x-y)$ plane perpendicular to the dipole moment $\mathbf{p} \equiv p \hat{\mathbf{z}}$ (see Figure 1).

To quantify this angular dependence of radiated power, we define the directivity $D$ as

## Directivity

$$
\begin{equation*}
D:=\frac{\text { Maximum radiated power }}{\text { Average radiated power }} \tag{3.53}
\end{equation*}
$$

[^17]We can first compute the total power radiated by simply integrating over some spherical surface of radius $r$, i.e.

$$
\begin{aligned}
P_{\text {total }}(t) & =\int_{\text {sphere }} \mathbf{S} \cdot \mathbf{d A} \\
& =\int_{\phi=0}^{2 \pi} \int_{\theta=0}^{\pi} \frac{\mu_{0}[\ddot{p}]^{2}}{(4 \pi r)^{2} c} \sin ^{2} \theta \hat{\mathbf{r}} \cdot\left(r^{2} \sin \theta d \theta d \phi \hat{\mathbf{r}}\right) \\
& =\frac{\mu_{0}[\ddot{p}]^{2}}{(4 \pi)^{2} c} \int_{\phi=0}^{2 \pi} d \phi \int_{\theta=0}^{\pi} d \theta \sin ^{3} \theta \\
& =\frac{\mu_{0}[\ddot{p}]^{2}}{(4 \pi)^{2} c} \cdot 2 \pi \cdot\left[\frac{1}{3} \cos ^{3} \theta-\cos \theta\right]_{0}^{\pi} \\
& =\frac{\mu_{0}[\ddot{p}]^{2}}{6 \pi c}
\end{aligned}
$$

Then, the average power will be given by the total power divided by $4 \pi r^{2}$. In particular, we have the directivity

$$
D=\frac{P_{\max }}{P_{\text {avg }}}=\frac{\mu_{0}[\ddot{p}]^{2}}{(4 \pi r)^{2} c} /\left[\frac{1}{4 \pi r^{2}}\left(\frac{\mu_{0}[\ddot{p}]^{2}}{6 \pi c}\right)\right]=\frac{3}{2}
$$

It is often useful to also include the angular dependence of $\mathbf{S}$ explicitly in the so-called angular gain. In this case, the angular gain is

$$
G(\theta, \phi)=\frac{3}{2} \sin ^{2} \theta
$$

Furthermore, assuming $\mathbf{p}$ to be sinusoidal (e.g. if it is of the form $p=p_{0} e^{i \omega t}$ as before), then

$$
\langle\ddot{p}\rangle=\omega^{2}\langle p\rangle=\omega^{2} p_{0} / \sqrt{2}
$$

Hence, the total time-averaged power is
Total time-averaged power from dipole radiation

$$
\begin{equation*}
\left\langle P_{\text {total }}\right\rangle=\frac{\mu_{0}\left\langle[\ddot{p}]^{2}\right\rangle}{6 \pi c}=\frac{\mu_{0} \omega^{4} p_{0}^{2}}{12 \pi c} \propto \frac{1}{\lambda^{4}} \tag{3.54}
\end{equation*}
$$

The key takeaway here is that $\left\langle P_{\text {total }}\right\rangle \propto \omega^{4} \propto \lambda^{-4}$. In particular, the radiation is skewed heavily towards shorter wavelengths. This leads to the infamous Rayleigh scattering!

### 3.4 Rayleigh Scattering: Blue Skies and Red Sunsets

Due to incoming solar radiation, diatomic molecules in the atmosphere (e.g. $N_{2}, O_{2}$ ) experience an oscillatory electric field, which induces molecular dipoles of the form

$$
\mathbf{p}(t)=\alpha \mathbf{E}(t)
$$

for some constant $\alpha \in \mathbb{R}$. From above, we know that the molecules in turn give out radiation with a time-averaged power $\langle P\rangle \propto \lambda^{-4}$. That is, incoming sunlight of all wavelengths are scattered, but shorter wavelengths are scattered more strongly, hence Mr. Blue Sky $\left.{ }^{\top \pi} \sqrt{32} \sqrt{33}\right]^{4}$

[^18]

Figure 2: Rayleigh scattering in opalescent glass ${ }^{36}$

During sunsets, due to the oblique angle, we see the sun through a much larger and denser proportion of the atmosphere near the Earth's surface, in which Rayleigh scattering removes a significant proportion of the shorter wavelength (blue/green) light from the direct path to the observer. The remaining unscattered light is therefore mostly of longer wavelengths, i.e. we get red sunsets. Note that this is also why we can safely look at sunsets directly in the first place - the sun's intense radiation is much attenuated by scattering along the way.

