1 Variational Calculus of Functionals

1.1 Hamilton's Principle

In physics, we often encounter problem of minimizing value of certain integral, that somehow combines different functions of different independent variables. Classical example would be for example minimization of action of a particle in a mechanical system.
Hamilton's principle predicts that the particle follows a trajectory $\vec{r}(t) = (x(t), y(t), z(t))$, where $\vec{r}$ is the position vector pointing to a point on the trajectory and t is time, such that the action of the particle is minimal. The action is defined as

$$S = \int_{t_i}^{t_f} \mathscr{L}(\vec{r}, \dot{\vec{r}}, t)dt \tag{1}$$

where $\dot{\vec{r}} = \frac{d}{dt}\vec{r}$ and $\mathscr{L}$ is called the Lagrangian and presents some combination of variables $\vec{r}$, $\dot{\vec{r}}$ and t. Since, $\vec{r}$ is also a function of independent variable t, $\mathscr{L}$ is not a function in the common sense. More importantly, even S is not a function in common sense, since its value depends on the choice of functions $\vec{r}$ as well. We call S a functional, to make this distinction.
We are interested in finding the form of the functions $\vec{r}$ which minimize the value of the integral S. There is a specific procedure we can take to determine the conditions on $\mathscr{L}$. If we then now exact form of $\mathscr{L}$, these conditions become conditions on $\vec{r}$, which will help us find minimal solutions, and therefore the trajectory of a particle.
Consider the situation where we found $\vec{r}$ which does minimize S. Now, consider some $\delta\vec{r}$ such that in every point t, $|\delta\vec{r}| \ll |\vec{r}|$. These functions (there is again 3 of them in physical space) are called variations of $\vec{r}$. Since $\vec{r}$ was such that S was minimal, adding the variation $\delta\vec{r}$ to the original function $\vec{r}$ and evaluating new action using $\vec{r} + \delta\vec{r}$ instead of $\vec{r}$ should lead to no real change in the action, as the variation gets smaller. Hence, we have

$$\lim_{|\delta\vec{r}|\to 0}\left(\int_{t_i}^{t_f} \mathscr{L}(\vec{r}+\delta\vec{r}, \dot{\vec{r}}+\delta\dot{\vec{r}}, t)dt\right) = S = \int_{t_i}^{t_f} \mathscr{L}(\vec{r}, \dot{\vec{r}}, t)dt \tag{2}$$

where $\delta\dot{\vec{r}} = \frac{d}{dt}\delta\vec{r}$. Using the Taylor expansion of $\mathscr{L}$ in $\vec{r}$ and $\dot{\vec{r}}$, we can express the left-hand side of (2) as

$$\lim_{|\delta\vec{r}|\to 0}\left(\int_{t_i}^{t_f} \mathscr{L}(\vec{r}+\delta\vec{r}, \dot{\vec{r}}+\delta\dot{\vec{r}}, t)dt\right) = \int_{t_i}^{t_f}\left(\mathscr{L}(\vec{r}, \dot{\vec{r}}, t) + \sum_i \frac{\partial\mathscr{L}}{\partial r_i}\delta r_i + \sum_i \frac{\partial\mathscr{L}}{\partial \dot{r}_i}\delta\dot{r}_i\right)dt$$

where r_i and $\dot{r}_i$ are components of $\vec{r}$ and $\dot{\vec{r}}$, respectivelly (and similarly for δr_i and $\delta\dot{r}_i$).
Compairing this to the right hand side of (2), we obtain

$$\int_{t_i}^{t_f}\left(\sum_i \frac{\partial\mathscr{L}}{\partial r_i}\delta r_i + \sum_i \frac{\partial\mathscr{L}}{\partial \dot{r}_i}\delta\dot{r}_i\right)dt$$

The terms in the second sum can be integrated by parts as

$$\int_{t_i}^{t_f} \frac{\partial\mathscr{L}}{\partial\dot{r}_i}\delta\dot{r}_i dt = \left[\frac{\partial\mathscr{L}}{\partial\dot{r}_i}\delta r_i\right]_{t_i}^{t_f} - \int_{t_i}^{t_f}\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{r}_i}\right)\delta r_i dt$$

Assuming that the variation of the path $\delta\vec{r}$ does not move the end points (we are searching for trajectory of the system between two points in space), we have $\delta\vec{r}(t_i) = \delta\vec{r}(t_f) = \vec{0}$. Hence, the first term in the equation above is zero. Since the integral of the sum can be split into the sum of the integrals, we can write integrate each of the terms separately to obtain

$$\int_{t_i}^{t_f}\sum_i\left[\frac{\partial\mathscr{L}}{\partial r_i} - \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{r}_i}\right)\right]\delta r_i dt = 0$$

Since we can choose δr_i arbitrarily along the time t, $t_i \leq t \leq t_f$, the only possibility how to ensure that this equality is satisfied for any value of any δr_i is by requiring

$$\forall i : \frac{\partial\mathscr{L}}{\partial r_i} = \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{r}_i}\right) \tag{3}$$

This is the Lagrange equation, famous in mechanics and first example of a differential equation derived based on minimisation of a certain functional.

1.2 General Functional Minimization

Consider an $F = F(\vec{y}, \frac{\partial \vec{y}}{\partial \vec{x}}, \frac{\partial^2 \vec{y}}{\partial \vec{x}^2}, ..., \frac{\partial^n \vec{y}}{\partial \vec{x}^n}, \vec{x})$, where $\vec{y}$ is some vector field defined on independent variable vector $\vec{x}$. By this, we mean that F depends on some set of functions $\vec{y}$, where each of the functions y_i (there is m of these functions) depends on all independent variables from the set $\vec{x}$ (on each x_j, which there is k of). The derivative notation $\frac{\partial \vec{y}}{\partial \vec{x}}$ signifies that F depends on all derivatives $\frac{\partial y_i}{\partial x_j}$ for all possible values of i and j. Same notation applies for higher order derivatives (indexing those by index l).
Lets start by generalizing problem from previous section without changing the boundary conditions - we will try to extremize some variable I, defined as

$$I = \int_{\vec{x}_i}^{\vec{x}_f} F(\vec{y}, \frac{\partial \vec{y}}{\partial \vec{x}}, \frac{\partial^2 \vec{y}}{\partial \vec{x}^2}, ..., \frac{\partial^n \vec{y}}{\partial \vec{x}^n}, \vec{x}) d^k x$$

where $\vec{x}_i$ and $\vec{x}_f$ are some fixed end points. We will again introduce a small variation $\delta \vec{y}$ to the functions, with boundary conditions $\delta \vec{y}(\vec{x}_i) = \delta \vec{y}(\vec{x}_f) = \vec{0}$, applying also for all the derivatives of $\vec{y}$. Assuming zero change in I close to the extremum of I, we obtain following expression

$$\int_{\vec{x}_i}^{\vec{x}_f} d^k x \sum_{i,j,l} \frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \delta \left(\frac{\partial^l y_i}{\partial x_j^l} \right) = 0$$

where, to be clear, i goes from 1 to m, j goes from 1 to k and l goes from 0 to n. Decomposing the integral of the sum as the sum of the integrals, each individual integral can be integrated by parts as

$$\int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \delta \left(\frac{\partial^l y_i}{\partial x_j^l} \right) = \left[\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \delta \left(\frac{\partial^{l-1} y_i}{\partial x_j^{l-1}} \right) \right]_{\vec{x}_i}^{\vec{x}_f} - \int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{d}{dx_j} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta \left(\frac{\partial^{l-1} y_i}{\partial x_j^{l-1}} \right)$$

We can recognize that the first term is zero, as the boundary conditions we defined required all derivatives of all functions y_i to be zero at the boundary. Hence, we obtained a new integral, which can be integrated by parts again, this time leading to (disregarding the zero term arising from boundary conditions)

$$\int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{d}{dx_j} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta \left(\frac{\partial^{l-1} y_i}{\partial x_j^{l-1}} \right) = - \int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{d^2}{dx_j^2} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta \left(\frac{\partial^{l-2} y_i}{\partial x_j^{l-2}} \right)$$

Which leads us to the generalization

$$\int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \delta \left(\frac{\partial^l y_i}{\partial x_j^l} \right) = (-1)^l \int_{\vec{x}_i}^{\vec{x}_f} d^k x \frac{d^l}{dx_j^l} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta y_i$$

Hence the condition for extremisation of F becomes

$$\int_{\vec{x}_i}^{\vec{x}_f} d^k x \sum_{i,j,l} (-1)^l \frac{d^l}{dx_j^l} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta y_i = 0$$

Since we can choose all y_i arbitrarily, as long as they satisfy our boundary conditions, the only possibility how to satisfy this condition is by requiring

$$\forall i : \sum_{j,l} (-1)^l \frac{d^l}{dx_j^l} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) = 0 \tag{4}$$

So, we have m differential equations in total (for each i), featuring k different independent variables, of order $n+1$. For example, for a F of type

$$F = F \left(z(x,y), \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, x, y \right)$$

we have a single equation (only one independent variable - z), of order 3, which has a form

$$\frac{\partial F}{\partial z} - \frac{d}{dx} \left(\frac{\partial F}{\partial \left(\frac{\partial z}{\partial x} \right)} \right) - \frac{d}{dy} \left(\frac{\partial F}{\partial \left(\frac{\partial z}{\partial y} \right)} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial \left(\frac{\partial^2 z}{\partial x^2} \right)} \right) + \frac{d^2}{dy^2} \left(\frac{\partial F}{\partial \left(\frac{\partial^2 z}{\partial y^2} \right)} \right) = 0$$

1.3 Useful Simplifications for Lagrange Equations

In physics, we most commonly come across the Lagrangian-type functionals, which in extremisation satisfy equation (3). Consider now two special cases.