Aside: Interestingly, Rayleigh-type $\propto \lambda^{-4}$ scattering can also be demonstrated using nanoporous material 35 , e.g. scattered light in a piece of opalescent glass makes the glass appear blue from the side, while longer-wavelength orange light shines through (see Figure 2).

### 3.5 The Short Dipole Antenna

As mentioned before, electric dipoles are commonly found in antennae, and are easily set up by passing through an AC current $I(t)$ from the centre of the dipole to two nearby points (forming the two opposite "point charges").

More specifically, suppose we have a current given by

$$
I\left(t_{R}\right)=I_{0} \sin \left(\omega t_{R}\right)
$$

Then, with dipole moment $\mathbf{p}=q\left(t_{R}\right) b \hat{\mathbf{z}} \Longrightarrow \dot{\mathbf{p}}=I\left(t_{R}\right) b \hat{\mathbf{z}}$, we have

$$
\ddot{\mathbf{p}}\left(t_{R}\right)=\frac{d I}{d t_{R}} b \hat{\mathbf{z}}=\omega I_{0} b \cos \left(\omega t_{R}\right) \hat{\mathbf{z}}=2 \pi c\left(\frac{b}{\lambda}\right) I_{0} \cos \left(\omega t_{R}\right) \hat{\mathbf{z}}
$$

where we note that $b \ll \lambda \Longrightarrow b / \lambda \ll 1$ for Hertzian dipoles.
From above, we know that this antenna generates radiation with an average power of

$$
\left\langle P_{r a d}\right\rangle=\frac{\mu_{0}\left\langle[\ddot{p}]^{2}\right\rangle}{6 \pi c}=\mu_{0} \frac{4 \pi^{2} c^{2}}{6 \pi c}\left(\frac{b}{\lambda}\right)^{2}\left\langle I_{0}^{2} \cos ^{2}\left(\omega t_{R}\right)\right\rangle=\left[\frac{2 \pi}{3}\left(\frac{b}{\lambda}\right)^{2} Z_{0}\right] I_{r m s}^{2}
$$

where $I_{r m s}^{2} \equiv\left\langle I_{0}^{2} \sin ^{2}\left(\omega t_{R}\right)\right\rangle=\left\langle I_{0}^{2} \cos ^{2}\left(\omega t_{R}\right)\right\rangle$, and $Z_{0}=\mu_{0} c=\sqrt{\mu_{0} / \epsilon_{0}} \approx 377 \Omega$ is the impedance of free space.

[^19]Relating this to the usual power of an electrical circuit $P=I^{2} R$, we define a new quantity known as the radiation resistance, given by

$$
R_{r a d}:=\frac{\left\langle P_{r a d}\right\rangle}{\left\langle I^{2}\right\rangle}
$$

This is a so-called effective resistance. Unlike conventional (Ohmic) resistance, radiation resistance is not due to the resistivity of the imperfect conducting materials the antenna is made of, but rather due to the power carried from the antenna as radiation (usually as radio waves).

For dipole antennae, we have

## Radiation resistance of a dipole antenna

$$
\begin{equation*}
R_{\text {rad }}=\frac{\left\langle P_{\text {rad }}\right\rangle}{\left\langle I^{2}\right\rangle}=\frac{2 \pi}{3}\left(\frac{b}{\lambda}\right)^{2} Z_{0} \tag{3.55}
\end{equation*}
$$

We can also speak of an antenna's radiation efficiency

$$
\eta:=\frac{P_{\text {rad }}}{P_{\text {in }}}=\frac{R_{\text {rad }}}{R_{\text {rad }}+R_{\mathrm{Ohm}}}
$$

where $R_{\text {Ohm }}$ is the Ohmic resistance of the antenna. So, for an efficient (high $\eta$ ) antenna, we want $R_{\text {rad }} \gg R_{\text {Ohm }}$.

## EXAMPLE 2.

For a short dipole with $b=1 \mathrm{~cm}, \lambda=1 \mathrm{~m}(\sim 300 \mathrm{MHz})$, we have

$$
R_{\text {rad }}=\frac{2 \pi}{3}\left(\frac{1}{100}\right)^{2}(377) \approx 80 \mathrm{~m} \Omega
$$

To radiate 1 W of power, we require a driving current of

$$
I=\frac{P}{R_{r a d}^{2}}=\frac{1}{0.08^{2}} \approx 156 \mathrm{~A}
$$

Notice here how the low $R_{\text {rad }}$ of Hertzian dipoles requires antennae to draw impractically high currents. In practice, we use longer antennae to increase $R_{\text {rad }}$, though this does mean that our original $b \ll \lambda$ assumption will no longer hold.