Functional is not explicitly dependent on the independent variable Then, we can find out that (using $\frac{\partial y_i}{\partial x} = y_i'$)

$$\frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right) = y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{d}{dx}(\frac{\partial F}{\partial y_i'})$$

Using the Euler-Lagrange equation

$$\frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right) = y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{\partial F}{\partial y_i}$$

We can also recognize now that the last part is the same as the total differential of F with respect to x

$$\frac{dF}{dx} = \left(\frac{\partial y_i}{\partial x}\frac{\partial}{\partial y_i} + \frac{\partial y_i'}{\partial x}\frac{\partial}{\partial y_i'} + \frac{\partial}{\partial x}\right)F$$

since F does not depend explicitly on x. Hence, we have

$$\frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right) = \frac{dF}{dx}$$

$$\frac{d}{dx}\left(F - y_i'\frac{\partial F}{\partial y_i'}\right) = 0$$

$$F - y_i'\frac{\partial F}{\partial y_i'} = \text{constant in x} \tag{5}$$

This equation is equivalent in some cases to Euler-Lagrange equations.

Functional does not explicitly depend on the independent variable According to Euler-Lagrange equation

$$\frac{\partial F}{\partial y_i} = 0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)$$

$$\frac{\partial F}{\partial y'} = \text{constant in x} \tag{6}$$

1.4 Boundary point on a line

For a special case of only one independent variable, we can also make a very useful generalization for the boundary conditions of the functionals. Suppose that we have a functional $F = F(\vec{y}, \frac{\partial \vec{y}}{\partial x}, x)$, where $\vec{y}$ is the vector of dependent variables, with m components.
Lets suppose that certain set of $\vec{y}$ does extremize functional I between certain end points a and b, while b has to lie on contour of $h(x, \vec{y})$. For simplicity, we can choose contour $h(x, \vec{y}) = 0$. Now, suppose that we add a small perturbation $\delta\vec{y}$ to $\vec{y}$. This moves the final point b to point $b + \Delta x$. The boundary conditions on $\delta\vec{y}$ become $\delta\vec{y}(a) = 0$ and $\delta\vec{y}(b + \Delta x) + \vec{y}(b + \Delta x) \in \{\vec{y} : h(x, \vec{y}) = 0\}$, i.e. the perturbation does not move the final point b away from the countour of h. Same applies for the original, extremized final point b. The integral for the unperturbed functional becomes

$$I = \int_a^b F(\vec{y}, \frac{\partial \vec{y}}{\partial x}, x)dx$$

and for the perturbed functional

$$I + \delta I = \int_a^{b+\Delta x} F(\vec{y} + \delta\vec{y}, \frac{\partial(\vec{y} + \delta\vec{y})}{\partial x}, x)dx$$

where δI is the perturbation in the functional. If the perturbation is small, we can approximate to the first order

$$\int_a^{b+\Delta x} F\left(\vec{y} + \delta\vec{y}, \frac{\partial(\vec{y} + \delta\vec{y})}{\partial x}, x\right)dx =$$

$$= \int_a^b F\left(\vec{y}+\delta\vec{y}, \frac{\partial(\vec{y}+\delta\vec{y})}{\partial x}, x\right) dx + \int_b^{b+\Delta x} F\left(\vec{y}+\delta\vec{y}, \frac{\partial(\vec{y}+\delta\vec{y})}{\partial x}, x\right) dx \approx$$

$$\approx \int_a^b F\left(\vec{y}+\delta\vec{y}, \frac{\partial(\vec{y}+\delta\vec{y})}{\partial x}, x\right) dx + F\left(\vec{y}(b)+\delta\vec{y}(b), \frac{\partial(\vec{y}+\delta\vec{y})}{\partial x}|_b, b\right)\Delta x \approx$$

$$\approx \int_a^b F\left(\vec{y}+\delta\vec{y}, \frac{\partial(\vec{y}+\delta\vec{y})}{\partial x}, x\right) dx + F\left(\vec{y}(b), \frac{\partial\vec{y}}{\partial x}|_b, b\right)\Delta x$$

The integral can be treated in the standard way - expanding in Taylor series of F

$$I+\delta I = \int_a^b \left[F\left(\vec{y}, \frac{d\vec{y}}{dx}, x\right) + \sum_{i=1}^m \frac{\partial F}{\partial y_i}\delta y_i + \sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\delta \dot{y}_i\right] dx + F(b)\Delta x$$

where I used shorthand $F(\vec{y}(b), \frac{\partial\vec{y}}{\partial x}|_{x=b}, b) = F(b)$ and $\dot{y}_i = \frac{dy_i}{dx}$ and the sums are over the components of $\vec{y}$ (and $\delta\vec{y}$).

The first term, we can recognize as the unperturbed integral I. The last term can be integrated by parts to obtain

$$I+\delta I = I + \int_a^b \sum_{i=1}^m \frac{\partial F}{\partial y_i}\delta y_i dx + \left[\sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\delta y_i\right]_a^b - \int_a^b \sum_{i=1}^m \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}_i}\right)\delta y_i dx + F(b)\Delta x$$

$$\delta I = \int_a^b \sum_{i=1}^m \left(\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}_i}\right)\right)\delta y_i dx + \sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\delta y_i\bigg|_{x=b} + F(b)\Delta x$$

When $\vec{y}$ extremizes I, $\delta I = 0$. If we want to allow consistency with the previous case (and effectively treat fixed boundaries as a special case of this type of boundary problem), we will still require the integral part to go to zero, so the Euler-Lagrange equations will still apply. There will be however an extra condition, which will enable us to calculate the extremising functions vector $\vec{y}$.

If $\vec{y}$ extremizes I and the Euler-Lagrange equations apply for the reason of consistency, we are left with.

$$\sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\delta y_i\bigg|_{x=b} = -F(b)\Delta x$$

From now on until the final equation (7), consider all derivatives evaluated at the end point b. Lets think about the value of Δx. If the end point is to stay on the line given by $h(x,\vec{y}) = 0$, the change in the value of h must be zero for a point close to b. Hence

$$\Delta h = 0 = \frac{\partial h}{\partial x}\Delta x + \sum_{i=1}^m \frac{\partial h}{\partial y_i}\Delta y_i$$

Consider the change that occurs in each y_i due to change in x. There is a change due to continuation of y_i beyond x, but there is also a change due to the perturbation δy_i. Since δy_i is small, we can approximate Δy_i to the first order as

$$\forall i : \Delta y_i \approx \delta y_i(b) + \frac{\partial y_i}{\partial x}\Delta x = \delta y_i + \dot{y}_i \Delta x$$

Hence, since h cannot change

$$0 = \frac{\partial h}{\partial x}\Delta x + \sum_{i=1}^m \frac{\partial h}{\partial y_i}(\delta y_i + \dot{y}_i\Delta x)$$

So, we can express Δx as

$$\Delta x = \frac{-\sum_{i=1}^m \frac{\partial h}{\partial y_i}\delta y_i}{\frac{\partial h}{\partial x} + \sum_{i=1}^m \frac{\partial h}{\partial y_i}\dot{y}_i}$$

Substituting this value back into the equation obtained from extremization, we arrive at

$$\sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\delta y_i = \frac{\sum_{i=1}^m F(b)\frac{\partial h}{\partial y_i}\delta y_i}{\frac{\partial h}{\partial x} + \sum_{i=1}^m \frac{\partial h}{\partial y_i}\dot{y}_i}$$

$$\sum_{i=1}^m \frac{\partial F}{\partial \dot{y}_i}\left(\frac{\partial h}{\partial x} + \sum_{j=1}^m \frac{\partial h}{\partial y_j}\dot{y}_j\right)\delta y_i = \sum_{i=1} F(b)\frac{\partial h}{\partial y_i}\delta y_i$$

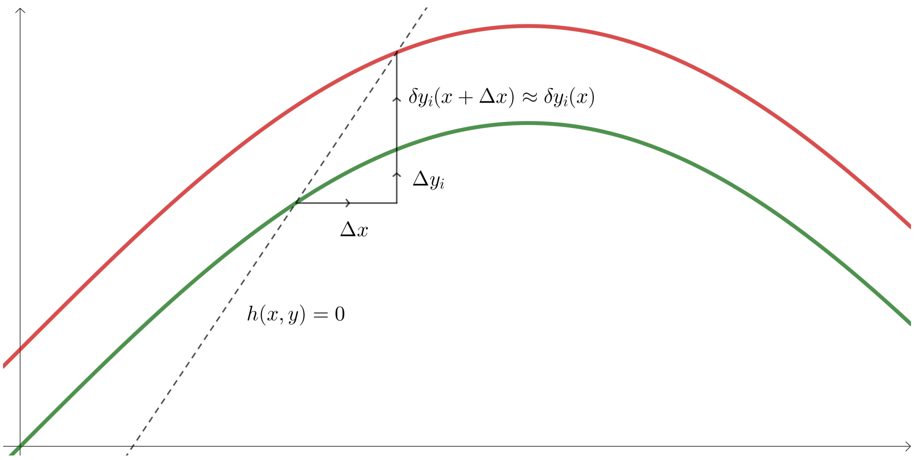

Figure 1: The variable end point visualisation in one of the independent variable y_i. The function $y_i(x)$ is in green, the variation of the function $\delta(y_i)$ is in red and is added on top of the function y_i for clarity. The intersection of the surface $h(x,\vec{y})$ with the xy_i plane is a curve, here shown as a dashed line.

$$\sum_{i=1} \delta y_i \left[F(b)\frac{\partial h}{\partial y_i} - \frac{\partial F}{\partial \dot{y}_i}\left(\frac{\partial h}{\partial x} + \sum_{j=1}^{m}\frac{\partial h}{\partial y_j}\dot{y}_j\right)\right] = 0$$

Since this equation has to apply for any δy_i we can choose, the only way to reach the equality is if the term inside the square brackets is zero for all i. Hence, we have

$$\forall i : F(b)\frac{\partial h}{\partial y_i} = \frac{\partial F}{\partial \dot{y}_i}\left(\frac{\partial h}{\partial x} + \sum_{j=1}^{m}\frac{\partial h}{\partial y_j}\dot{y}_j\right) \tag{7}$$

Example : Linearly moving line boundary Consider a minimization of action of a free particle in 2D, which moves between point $\vec{r}_0 = (x_0, y_0)$ and a variable boundary, which is a line satisfying equation $h(t,x,y) = ax+by+ct = 0$. By Euler-Lagrange equations, we find the general form of minimized dependent variables $x(t)$ and $y(t)$, and using (7) will allow us to find the global minimal action for the particle to travel from $\vec{r}_0$ to some point on the boundary.
Solving the Euler-Lagrange equations (in 2D, free particle has Lagrangian $\mathscr{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$)

$$\frac{\partial \mathscr{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}}$$

$$0 = \frac{d}{dt}(m\dot{x})$$

For constant mass m in time, we then obtain

$$\dot{x} = \text{const. in time} = u_0$$

$$x = u_0(t_b - t_0) + x_0$$

where t_0 is the time when particle is at the starting point $\vec{r}_0$ and t_b is the time when the particle arrives at the boundary. Since $\mathscr{L}$ is symmetrical in exchange of $\dot{x}$ and $\dot{y}$, analogous equation applies for y

$$y = v_0(t_b - t_0) + y_0$$

Now, we need to use the boundary conditions (7). In x

$$\frac{1}{2}m(u_0^2+v_0^2)\frac{\partial h}{\partial x}\Big|_b = \frac{\partial \mathscr{L}}{\partial \dot{x}}\Big|_b\left(\frac{\partial h}{\partial t}\Big|_b + \frac{\partial h}{\partial x}\Big|_b u_0 + \frac{\partial h}{\partial y}\Big|_b v_0\right)$$

$$\frac{1}{2}ma(u_0^2+v_0^2) = mu_0\left(c + au_0 + bv_0\right)$$

Similarly, in y

$$\frac{1}{2}mb(u_0^2+v_0^2) = mv_0\left(c + au_0 + bv_0\right)$$

Assuming that $b \neq 0$ and $v_0 \neq 0$, we can divide each side of the first equation by corresponding sides of the second equation, which leads to

$$\frac{a}{b} = \frac{u_0}{v_0}$$

Or $u_0 = \frac{a}{b} v_0$. Substituting this into the first equation

$$\frac{1}{2} ma \left(\frac{a^2}{b^2} v_0^2 + v_0^2 \right) = m \frac{a}{b} v_0 \left(c + \frac{a^2}{b} v_0 + b v_0 \right)$$

$$\frac{1}{2} a \left(\frac{a^2}{b^2} + 1 \right) v_0^2 = \frac{ac}{b} v_0 + a \left(\frac{a^2}{b^2} + 1 \right) v_0^2$$

$$v_0 \left[\frac{1}{2} a \left(\frac{a^2}{b^2} + 1 \right) v_0 + \frac{ac}{b} \right] = 0$$

We therefore have two solutions. One, $v_0 = 0$, which leads to $u_0 = 0$. This solution is applicable if at time t_0, the point lies infront of the boundary $h = 0$ in the sense that as time passes, boundary approaches the point $\vec{r}_0$. Then, the particle can reach $h = 0$ by simply resting in the original position.
The second solution, corresponding to case when the particle has to catch up to the boundary, has

$$v_0 = -\frac{\frac{ac}{b}}{\frac{1}{2} a \left(\frac{a^2}{b^2} + 1 \right)} = -\frac{2cb}{a^2 + b^2}$$

and

$$u_0 = -\frac{2ca}{a^2 + b^2}$$

To interpret these results, consider the vector normal to the line $h = 0$ at all times, which has components $\vec{n} = (a, b)$. By calculating the component of vector product in the additional dimension z, which is not present, we can find out whether $\vec{n}$ and $\vec{v} = (u_0, v_0)$ are parallel.

$$\vec{n} \times \vec{v} = -a \frac{2cb}{a^2 + b^2} + b \frac{2ca}{a^2 + b^2} = 0$$

Hence the vectors are indeed parallel - the particle moves in direction perpendicular to the line. Furthermore, we can notice that the condition $h = 0$ can be rewritten as

$$\vec{n} \cdot \vec{r} = -ct$$

Since $\vec{n}$ is independent of time, taking time derivative of both sides leads to

$$\vec{n} \cdot \vec{v}_b = -c$$

where $\vec{v}_b$ is the speed of points on the boundary. Hence

$$\vec{n} \cdot \vec{v}_b = |\vec{n}| v_{b,\perp} = -c$$

$$v_{b,\perp} = \frac{-c}{\sqrt{a^2 + b^2}}$$

The magnitude of the speed of the particle is

$$|\vec{v}| = \sqrt{\frac{4c^2 a^2}{(a^2 + b^2)^2} + \frac{4cb^2}{(a^2 + b^2)^2}} = \frac{2c}{\sqrt{a^2 + b^2}}$$

We have already shown that $\vec{v} \parallel \vec{n}$, and comparing the sign of $v_{b,\perp}$ to signs of u_0 and v_0 shows that particle moves in the same direction as the line $h = 0$, but with twice the velocity.

1.5 Constrained Optimization

Often, we might be interested in optimizing certain functional I

$$I = \int_a^b d^k x F(\vec{y}, \frac{\partial \vec{y}}{\partial \vec{x}}, ...)$$

under some constraint on properties of the independent variables, which itself can be expressed as a functional (for example, we would require a normalization of some of the independent variables). Lets consider constraint given by functional J

$$J = \int_a^b d^k x G(\vec{y}, \frac{\partial \vec{y}}{\partial \vec{x}}, ...)$$

which we require to be equal to some constant value for allowed solutions to minimization of I.
To solve this problem, we need to realize that both I and J are essentialy functions of different possible functions y_i, which we try for the evaluation of the integral. Since functions y_i can exist in some vector space (although infinite dimensional), we can approach this problem similarly as optimization of scalar functions on vector spaces of finite dimensions under some constraints. This type of optimization is solved using Lagrange multipliers.
Consider a function $f = f(\vec{r})$, where $\vec{r}$ is some vector in a real space. We try to find extremal value of f at some $\vec{r}$ such that some condition $g(\vec{r}) = 0$ is satisfied at that point $\vec{r}$. This means that the contours of the two curves must exactly touch at this point, leading to requirement

$$\nabla f = \lambda \nabla g$$

where λ is the Lagrange multiplier. We could also take the small difference of the functions in some common direction $\delta\vec{r}$ and the equivalent condition would be that

$$\nabla f \cdot \delta\vec{r} = \delta f = \lambda \nabla \cdot \delta\vec{r} = \lambda \delta g$$

Hence $\delta f = \lambda \delta g$. In the case of our functionals, considering the effect of small variations δy_i in the functions y_i on I or J is exactly analogous. Therefore, at the point where I is extremal and J is still a specified constant value, it follows that

$$\delta I = \lambda \delta J$$

We could have as well chosen $\lambda = -\lambda'$ as the Lagrange multiplier, without any loss of generality. If we do this choice, we then arrive at expression

$$\delta I + \lambda' \delta J = 0$$

Recalling the expressions for δI and δJ as changes due to small variations δy_i, we are left with (replacing λ' with λ, just a change of symbols)

$$\delta I + \lambda \delta J = \int_a^b d^k x \sum_{i,j,l} (-1)^l \frac{d^l}{dx_j^l} \left(\frac{\partial F}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta y_i + \lambda \int_{\vec{x}_i}^{\vec{x}_f} d^k x \sum_{i,j,l} (-1)^l \frac{d^l}{dx_j^l} \left(\frac{\partial G}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta y_i =$$

$$= \int_{\vec{x}_i}^{\vec{x}_f} d^k x \sum_{i,j,l} (-1)^l \frac{d^l}{dx_j^l} \left(\frac{\partial (F + \lambda G)}{\partial \left(\frac{\partial^l y_i}{\partial x_j^l} \right)} \right) \delta y_i = 0$$

Therefore, we obtain the result that the extremum of F constrained by G occurs for the same set of functions y_i that extremize new functional

$$K = \int_a^b d^k x (F + \lambda G)$$

without any constraints. Therefore, we can apply our previous results as Euler-Lagrange equations to solve for K to find the required set of functions y_i.
Importantly, our new functions y_i will not be completly determined, because we introduce the new unknown, λ, into the system. To get rid of this λ, we need to check for which λ does J have the required value. This can be somewhat simplified by considering

$$K = \int_a^b d^k x (F + \lambda G) = \int_a^b d^k x F + \lambda \int_a^b d^k x G = I + \lambda J$$

Hence

$$J = \frac{\partial K}{\partial \lambda}$$

So, the steps to find the solution are

1. Determine unconstrained minima of $K = I + \lambda J$ by solving equations (4) for function $H = F + \lambda G$.
2. Resultant set of functions y_i depends on the Lagrange multiplier λ. To find its value, calculate $K = \int_a^b d^k x H$
3. The resultant integral K depends on λ. By solving $\frac{\partial K}{\partial \lambda} = J$, we obtain the value of Lagrange multiplier.

Alternatively, one can solve the constraint directly, by evaluating

$$J = \int_a^b d^k x G$$

for the given set of functions y_i, dependent on λ.

1.5.1 Other Combined Functionals

We have seen that combining functionals and minimizing these combined functionals can help us solve specific constrained problems. One specific combined functional for problem of extremizing I under constraint J is the functional $\Lambda = \frac{I}{J}$. The variation in Λ can be approximated as (using derivatives)

$$\delta\Lambda = \delta\left(\frac{I}{J}\right) = \frac{\delta I J - I \delta J}{J^2} = \frac{\delta I - \frac{I}{J}\delta J}{J} = \frac{\delta I - \Lambda \delta J}{J}$$

Minimizing this functional follows exactly the same procedure as minimizing K, as long as $J \neq 0$, with the single difference that Λ has opposite sign to λ, which is not that important. Therefore, if instead of K we minimize Λ, the resultant minimal value of Λ automatically corresponds to the Lagrange multiplier $-\lambda$.