### 3.6 A Longer Antenna: The Half-Wave Dipole Antenna/Aerial

Suppose now we have a longer dipole, i.e. we remove the assumption that $b \ll \lambda$ (while keeping the far field approximation $r \gg \lambda$ ). In this case, the retarded time $t_{R}$ is no longer uniform across the dipole, but rather given by

$$
t_{R}(z)=t-\frac{r-z \cos \theta}{c}
$$

where $z$ is the relative height of a point on the antenna from the centre of the dipole. Furthermore, by requiring that $I \rightarrow 0$ at $z= \pm b / 2$ (due to the physical configuration of the wire), the current varies along the antenna (along $\hat{\mathbf{z}}$ ) as

$$
I\left(t_{R}, z\right)=I_{0} \cos \left(\frac{\pi z}{b}\right) e^{i \omega t_{R}}=I_{0} \cos \left(\frac{\pi z}{b}\right) \exp \left[i \omega\left(t-\frac{r-z \cos \theta}{c}\right)\right]
$$

for some maximum current $I_{0}$.


Figure 3: Radiation pattern of a half-wave dipole (solid line) vs a Hertzian dipole (dashed line) ${ }^{37}$

The retarded potential (with $r \gg b$ ) is then given by

$$
\begin{aligned}
A^{\mu}(t, r) & =\frac{\mu_{0}}{4 \pi} \int_{-b / 2}^{+b / 2} d z \frac{I\left(t_{R}, z\right)}{|r-z \cos \theta|} \\
& =\frac{\mu_{0} I_{0}}{4 \pi r} \int_{-b / 2}^{+b / 2} d z \cos \left(\frac{\pi z}{b}\right) \exp \left[i \omega\left(t-\frac{r-z \cos \theta}{c}\right)\right] \\
& =\frac{\mu_{0} I_{0}}{4 \pi r} \exp \left[i \omega\left(t-\frac{r}{c}\right)\right] F(\theta)
\end{aligned}
$$

where

$$
F(\theta):=\int_{-b / 2}^{+b / 2} d z \cos \left(\frac{\pi z}{b}\right) \exp \left(i \omega \frac{z \cos \theta}{c}\right)=\frac{\cos \left(\frac{\pi b}{\lambda} \cos \theta\right)-\cos \left(\frac{\pi b}{\lambda}\right)}{\sin ^{2} \theta}
$$

For the half-wave dipole (HWD), we choose $b=\lambda / 2$ such that

$$
F(\theta)=\frac{\cos \left(\frac{\pi}{2} \cos \theta\right)}{\sin ^{2} \theta}
$$

Substituting this gives a radiation field of

$$
E_{\theta}=c A \sin \theta=c \frac{\mu_{0} I_{0}}{4 \pi r} \frac{\cos \left(\frac{\pi}{2} \cos \theta\right)}{\sin ^{2} \theta} \exp \left[i \omega\left(t-\frac{r}{c}\right)\right]
$$

Hence, we get a radiated power of

$$
P \propto E^{2} \propto\left[\frac{\cos \left(\frac{\pi}{2} \cos \theta\right)}{\sin ^{2} \theta}\right]
$$

We can also compute the directivity of a half-wave dipole via

$$
D_{\mathrm{HWD}}=\frac{P_{\max }}{P_{\text {avg }}}=\frac{4 \pi}{\iint d \theta d \phi \frac{\cos \left(\frac{\pi}{2} \cos \theta\right)}{\sin ^{2} \theta} \sin \theta} \approx 1.66>1.5=D_{\text {Hertzian }}
$$

Plotting out the radiation pattern of both the HWD and the Hertzian dipole (see Figure 3) confirms that radiation from the HWD is indeed more focused radially (along the $x-y$ plane).

[^20]
[^0]:    ${ }^{1}$ For example, I often denote four-vectors with capital letters instead of lowercase to make things clearer (the latter of which was done in PX3A3 lectures), I highly encourage you to use the lecturer's conventions when taking the exam (in other words, please do not blame me for any marks lost). I also added plenty of footnotes and references to provide interesting physical and mathematical insights to topics which were glossed over in lectures.
    ${ }^{2}$ Rabbit hole for my fellow Physicists: http://www.damtp.cam.ac.uk/user/tong/teaching.html

[^1]:    ${ }^{3}$ In fact, for any "physical" divergence-free vector fields (smooth in $\mathbb{R}^{3}$ with compact support, in this case a magnetic field B), Helmholtz's theorem guarantees that one can always define such a vector potential $\mathbf{A}$.
    ${ }^{4}$ The existence of $\phi$ is again guaranteed by Helmholtz's theorem.