1.6 Sturm-Liouville Equations

We have seen that extremising functionals leads to sets of differential equations which need to be solved. Inversly, we could search for functionals such that when these are extremised, the known set of differential equations is obtained. The resultant form of the found functionals might help us learn more about the system. One particular type of differential equations for which it is useful to search for the form of the functionals are the Sturm-Liouville equations. These differential equations have form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y - \lambda\rho(x)y = 0$$

where $y(x)$ is some function defined on an interval $x \in (a, b)$ which vanishes at the boundaries of this interval and $\rho(x)$ is positive definite function on this interval.
To find the functional whose extremisation leads to these equations, start from the assumption that we will be searching for a simple $I = \int F(x, y, \frac{dy}{dx})$. The extremisation equations are then simply Euler-Lagrange equations

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)$$

We can see that a total differential of some more complex function is present in both expressions. Lets then start deriving the functional from assumption that these differentials are identical, i.e.

$$\frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right) = \frac{d}{dx}\left(p\frac{dy}{dx}\right)$$

$$\frac{\partial F}{\partial \frac{dy}{dx}} = p\frac{dy}{dx}$$

$$F = \frac{1}{2}p\left(\frac{dy}{dx}\right)^2 + Z(y, x)$$

where Z is some unknown function of y and x. Other parts of F can be derived upon considering that the requirement $\rho(x) > 0$ on (a, b) is similar to requirement for weight functions on some intervals (such as probability distributions etc.). We could then try to write F as combined function $F = G + \lambda H$ and do constrained minimization of F with constraining function H, corresponds to the term with the weight function ρ. The extremisation of H alone would then lead to

$$\frac{\partial H}{\partial y} = \frac{d}{dx}\frac{\partial H}{\partial \frac{dy}{dx}} = \lambda\rho(x)y$$

$$H = \frac{1}{2}\lambda\rho y^2 + Z'\left(x, \frac{dy}{dx}\right)$$

Finally, the remaining term qy can be inserted into the G part of the functional F. Since we have already assigned the total derivative part of G, we have to assign the term to the other term in Euler-Lagrange equations, i.e.

$$\frac{\partial G}{\partial y} + qy = 0$$

(since we assigned the total derivative on the opposite side to the side of the total differential in E-L equations, we need assign the term qy to the same side as the derivative in y)

$$G = -\frac{1}{2}qy^2 + Z''\left(x, \frac{dy}{dx}\right)$$

Combining all functions together, we have our guess for the functional form

$$K = \int_a^b dx \left(\frac{1}{2}p\left(\frac{dy}{dx}\right)^2 - \frac{1}{2}qy^2 + \frac{1}{2}\lambda\rho y^2\right) = \frac{1}{2}\int_a^b dx \left(p\left(\frac{dy}{dx}\right)^2 - qy^2 + \lambda\rho y^2\right) = \frac{1}{2}\int_a^b F dx$$

K will be extremal even when $2K$ will be extremal, so we can disregard the factor of one half before the integral. To check whether this functional is indeed a correct form of the functional, we apply Euler-Lagrange conditions

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \frac{dy}{dx}} = 0$$

$$-2qy + 2\lambda\rho y - \frac{d}{dx}\left(2p\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy - \lambda\rho y = 0$$

Therefore, the functional of form

$$K = \int_a^b dx \left(p\left(\frac{dy}{dx}\right)^2 - qy^2 + \lambda\rho y^2\right)$$

is the correct combined functional whose unconstrained extremisation leads to Sturm-Liouville equations as written above. The combined functional consists of

$$K = I + \lambda J$$

where

$$I = \int_a^b dx \left(p\left(\frac{dy}{dx}\right)^2 - qy^2\right)$$

and

$$J = \int_a^b \rho y^2 dx$$

Now, we might be interested in searching for the form of functional $\Lambda = \frac{I}{J}$, as its extremisation provides us automatically with values λ. To do this, consider multiplying Sturm-Liouville equations by y and integrating over the domain (a, b)

$$\int_a^b dx \left(y\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy^2 - \lambda\rho y^2\right) = \int_a^b (0)dx = 0$$

$$\int_a^b y\frac{d}{dx}\left(p\frac{dy}{dx}\right)dx + \int_a^b (qy^2 - \lambda\rho y^2)dx = 0$$

Integrating first integral by parts

$$\left[yp\frac{dy}{dx}\right]_a^b - \int_a^b p\left(\frac{dy}{dx}\right)^2 dx + \int_a^b (qy^2 - \lambda\rho y^2)dx = 0$$

Since y vanishes at the boundaries a and b, the first term goes to zero. Hence

$$-\lambda\int_a^b \rho y^2 dx = \int_a^b \left(p\left(\frac{dy}{dx}\right)^2 - qy^2\right)dx$$

$$-\lambda = \frac{\int_a^b \left(p\left(\frac{dy}{dx}\right)^2 - qy^2\right)dx}{\int_a^b \rho y^2 dx} = \frac{I}{J} = \Lambda$$

Hence, we obtained the expected form for Λ, with opposite sign to λ.

2 Complex Differentiation

Complex numbers are very useful construct enabling elegant description of real systems. Furthermore, some properties of the complex plane make some calculations in this plane relatively easy, with applications to calculations of real variables and unknowns. For example, calculation of certain real integrals is much easier when considering integration in the complex plane.
To understand why this is the case, we need to study the topology and calculus of complex numbers in more detail. We will start by the easiest part of the calculus - differentiation.

2.1 Complex Limit

For real numbers, we would define a limit of a function as a function value at a number, which is approaching some target number either in the positive (from the left) or the negative (from the right direction). But, in complex plane, there is no natural ordering of the elements, so we cannot define a definite positive or negative direction.
Therefore, the definition of derivative is somewhat stronger in complex plane. While the real derivative is defined to exist when the certain limit in the positive direction and certain limit in the negative direction are equal, in complex plane, a derivative of a function is only defined when the limit in any possible direction is the same.
Consider a complex function $f(z)$ of a complex variable z. We define the derivative of f as

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0}$$

When we approach z_0 along certain path, we can assume that close to z_0 path is linear. Let $\vec{a} = (a_1, a_2)$ be the unit vector parallel to the direction of the path close to z_0 in the complex plane. Lets also write the complex function in terms of its real and complex parts

$$f(z) = u(z) + iv(z)$$

where u and v are real. Lets also define $z = x + iy$, so we have

$$f(z) = u(x + iy) + iv(x + iy) = u(x, y) + iv(x, y)$$

Hence, the derivative becomes

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0 + a_1h + ia_2h) - f(z_0)}{a_1h + ih_2} =$$

$$= \lim_{h\to 0}\frac{[u(x_0 + a_1h, y_0 + a_2h) + iv(x_0 + a_1h, y_0 + a_2h) - u(x_0, y_0) - iv(x_0, y_0)](a_1h - ia_2h)}{a_1^2h^2 + a_2^2h^2} =$$

$$= \lim_{h\to 0}\frac{[u(x_0 + a_1h, y_0 + a_2h) - u(x_0, y_0)](a_1 - ia_2)}{(a_1^2 + a_2^2)h} + i\lim_{h\to 0}\frac{[v(x_0 + a_1h, y_0 + a_2h) - v(x_0, y_0)](a_1 - ia_2)}{(a_1^2 + a_2^2)h}$$

Using the fact that $\vec{a}_1$ is a unit vector, $(a_1^2 + a_2^2) = 1$. Furthermore, since u and v are real functions, we can use the definition of a derivative of a real function

$$f'(z_0) =$$

$$= (a_1 - ia_2) \lim_{h\to 0} \frac{u(x_0 + a_1 h, y_0 + a_2 h) - u(x_0, y_0)}{h} + i(a_1 - ia_2) \lim_{h\to 0} \frac{v(x_0 + a_1 h, y_0 + a_2 h) - v(x_0, y_0)}{h} =$$

$$= (a_1 - ia_2) \left[\frac{\partial u}{\partial x}\bigg|_{z_0} a_1 + \frac{\partial u}{\partial y}\bigg|_{z_0} a_2 \right] + (a_2 + ia_1) \left[\frac{\partial v}{\partial x}\bigg|_{z_0} a_1 + \frac{\partial v}{\partial y}\bigg|_{z_0} a_2 \right] =$$

$$= a_1^2 \frac{\partial u}{\partial x}\bigg|_{z_0} + a_2^2 \frac{\partial v}{\partial y}\bigg|_{z_0} + a_1 a_2 \left(\frac{\partial u}{\partial y}\bigg|_{z_0} + \frac{\partial v}{\partial x}\bigg|_{z_0} \right) + i \left[a_1^2 \frac{\partial v}{\partial x}\bigg|_{z_0} - a_2^2 \frac{\partial u}{\partial y}\bigg|_{z_0} + a_1 a_2 \left(\frac{\partial v}{\partial y}\bigg|_{z_0} - \frac{\partial u}{\partial x}\bigg|_{z_0} \right) \right]$$

Now, in order for f to have the derivative at z_0, this must be independent of the choice of a_1 and a_2. This imposes certain conditions on f (or equivalently on u and v). To see how these conditions apply, it is more clear to consider to special cases of directions to derive some conditions, that, as we will see, guarantee that the derivative is independent of the direction.
Consider first a special case when $a_1 = 1$ and $a_2 = 0$ (approach from the left along a line parallel to real axis). Then

$$f'(z_0) = \frac{\partial u}{\partial x}\bigg|_{z_0} + i \frac{\partial v}{\partial x}\bigg|_{z_0}$$

Second special case is for approach from the bottom along line parallel to the imaginary axis, which corresponds to $a_1 = 0$ and $a_2 = 1$. Then

$$f'(z_0) = \frac{\partial v}{\partial y}\bigg|_{z_0} - i \frac{\partial u}{\partial y}\bigg|_{z_0}$$

Equating the derivative for these two special paths leads to famous Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x}\bigg|_{z_0} = \frac{\partial v}{\partial y}\bigg|_{z_0}, \quad \frac{\partial u}{\partial y}\bigg|_{z_0} = -\frac{\partial v}{\partial x}\bigg|_{z_0} \tag{8}$$

Now, consider applying applying these special conditions for very special set of paths in our expression for general approach part. It becomes

$$f'(z_0) = a_1^2 \frac{\partial u}{\partial x}\bigg|_{z_0} + a_2^2 \frac{\partial u}{\partial x}\bigg|_{z_0} + a_1 a_2 \left(\frac{\partial u}{\partial y}\bigg|_{z_0} - \frac{\partial u}{\partial y}\bigg|_{z_0} \right) + i \left[-a_1^2 \frac{\partial u}{\partial y}\bigg|_{z_0} - a_2^2 \frac{\partial u}{\partial y}\bigg|_{z_0} + a_1 a_2 \left(\frac{\partial u}{\partial x}\bigg|_{z_0} - \frac{\partial u}{\partial x}\bigg|_{z_0} \right) \right]$$

$$f'(z_0) = \frac{\partial u}{\partial x}\bigg|_{z_0} (a_1^2 + a_2^2) - i \frac{\partial u}{\partial y}\bigg|_{z_0} (a_1^2 + a_2^2) = \frac{\partial u}{\partial x}\bigg|_{z_0} - i \frac{\partial u}{\partial y}\bigg|_{z_0}$$

where I used the fact that $\vec{a}_1$ is the unit vector, and therefore $a_1^2 + a_2^2 = 1$. We can see that this expression is independent of the choice of a_1 or a_2. This means that the conditions (8) are not exclusive to a special two paths, but are rather sufficient conditions for the derivative of f to be identical for all possible approach paths.
When a function satisfies these conditions, it is said to be holomorphic. In our course, it is equivalent with saying that the function is analytic. Since these conditions apply at a point, the function is also holomorphic only at those points where the Cauchy-Riemann conditions apply.
It should be noted that the Cauchy-Riemann equations can be usefully rewritten in a fully complex form as

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \tag{9}$$

2.1.1 Rules for Calculating derivatives

Since the definition of the complex derivative was similar in form to the definition of the real derivative, most of the properties of real derivatives (specifically, those following from the linearity of the definition) apply for complex derivatives as well. Specifically, denoting the complex derivative as a prime, for complex functions of complex variables f and g $(af + bg)' = af' + bg'$, where a and b are complex constants, $(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, $f = g^n : f' = ng^{n-1}g'$ where n is an integer, $(z^n)' = nz^{n-1}$ where z is the complex variable.

2.2 Special Cases of Complex Derivative

To appreciate the difference of complex derivative of complex function, we will consider several different special cases of function f.

Complex Derivative of Real Function Consider f being real, i.e. $v = 0$. Then, the Cauchy-Riemann equations imply

$$\frac{\partial u}{\partial x} = 0\,, \; \frac{\partial u}{\partial y} = 0$$

therefore, since for real f, $f = u$, any real function of complex variables that is to be holomorphic has to be a real constant.

Complex Derivative of Function of Only Real Variable Consider now that $f = u(x) + iv(x)$, i.e. there is no dependence on the imaginary part of the number z. Then, the Cauchy-Riemann equations dictate that

$$\frac{\partial u}{\partial x} = 0\,, \; \frac{\partial v}{\partial x} = 0$$

Hence, again, if this type of function is to be holomorphic, it has to be a constant.

Complex Function of Two Real Variables Consider that now we would define the variables x and y to be the real independent variables instead of a part of the complex variable z. We can actually derive x and y from the definition of $z = x + iy$ as $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$ where $\bar{z}$ is the complex conjugate of z. Then, consider that instead of f being a function of z, let $f = f(x, y)$. Then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

For complex function of two real variables, Cauchy-Riemann conditions do not apply, hence both of these expressions are generally non-zero and therefore

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}$$

However, if f was a function of complex variable, where Cauchy-Riemann conditions apply, then

$$\frac{\partial f}{\partial \bar{z}} = 0$$

And so

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

These two expressions are generally different. The reason for this discrepancy is essentially the fact that the geometry/metric in the complex plane is fundamentaly different from the 2D real plane. In the definition of the derivative of function of two real derivatives, we would always divide by small real factor h, while for the complex derivative, the factor can be imaginary/complex.
As an aside, we should note that the expression $\frac{\partial f}{\partial \bar{z}} = 0$ implies that f cannot depend on $\bar{z}$ if it is holomorphic. Therefore, we have to be able to express f using z only, if we translate from x and y to z and $\bar{z}$.

2.3 Laplace's Equation in 2D

Consider a real part of the complex function $f = u(x, y) + iv(x, y)$ where u and v are real. What is $\nabla^2 u$ equal to? We can actually evaluate this expression, as we know that u and v must satisfy Cauchy-Riemann equations. So

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = 0$$

Hence, u satisfies the Laplace's equation $\nabla^2 u = 0$. Furthermore

$$\nabla^2 v = \frac{\partial}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = 0$$

Hence, both components of the complex function satisfy the Laplace's equation, hence any complex function f satisfies $\nabla^2 f = 0$.
Specific application of this result is calculating other solutions of the Laplace's equation given some known solutions. For example, lets say we know that $h(x, y)$ satisfies the Laplace's equation. We can than search for $\tilde{h}(x, y)$, which also satisfies Laplace's equation, which will correspond to the other component of the complex function g. For example, we could assign that h is the real part of g, and then $g = h(x, y) + i\tilde{h}(x, y)$. Then, instead of solving second order partial differential equation again with search for a different solution, we can just apply Cauchy-Riemann conditions to find $\tilde{h}$ from h. We call $\tilde{h}$ the harmonic conjugate of h. Similarly, h is the harmonic conjugate of $\tilde{h}$, since Cauchy-Riemann conditions are equalities and hence apply in both directions.

Finding Harmonic Conjugates For example, consider $f = e^x \cos(y)$. We can see that f satisfies the Laplace's equation. How to find harmonic conjugate of f, g? We start by assuming that f is the real part of some complex function. Then, Cauchy-Riemann conditions between f and g are

$$\frac{\partial f}{\partial x} = e^x \cos(y) = \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial y} = -e^x \sin(y) = -\frac{\partial g}{\partial x}$$

From the first equation, it follows that

$$g = e^x \sin(y) + F(x)$$

where $F(x)$ is some function of x. From the second equation, it follows that

$$g = e^x \sin(y) + F'(y)$$

where $F'(y)$ is some function of y. By comparing these expressions, we determine that

$$g = e^x \sin(y)$$

which also clearly satisfies the Laplace's equation. Hence, we have found our harmonic conjugate. The complex function in this case would be $f + ig = e^x(\cos(y) + i\sin(y)) = e^x e^{iy} = e^{x+iy} = e^z$.

2.4 Conformal Mappings

Conformal mappings are maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that any two curves in the domain of F that intersect at certain point $\vec{x}_0$ at angle α intersect at that same angle α in the image of F at $F(\vec{x}_0)$. To put more meaning to these conditions, consider two curves γ_1 and γ_2. The coordinates of the points on these lines are parametrized by some real t as $x_1(t)$ and $y_1(t)$ for γ_1 and $x_2(t)$ and $y_2(t)$ for γ_2. Then, the tangential vector to γ_1 at $\vec{x}_0$ is

$$\vec{s}_1 = \left(\frac{dx_1}{dt}\bigg|_{\vec{x}_0}, \frac{dy_1}{dt}\bigg|_{\vec{x}_0} \right)$$

Similarly

$$\vec{s}_2 = \left(\frac{dx_2}{dt}\bigg|_{\vec{x}_0}, \frac{dy_2}{dt}\bigg|_{\vec{x}_0} \right)$$

is the tangential vector to γ_2 at the intersection point $\vec{x}_0$.
For the sake of brevity, I will drop the explicit marking of the point at which the derivative is taken and replace total derivatives with t by the dot notation, i.e. $\frac{dx_1}{dt}\Big|_{\vec{x}_0} = \dot{x}_1$.
The angle at which these two curves intersect is given by the dot product of the tangential vectors as

$$\cos\alpha = \frac{\vec{s}_1 \cdot \vec{s}_2}{|\vec{s}_1||\vec{s}_2|}$$

This can be determined to be, using the expressions for tangential vectors

$$\cos\alpha = \frac{\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2}{\sqrt{(\dot{x}_1^2 + \dot{y}_1^2)(\dot{x}_2^2 + \dot{y}_2^2)}}$$

Consider now general mapping F, which maps each point (x, y) to point (x', y') as

$$x' = u(x, y)$$

$$y' = v(x, y)$$

Consider now the images of γ_1 and γ_2 and their intersection at $F(\vec{x}_0)$. The tangential vector to the image of γ_1 (which I will denote as γ_1') is

$$(\vec{s}_1)' = \left(\frac{dx_1'}{dt}, \frac{dy_1'}{dt}\right)$$

where the functions $\frac{dx_1'}{dt}$ and $\frac{dy_1'}{dt}$ can be found via chain rule as

$$\frac{dx_1'}{dt} = \frac{du(x_1, y_1)}{dt} = \frac{\partial u}{\partial x}\frac{dx_1}{dt} + \frac{\partial u}{\partial y}\frac{dy_1}{dt}$$

For more brevity, I will denote the partial derivatives in the index notation, as $\frac{\partial u}{\partial x} = u_x$. Hence

$$\frac{dx_1'}{dt} = u_x\dot{x}_1 + u_y\dot{y}_1$$

Similarly

$$\frac{dy_1'}{dt} = v_x\dot{x}_1 + v_y\dot{y}_1$$

Hence

$$(\vec{s}_1)' = (u_x\dot{x}_1 + u_y\dot{y}_1, v_x\dot{x}_1 + v_y\dot{y}_1)$$

We could similarly show that the tangetial vector to the image of γ_2 is

$$(\vec{s}_2)' = (u_x\dot{x}_2 + u_y\dot{y}_2, v_x\dot{x}_2 + v_y\dot{y}_2)$$

Then, the angle at which these images intersect is given again by the dot product of these vectors as

$$\cos\alpha' = \frac{(\vec{s}_1)' \cdot (\vec{s}_2)'}{|(\vec{s}_1)'||(\vec{s}_2)'|} = \frac{(u_x\dot{x}_1 + u_y\dot{y}_1)(u_x\dot{x}_2 + u_y\dot{y}_2) + (v_x\dot{x}_1 + v_y\dot{y}_1)(v_x\dot{x}_2 + v_y\dot{y}_2)}{\sqrt{(u_x\dot{x}_1 + u_y\dot{y}_1)^2 + (v_x\dot{x}_1 + v_y\dot{y}_1)^2}\sqrt{(u_x\dot{x}_2 + u_y\dot{y}_2)^2 + (v_x\dot{x}_2 + v_y\dot{y}_2)^2}} =$$

$$= \frac{\dot{x}_1\dot{x}_2(u_x^2 + v_x^2) + \dot{y}_1\dot{y}_2(u_y^2 + v_y^2) + (\dot{x}_1\dot{y}_2 + \dot{x}_2\dot{y}_1)(u_xu_y + v_xv_y)}{\sqrt{\dot{x}_1^2(u_x^2 + v_x^2) + \dot{y}_1^2(u_y^2 + v_y^2) + 2\dot{x}_1\dot{y}_1(u_xu_y + v_xv_y)}\sqrt{\dot{x}_2^2(u_x^2 + v_x^2) + \dot{y}_2^2(u_y^2 + v_y^2) + 2\dot{x}_2\dot{y}_2(u_xu_y + v_xv_y)}}$$

We can see that if all derivatives of u and v are zero at a given point, the angle at this point is not defined. Now, we will show that if u and v satisfy the Cauchy-Riemann conditions, then the new angle α' is the same as the angle α in the domain of F. It can be in fact shown that the implication works the other way as well (i.e. that the requirement that these angles are equal leads to Cauchy-Riemann conditions on u and v), but it is harder to do (and was not presented in lectures).
The Cauchy-Riemann conditions in this index notation become $u_x = v_y$ and $u_y = -v_x$. Immediately, by the multiplication of these equations, we can determine that $u_xu_y = -v_xv_y$. Furhermore, by taking square and adding the equations together, $u_x^2 + v_x^2 = u_y^2 + v_y^2$. Therefore, the formula for $\cos\alpha'$ substantially reduces to

$$\cos\alpha' = \frac{(u_x^2 + v_x^2)(\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2)}{(u_x^2 + v_x^2)\sqrt{\dot{x}_1^2 + \dot{y}_1^2}\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} = \frac{\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2}{\sqrt{(\dot{x}_1^2 + \dot{y}_1^2)(\dot{x}_2^2 + \dot{y}_2^2)}} = \cos\alpha$$

Hence, we have shown that if the functions u and v satisfy Cauchy-Riemann conditions, the mapping F is conformal.