[^2]:    ${ }^{5}$ NOT Lorentz, typo in Chapman's Core Electrodynamics!
    ${ }^{6}$ In electrostatics this reduces to the Coulomb gauge $\nabla \cdot \mathbf{A}=0$ (useful for plane waves), which gives Poisson equations for both $\psi$ and $\mathbf{A}$ :

    $$
    -\nabla^{2} \phi=\frac{\rho}{\epsilon_{0}} \quad \text { and } \quad-\nabla^{2} \mathbf{A}=\mu_{0} \mathbf{J}
    $$

[^3]:    ${ }^{7}$ Einstein's PhD supervisor!
    ${ }^{8}$ Topologically $\mathbb{R}^{4}$, endowed with the Minkowski inner product $v \cdot w=\eta(v, w):=\eta_{\mu \nu} v^{\mu} w^{\nu}$.

[^4]:    ${ }^{9} \alpha$ here is known as the rapidity.
    ${ }^{10}$ In fact, just like regular rotations (which form the special orthogonal groups $S O(n)$ ), the translations, rotations and Lorentz boosts altogether form what's known as the Poincaré group.
    ${ }^{11}$ Mathematical note for those who took MA251: More preciesly, the Minkowski inner product is a pseudoinner product, which one does not require to be positive definite, i.e. we allow $\eta(u, v)<0$. Furthermore, the product satisfies linearity in first argument, symmetry, and non-degeneracy (i.e. $\eta(u, v)=0 \forall v \in M \Longrightarrow u=0$ ); note that the first two properties automatically imply bilinearity, hence the "inner product" is well-defined via this (non-degenerate) symmetric bilinear form.

[^5]:    ${ }^{12} \mathrm{Ngl}$ this guy just has the best lecture notes on the internet (plus they are completely free!!), could've probably studied all of undergraduate Physics from his website alone if I wasn't this lazy.
    ${ }^{13}$ Not everything with indices are tensors! Tensors have to transform properly under a change of coordinates. Even in GR, we deal with Christoffel symbols (more about them in PX436 or MA3D9), which usually look something like $\Gamma^{i}{ }_{j k}$, but they do not transform like tensors under a change of coordinates, hence are not tensors.
    Aside: More precisely, in differential geometry, given a smooth manifold, one may choose a coordinate basis for the tangent vector space at a point, and consider a tensor with components that are functions of points on the manifold. Then we can speak of the transformation law of tensors, which can be expressed explicitly in terms of partial derivatives of the chosen coordinate functions. In our case, the $\Lambda$ 's of LT are exactly the said laws:

    $$
    \Lambda_{\nu}^{\mu} \equiv \frac{\partial x^{\mu}}{\partial x^{\prime \nu}} \Longrightarrow a^{\mu}=\frac{\partial x^{\mu}}{\partial x^{\prime \nu}} a^{\nu}=\Lambda_{\nu}^{\mu} a^{\nu}
    $$

    More about this in MA3H5 Manifolds (one of the most difficult third year modules imo) and PX408 Relativistic Quantum Mechanics (only briefly mentioned here without much deeper explanation).
    ${ }^{14}$ Potential confusion: We use the $(+---)$ convention in this module (and for particle physics in general). However, in other fields (e.g. GR), the ( -+++ ) convention may be used instead, where $d s^{2} \equiv-(c t)^{2}+x^{2}+y^{2}+z^{2}$.

[^6]:    ${ }^{15}$ Credit to Crentist_the-Dentist from Reddit r/Jokes.
    ${ }^{16}$ This is always true for a physical particle travelling inside the light cone (ignoring so-called "tachyons", which are now commonly interpreted as instabilities in quantum fields rather than particles doomed to travel on space-like trajectories anyway).

[^7]:    ${ }^{17}$ Derivation: $U_{\mu} U^{\mu}=\gamma^{2}\left(c^{2}-v^{2}\right)=\left(1-\beta^{2}\right)^{-1}\left(1-\beta^{2}\right) c^{2}=c^{2}$
    ${ }^{18}$ Similarly, this is defined such that $P_{\mu} P^{\mu}=\gamma^{2} m^{2}\left(c^{2}-v^{2}\right)=m^{2} c^{2}$ is oberserver-independent.