2.4.1 Complex Representation of Conformal Mappings

Since conformal mappings satisfy Cauchy-Riemann conditions, we can express them as a fully complex functions, taking the points in complex plane to other points in complex plane, where the mapping is a holomorphic function. Let $f(z)$ be a holomorphic function, representing our mapping, γ_1 and γ_2 our paths in the complex plane, parametrized by $z_1(t)$ and $z_2(t)$. The paths in the image are w_1 and w_2, and satisfy

$$i \in \{1, 2\} : w_i(t) = f(z(t))$$

The tangent vector to a path in a complex plane can be represented by a single complex number, analogously with definition in real plane, as

$$s_i = \frac{dz_i}{dt}$$

If the paths intersect at t_0, the angle under which they intersect is

$$\alpha = \text{Arg}\left(\frac{dz_2}{dt}\right) - \text{Arg}\left(\frac{dz_1}{dt}\right) = \text{Arg}\left(\frac{\frac{dz_2}{dt}}{\frac{dz_1}{dt}}\right)$$

evaluated at t_0 (from now on, implied).
In the image of f, the angle of intersection is

$$\alpha' = \text{Arg}\left(\frac{dw_2}{dt}\right) - \text{Arg}\left(\frac{dw_1}{dt}\right) = \text{Arg}\left(\frac{\frac{dw_2}{dt}}{\frac{dw_1}{dt}}\right)$$

where

$$\frac{dw_i}{dt} = \frac{\partial f}{\partial z}\frac{dz_i}{dt}$$

Hence, we clearly see that

$$\alpha' = \text{Arg}\left(\frac{\frac{\partial f}{\partial z}\frac{dz_2}{dt}}{\frac{\partial f}{\partial z}\frac{dz_1}{dt}}\right) = \text{Arg}\left(\frac{\frac{dz_2}{dt}}{\frac{dz_1}{dt}}\right) = \alpha$$

It may seem like we did not need f to be holomorphic, but we used the holomorphicity of f when we stated that the derivatives of f at $f(z(t_0))$ are the same for following path γ_1' and γ_2'. Hence, the mapping in between complex planes mediated by a holomorphic function is always conformal.
However, we should note that if $\frac{\partial f}{\partial z} = 0$, the angle in the image is undefined - hence the mapping is not conformal at these points. This is the more precise reformulation of the requirement that "all u_x, u_y, v_x and v_y are non-zero" from original derivation of conformal mappings.

2.4.2 Conformal Mappings of Solutions of Laplace's Equations

Since the conformal mappings correspond to holomorphic functions in complex plane and solutions of Laplace's equations are all holomorphic functions in complex plane, we can transform solutions of Laplace's equation on certain domain to solutions on other domain via the conformal mappings, since the holomorphic function of a holomorphic function is still a holomorphic function.
To present an example of this, consider a square region in real plane, $0 \le x \le a$ and $0 \le y \le a$ for some real a. In complex plane, this region is a square along the diagonal from $z = 0$ to $z = a + ai$. We might have a solution of Laplace's equation in this region, satisfying certain boundary conditions on the edges of the square. For example, for solution vanishing on the horizontal boundaries (boundaries parallel with x axis), we could have solution

$$f(x, y) = e^{kx}\sin(ky)$$

where $k = \frac{\pi}{a}$. It is important to note that generally, we might be doing some non-linear transformation map, and in such case, we should really map the dimensionless variables x and y such that $kx \to x$ and $ky \to y$. After the transformation, we can substitute back the dimensionality.
The complex function representing this solution in the complex plain in terms of dimensionless variables will be

$$F(z) = e^z = e^{x+iy}$$

with f being the imaginary part of F.
Now, consider applying general linear conformal map $z \to \zeta(z) = \alpha z + \beta$, where α and β are complex numbers. The inverse transformation is $z = \frac{\zeta - \beta}{\alpha}$, and is defined for any $\alpha \neq 0$, which is also the condition for $\zeta(z)$ to be conformal in any point on the complex plane.
The bottom boundary of the original domain (square) in complex plane can be parametrized as $z_b = at, t \in [0, 1]$. Similarly, left boundary is $z_l = iat$, right boundary is $z_r = a + iat$ and the top boundary is $z_t = ia + at$. Hence, the transformed boundaries are

$$\tilde{z}_b = \zeta(z_b) = \alpha z_b + \beta = (\alpha_r + i\alpha_i)(at) + \beta_r + i\beta_i = \alpha_r at + \beta_r + i(\alpha_i at + \beta_i)$$

where α_r is the real part of α and α_i is the imaginary part of α, and similarly for β. In the same spirit

$$\tilde{z}_l = -\alpha_i at + \beta_r + i(\alpha_r at + \beta_i)$$

$$\tilde{z}_r = \alpha_r a - \alpha_i a t + \beta_r + i(\alpha_r a t + \alpha_i a + \beta_i)$$
$$\tilde{z}_t = \alpha_r a t - \alpha_i a + \beta_r + i(\alpha_r a + \alpha_i a t + \beta_i)$$

This region is depicted in Fig. 2

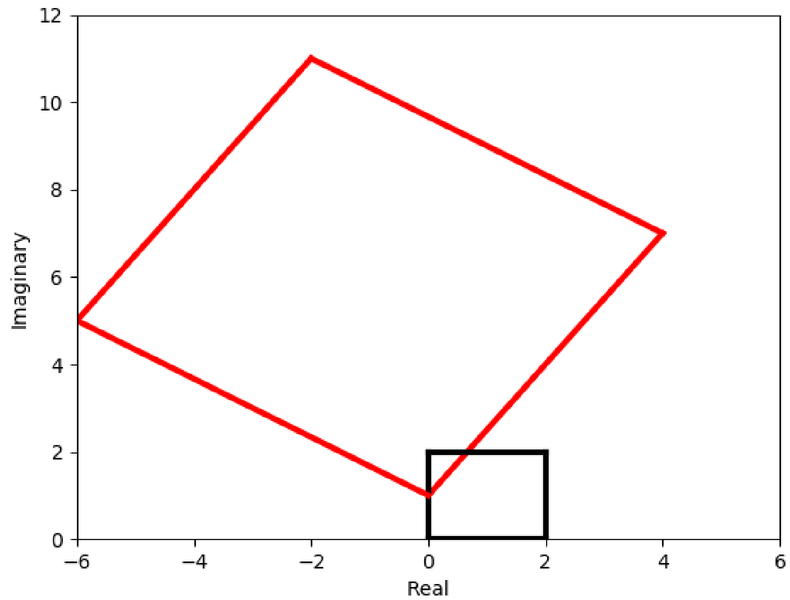

Figure 2: A conformal mapping of a square of side length 2 by a map $\zeta = (2+3i)z + i$ in the complex plane. The mapped region boundaries are in red.

The solution of the Laplace's equation on this new region with the same boundary conditions will be then

$$\tilde{F}(\zeta) = F(\zeta(z)) = e^{\alpha z + \beta} = e^{(\alpha_r + i\alpha_i)(x+iy)+\beta_r + i\beta_i} = e^{\alpha_r x - \alpha_i y + \beta_r} e^{i(\alpha_i x + \alpha_r y + \beta_i)} =$$
$$= e^{\alpha_r x - \alpha_i y + \beta_r} \left(\cos\left(\alpha_i x + \alpha_r y + \beta_i\right) + i \sin\left(\alpha_i x + \alpha_r y + \beta_i\right)\right)$$

Which has imaginary part

$$\tilde{f} = e^{\alpha_r x - \alpha_i y + \beta_r} \sin\left(\alpha_i x + \alpha_r y + \beta_i\right)$$

or converted to dimensional units

$$\tilde{f} = e^{\alpha_r k x - \alpha_i k y + \beta_r} \sin\left(\alpha_i k x + \alpha_r k y + \beta_i\right)$$

Consider now the α in the radial form, $\alpha = |\alpha|(\cos(\theta) + i\sin(\theta))$, where θ is the argument of α. Then

$$\tilde{f} = e^{|\alpha|(\cos(\theta)kx - \sin(\theta)ky) + \beta_r} \sin\left(|\alpha|(\sin(\theta)kx + \cos(\theta)ky) + \beta_i\right)$$

which we can recognize as a combination of rotation, linear scaling and translation. Therefore, any combination of these can be expressed as a linear conformal map in the complex plane. We should note that pure rotation occurs when $\beta = 0$ and $|\alpha| = 1$, pure scaling occurs when $\theta = 0$, i.e. α is real and $\beta = 0$, and pure translation occurs when $\alpha = 1$ and $\beta \neq 0$.

3 Complex Integration

When approaching the integration in complex plane, it is useful to first define what we mean by an integral. In a simplified manner, the integral is related to the anti-derivative $F(z)$ of a function $f(z)$ by the fundamental theorem of calculus

$$\int_a^b f(z)dz = F(b) - F(a)$$

where $f(z) = \frac{dF}{dz}$. However, in complex plane, the path between points a and b is not obvious, as there is no natural ordering of the complex numbers.

We can also interpret the integral as a limit of a sum

$$\int_a^b f(z)dz = \lim_{dz_i \to 0} \lim_{N\to\infty} \sum_n^N f(a + \sum_{j=1}^{n-1} dz_j)dz_n$$

This leads to interpretation of the complex integral as sort of a line integral in the complex plane. As for any line integral, we must specify the path along which to take the integral. In complex calculus, we often use term contour for the path in complex plane.
Then, we write that for contour $\Gamma(t)$ which is parametrized by some real parameter t

$$\int_\Gamma f(z)dz = F(\Gamma(t_f)) - F(\Gamma(t_i))$$

where $\Gamma(t_f) = z_f$ is the final point of the path and $\Gamma(t_i) = z_i$ is the starting point of the path. Most commonly, we will not try to discover the anti-derivative F, but rather we will try to parametrize Γ by some real t, so that we get $z = \Gamma(t) = z(t)$ and then we calculate the integral. The reasons for this will become apparent in the next subsection.
But first, we need to prove a useful theorem for complex integration. We are about to prove that

$$\left| \int_\Gamma f(z)dz \right| \le \int_\Gamma |f(z)dz|$$

To prove this, lets rewrite this in the summation limit

$$\left| \lim_{dz_i \to 0} \lim_{N\to\infty} \sum_n^N f(a + \sum_{j=1}^{n-1} dz_j)dz_n \right| \le \lim_{dz_i \to 0} \lim_{N\to\infty} \sum_n^N \left| f(a + \sum_{j=1}^{n-1} dz_j)dz_n \right|$$

We can move the absolute value inside the limit on the left hand side. Shortening $\lim_{dz_i\to 0} \lim_{N\to\infty} \sum_n^N f(a+ \sum_{j=1}^{n-1} dz_j)dz_n \iff \lim$ we than have

$$\lim \left| \sum_n^N a_n \right| \le \lim \sum_n^N |a_n|$$

where $a_n = f(a + \sum_{j=1}^{n-1} dz_j)dz_n$. But, we can prove that for any finite set of complex numbers $\{a_i\}$

$$\left| \sum_i a_i \right| \le \sum_i |a_i|$$

Rewriting $a_i = r_r e^{i\theta_i}$ in polar coordinates

$$\left| \sum_i r_i e^{i\theta_i} \right| \le \sum_i \left| r_i e^{i\theta_i} \right| = \sum_i r_i$$

Taking a square of both sides, since both sides are positive numbers

$$\left| \sum_i r_i e^{i\theta_i} \right|^2 \le \left(\sum_i r_i \right)$$

Using that the absolute value squared of a complex number is the product of the complex number and its complex conjugate, we can rewrite the left hand side. Also, expanding the square of the sum

$$\left(\sum_i r_i e^{i\theta_i} \right) \left(\sum_j r_j e^{-i\theta_j} \right) \le \left(\sum_i r_i \right) \left(\sum_j r_j \right)$$

Hence, connecting the sums and moving everything to the right hand side, we have

$$0 \le \sum_i \sum_j r_i r_j \left(1 - e^{i(\theta_i - \theta_j)} \right)$$

We can see that when $i = j$, $e^{i(\theta_i - \theta_j)} = e^0 = 1$, and therefore $1 - e^{i(\theta_i - \theta_j)} = 0$ and these terms dissappear from the summations. Furthermore, we can see that for every $i = k$ and $j = l$ pair, there exists $i = l$ and $j = k$ pair. Since the exponent is antisymmetric with exchange $i \iff j$ and the r multiplication is symmetric, we can rewrite the sums to sum over without double counting, i.e.

$$0 \leq \sum_i \sum_{j>i} r_i r_j (1 - e^{i(\theta_i - \theta_j)}) + r_j r_i (1 - e^{i(\theta_j - \theta_i)}) =$$

$$= \sum_i \sum_{j>i} 2 r_i r_j \left(1 - \frac{e^{i(\theta_i - \theta_j)} - e^{-i(\theta_i - \theta_j)}}{2}\right) = \sum_i \sum_j 2 r_i r_j (1 - \cos(\theta_i - \theta_j))$$

But since $r_i > 0$, $r_j > 0$ and $(1 - \cos(\theta_i - \theta_j)) \geq 0$, the whole expression is greater than zero, which is what we wanted to show.
Hence, taking the limit of the sum, we have our theorem,

$$\left| \int_\Gamma f(z) dz \right| \leq \int_\Gamma |f(z) dz| \tag{10}$$

3.1 Parametrization of Complex Integral

Since we integrate along a certain contour Γ in the complex plane, we can parametrize the points $z \in \Gamma$ as $z = z(t)$ where t is some real number, taken from a certain interval in certain order. For example, a straight line in a complex plane from 0 to $1 + i$ could be parametrized as $z = (1 + i)t$ where t runs from 0 to 1. Importantly, the order of the parametrization generally matters, so we must be careful about orientations/minus signs etc.
But, the existence of contour Γ does not really depend on the form of the parametrization. For example, we could have just as well chosen $z = (1 + i)t^2$ for the previous exercise, and the same path would have been covered. Does the complex integral change when we use different parametrizations?
Suppose we have two parametrizations of z along Γ, $z = w(t)$ and $z = v(q)$, where q and t are the real parameters, running from t_0 and q_0 to t_1 and q_1, respectively. The integral of some function $f(z)$ in parametrization w becomes

$$\int_\Gamma f(z) dz = \int_{t_0}^{t_1} f(w(t)) \frac{dw}{dt} dt$$

Suppose now that there exists a function that maps the interval of t onto the interval of q, $\lambda : [t_0, t_1] \to [q_0, q_1]$, that creates one-to-one correspondence. If it does not create one-to-one correspondence, than some parts of the path in the complex plane might not be covered, or might be covered multiple times.
Then, we can write

$$\int_{t_0}^{t_1} f(v(\lambda(t))) \frac{dv(\lambda(t))}{dt} dt = \int_{t_0}^{t_1} f(v(\lambda(t))) \frac{dv}{d\lambda} \frac{d\lambda}{dt} dt$$

We can than change variables to λ

$$\int_{\lambda(t_0)}^{\lambda(t_1)} f(v(\lambda)) \frac{dv}{d\lambda} d\lambda$$

And since λ here become the dummy variable, and $\lambda(t_0) = q_0$ and $\lambda(t_1) = q_1$, by substituting $\lambda(t) = v$,we have reached to the conclusion that

$$\int_{t_0}^{t_1} f(w(t)) \frac{dw}{dt} dt = \int_{q_0}^{q_1} f(v(q)) \frac{dv}{dq} dq$$

Hence any two parametrizations between the intervals of which there is a one-to-one correspondence lead to the same value of the complex integral. Therefore, the choice of the parametrization is arbitrary, and we should always try to search for the most convenient one.
We should also note that by parametrization of the integral, we effectively changed the complex integral into a vector integral in the complex plane. This means that common theorems of real integrals apply, namely linearity

$$\int_\Gamma (\alpha f(z) + \beta g(z)) dz = \alpha \int_\Gamma f(z) dz + \beta \int_\Gamma g(z) dz$$

, and opposite direction of integration produces

$$\int_{t_0}^{t_1} f(w(t)) \frac{dw}{dt} dt = -\int_{t_1}^{t_0} f(w(t)) \frac{dw}{dt} dt$$

Also, the Green's theorem in plane applies to complex integrals, with interesting consequences.

3.2 Cauchy's Integral Theorem

Since the complex integral is essentially integral along a line in a complex plane, we can use Green's theorem in plane (2D Stokes' theorem) to determine more about the integral, without specifying the function f. Green's theorem states that for a vector field $\vec{f}(x,y) = u(x,y)\hat{i} + v(x,y)\hat{j}$, the integral along a closed line in the x, y plane in the anti-clockwise direction leads to

$$\oint_{\partial S} \vec{f} \cdot d\hat{l} = \oint_{\partial S} udx + vdy = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{y} \right) dS$$

where ∂S is the closed boundary of simple connected area S, $d\hat{l}$ is the vector element along this boundary and dS is the element of the area. If the direction is clockwise, the signs on the result are changed.
Our complex integral can be then rewritten as (using ∂S instead of Γ)

$$I = \oint_{\partial S} f(z)dz = \oint_{\partial S} (u+iv)(dx+idy) = \oint_{\partial S} (udx - vdy) + i(udy + vdx) =$$

$$= \oint_{\partial S} (udx - vdy) + i \oint_{\partial S} (udy + vdx)$$

These are both real integrals, and therefore, we can apply the Green's theorem in plane, as stated above.

$$I = \iint_S \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \right) dS + i \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dS$$

But, comparing this to the Cauchy-Riemann conditions, we obtain that IF the function is analytical everywhere across the region S (i.e. Cauchy-Riemann conditions apply at every point of S), then

$$I = \iint_S \left(-\frac{\partial v}{\partial x} - \left(-\frac{\partial v}{\partial x} \right) \right) dS + i \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dS = 0$$

Therefore, we have arrived at Cauchy's integral theorem, which states that for a function f analytical across the simply connected region S

$$\oint_{\partial S} f(z)dz = 0 \tag{11}$$

Important condition for this integrar equation to apply is the analyticity of the f everywhere across the region S and the fact that S has to be simply connected. If either of these conditions is not satisfied, then the theorem does not apply

3.2.1 Integral of 1/z

Consider now a specific integral for $f(z) = \frac{1}{z - z_0}$ along a contour Γ that encloses point z_0, where f is not defined. Besides this point, f is always analytical.
We can separate the integral into three integrals, as shown in Fig. 3
Hence

$$\oint_\Gamma f(z)dz = \oint_{\Gamma_1} f(z)dz + \oint_{\Gamma_2} f(z)dz + \oint_{\Gamma_3} f(z)dz$$

Since Γ_1 and Γ_2 are closed contours and f is analytical at every point closed by Γ_1 and Γ_2 respectively, the integrals along these contours go to zero. Hence, we are left with

$$\oint_\Gamma f(z)dz = \oint_{\Gamma_3} f(z)dz$$

Parametrizing $z = z_0 + Re^{i\theta}$, where R is the radius of the circle Γ_3 and θ runs from 0 to 2π, we have

$$\oint_0^{2\pi} \frac{1}{z_0 + Re^{i\theta} - z_0} Rie^{i\theta} d\theta = \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

Hence

$$\oint_\Gamma \frac{1}{z - z_0} dz = 2\pi i \tag{12}$$

This is not zero, which is contributed to the fact that f is not analytical within Γ_3. Note that, if we carried out the integral in opposite (clockwise) direction, we would obtain simply $-2\pi i$.