[^8]:    ${ }^{19}$ We also used the vector identity $\mathbf{a} \cdot(\mathbf{a} \times \mathbf{b})=0$

[^9]:    ${ }^{20}$ We can always do this by changing reference frames. In particular, we shall later see that Lorentz Transformations leave the parallel components $\mathbf{E}_{\|}$and $\mathbf{B}_{\|}$unchanged, thereby justifying our (Lorentz boosted) calculations.

[^10]:    ${ }^{21}$ We can always do this by choosing the $+z$ direction to be perpendicular to $\mathbf{v}(\tau=0)$.

[^11]:    ${ }^{22}$ Note: Matrices are fundamentally different from rank-2 tensors, they are merely a convenient representation we use in this module (and in most other circumstances)!

[^12]:    ${ }^{23}$ See p. 112 of David Tong's lecture notes on electromagnetism.
    ${ }^{24} \tilde{F}^{\mu \nu}$ is sometimes also written as $\star F^{\mu \nu}$. This alternative (in fact more common) notation comes from the fact that $\star F^{\mu \nu}$ is the so-called Hodge dual of the Faraday tensor $F^{\mu \nu}$ (whatever this means).
    ${ }^{25}$ We can derive $\tilde{F}_{\mu \nu}$ with the usual raising and lowering of indices (try this yourself!):

    $$
    \tilde{F}_{\mu \nu}=\eta_{\mu \alpha} \eta_{\nu \beta} \tilde{F}^{\alpha \beta}=\frac{1}{2} \eta_{\mu \alpha} \eta_{\nu \beta} \epsilon^{\alpha \beta \gamma \tau}\left(\eta_{\gamma \rho} \eta_{\tau \sigma} F^{\rho \sigma}\right)=\frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}
    $$

[^13]:    ${ }^{27}$ Mathematically, the 4-potential, $A$, is known as a differential 1-form. $F$ is then defined as the exterior derivative of $A$, i.e. $F:=d A$, or in terms of local coordinates, $F_{\mu \nu}=(d A)_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$ (as defined above), which is therefore a differential 2-form. Again, more about this in MA3H5 Manifolds.

[^14]:    ${ }^{28}$ Some justification for why radiation comes up here: Recall that radiation is merely some fluctuation in the electric and magnetic fields (aka EM waves). With time-varying charges/currents, we have a time-varying E-field, and can reasonably expect the generation of a B-field (e.g. by Lenz's Law). In other words, we have radiation, which must travel (in vacuum) with speed $c$ due to Special Relativity. More on this in the next few sections.

[^15]:    ${ }^{29}$ This can be done by introducing appropriate Green functions. For more details, see J.D.Jackson Classical Electrodynamics, Second Edition, Section 6.6, p. 223 (please don't, it really is just 4 pages of painful maths).

[^16]:    ${ }^{30}$ More precisely, whenever we speak of "Radiation Fields", we are referring to fields in the limit $r \gg$ (alongside the usual $\lambda \gg b$ for Hertzian dipoles), as shown here.

[^17]:    ${ }^{31}$ Images taken from Photonics 101 (this website also comes with a rather handy and comprehensive introduction to electrodynamics).

[^18]:    ${ }^{32}$ Absolute banger from the Electric Light Orchestra (ELO), a band quite fittingly named for this module :)
    ${ }^{33}$ But why isn't the sky violet? The Sun, like every other star, has its own radiation spectrum, in this case the intensity peaks at around 500nm (green) and falls off in the violet region (as seen from, e.g. Wien's displacement $l a w)$. Additionally, oxygen in the Earth's atmosphere absorbs photons of near-ultraviolet wavelengths. The resulting colour, which appears pale blue, is therefore a mixture of all the scattered colours (mainly blue and green).
    ${ }^{34} \mathrm{~A}$ fun fact: In locations with little light pollution, the moonlit night sky is also blue since moonlight is just reflected sunlight. The moonlit sky is not perceived as blue, however, because at low light levels, human vision comes mainly from rod cells which do not produce any colour perception (this is knows as the Purkinje effect).

[^19]:    ${ }^{35}$ In this case, the strong scattering is due instead to the large difference in refractive index between pores and solid parts within the material.
    ${ }^{36}$ Image (along with a long discussion on why the sky is blue) found on https://www.flickr.com/photos/ optick/112909824/

[^20]:    ${ }^{37}$ Image taken from https://commons.wikimedia.org/wiki/File:L-over2-rad-pat.svg