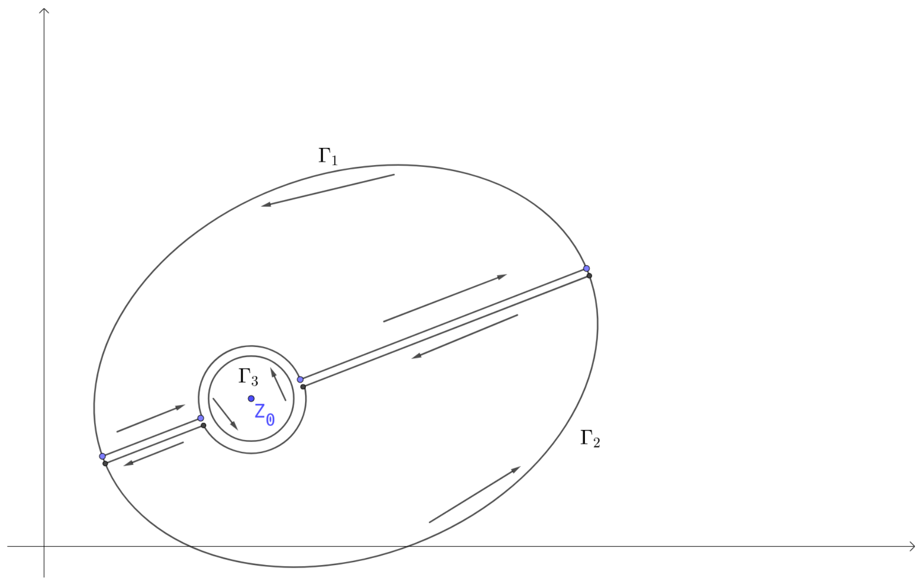

Figure 3: A general contour around point z_0 is separated into a circle Γ_3, centered on z_0, and two other closed contours, Γ_1 and Γ_2. In the limit when the separation between these contours goes to zero, the sum of the integral along these contours is the integral over the big contour Γ, in this figure represented by the big ellipse.

3.2.2 Relation to Primitive Functions

The previous result might look fairly obscure given the fundamental theorem of calculus, which states that

$$\int_a^b f(z)dz = F(b) - F(a)$$

Hence, for the same starting and ending point, this seems like it should be always zero, not just in the case of analytical functions. The complication here is the somewhat non-uniqueness of some primitive functions in complex plane. For the discussed example of $f(z) = \frac{1}{z-z_0}$, the primitive function is natural logarithm in the complex plane, which is not uniquely defined - consider number $z = re^{i\theta}$. Same number is clearly represented even by $z = re^{i(\theta+2n\pi)}$ where n is natural number, as $e^{i\theta}$ is periodic in θ with period length 2π. Now, taking logarithm of $\frac{z}{r}$

$$\log\left(\frac{z}{r}\right) = \log\left(e^{i\theta}\right) = i\theta = \log\left(e^{i(\theta+2\pi)}\right) = i\theta + 2n\pi i$$

We therefore see that the logarithm is defined up to a constant addition of $2n\pi i$. We than say that the logarithm function has branches/is not single valued. We can then see that the integral over a closed contour corresponds to the difference of the primitive functions after travelling along this closed contour. For the logarithm, after travelling one circle, we effectively added 2π to the angle θ, and hence

$$\log\left(re^{i(\theta+2\pi)}\right) - \log\left(re^{i\theta}\right) = i\theta + 2\pi i - i\theta = 2\pi i$$

, which is what we have found before.

3.3 Cauchy's Integral Formula

Important result follows from the integral of $frac1z - z_0$. Starting from the equation for evaluation of this integral

$$\oint_\Gamma \frac{1}{z-z_0}dz = 2\pi i$$

we can multiply both sides by value of some analytical single valued function f (analytical across region enclosed by Γ) at point z_0

$$f(z_0)\oint_\Gamma \frac{1}{z-z_0}dz = 2\pi i f(z_0)$$

$$\oint_\Gamma \frac{f(z_0)}{z-z_0}dz = 2\pi i f(z_0)$$

$$2\pi i f(z_0) = \oint_\Gamma \frac{f(z_0)+f(z)-f(z)}{z-z_0}dz = \oint_\Gamma \frac{f(z)}{z-z_0}dz - \oint_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz$$

Since $f(z)$ is analytical across region bounded by Γ, $\frac{f(z)-f(z_0)}{z-z_0}dz$ is also analytical everywhere except for the point z_0. Therefore, we can divide the second integral exactly as in Fig. 3. The radius of the circle can be arbitrary, so we can choose a very small radius, effectively setting taking the limit $z \to z_0$. But, we can notice that the integrand of the second integral has a form of derivative. Therefore, for a very small circle

$$\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz = \oint_{\Gamma_3} \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}dz = \oint_{\Gamma_3} f'(z)dz$$

But, since f is single valued, the integral from some starting point a

$$\oint_{\Gamma_3} f'(z)dz = f(a) - f(a) = 0$$

Therefore, we are left with

$$2\pi i f(z_0) = \oint_\Gamma \frac{f(z)}{z-z_0}dz$$

Now, consider taking the n-th derivative with respect to z_0 of both sides. Then

$$2\pi i \frac{d^n f(z_0)}{dz_0^n} = \frac{d^n}{dz_0^n}\oint_\Gamma \frac{f(z)}{z-z_0}dz$$

Since we are integrating with respect to z, not z_0, and since we the variable in the derivative is a dummy variable, we can rewrite this as

$$2\pi i \frac{d^n f}{dz^n}\Big|_{z_0} = \oint_\Gamma \frac{d^n}{dz_0^n}\left(\frac{f(z)}{z-z_0}\right)dz = \oint_\Gamma \frac{d^{n-1}}{dz^{n-1}}\left(\frac{f(z)}{(z-z_0)^2}\right)dz =$$

$$= \oint_\Gamma \frac{d^{n-2}}{dz^{n-2}}\left(\frac{2f(z)}{(z-z_0)^3}\right)dz = ... = \oint_\Gamma \frac{n!f(z)}{(z-z_0)^{n+1}}dz = n!\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Hence, we have

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!}\frac{d^n f}{dz^n}\Big|_{z_0} \tag{13}$$

This is the Cauchy integral formula, a major result of calculus of complex functions.

3.3.1 Self Consistency with Taylor Series

Suppose that we expand $f(z)$ in proximity of z_0 into Taylor series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

where a_j are the constant Taylor coefficients. Substituting this into the integral in Cauchy integral formula leads to

$$\oint_\Gamma \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2}{(z-z_0)^{n+1}}dz = \oint_\Gamma \left(\sum_{j=0}^{\infty} \frac{a_j}{(z-z_0)^{n+1-j}}\right)dz =$$

$$= \oint_\Gamma \left(\sum_{j=0}^{n} \frac{a_j}{(z-z_0)^{n+1-j}}\right)dz + \oint_\Gamma \left(\sum_{j=n+1}^{\infty} a_n(z-z_0)^{j-n-1}\right)dz$$

The integrand in the second integral is a polynomial, which is analytical everywhere, and therefore the integral goes to zero. The first integral can be rewritten as sum of integrals

$$\oint_\Gamma \left(\sum_{i=0}^{n} \frac{a_j}{(z-z_0)^{n+1-j}}\right)dz = \sum_{j=0}^{n}\oint_\Gamma \frac{a_j}{(z-z_0)^{n+1-j}}dz$$

Each of these integrals is equivalent to the integral along a circle centered on z_0, as shown before in Fig. 3. Parametrizing $z = z_0 + re^{i\theta}$, this becomes

$$\sum_{j=0}^{n} \oint_0^{2\pi} \frac{a_j}{r^{n+1-j} e^{i(n+1-j)\theta}} rie^{i\theta} d\theta = \sum_{j=0}^{n} \frac{ia_j}{r^{n-j}} \oint_0^{2\pi} e^{-i(n-j)\theta} d\theta =$$

$$= \sum_{j=0}^{n-1} \frac{ia_j}{r^{n-j}} \left[\frac{-1}{i(n-j)} e^{-i(n-j)\theta} \right]_0^{2\pi} + 2\pi i a_n$$

where δ_{n0} is the Kronecker delta (at $j = n$, the integral evaluates to 2π) But since $e^{i\theta}$ is periodic in θ with period 2π and $n - j$ is a natural number, the terms in square brackets evaluate to zero. We are then left with

$$\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz = 2\pi i a_n$$

From the Taylor series, we know that $a_n = \frac{1}{n!} \frac{d^n f}{dz^n}\Big|_{z_0}$, and thus we have shown that

$$\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz = 2\pi i a_n = \frac{2\pi i}{n!} \frac{d^n f}{dz^n}\Big|_{z_0}$$

which is consistent with our previous result.

3.3.2 Cauchy's bound on function derivatives

Consider now Γ to be a circle of radius R centered on z_0. The absolute value of the nth derivative of function f can be estimated from Cauchy's integral formula

$$\left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| = \left| \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \right| = \frac{n!}{2\pi} \left| \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \right|$$

Using (10)

$$\left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| \le \frac{n!}{2\pi} \int_\Gamma \frac{|f(z)|}{|z - z_0|^{n+1}} |dz|$$

Since Γ is a circle of radius R centered on z_0, we can parametrize $z = z_0 + Re^{i\theta}$, and thus $|z - z_0| = R$ and $|dz| = Rd\theta$

$$\left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| \le \frac{n!}{2\pi} \int_\Gamma \frac{|f(Re^{i\theta})|}{R^{n+1}} R d\theta \le \frac{n!}{2\pi} \int_\Gamma \frac{\max_\Gamma |f(Re^{i\theta})|}{R^n} d\theta$$

where $\max_\Gamma |f(Re^{i\theta})|$ is the maximum value of $|f(Re^{i\theta})|$ along Γ (for θ from 0 to 2π). Since this is only some number, we can assign $M = \max_\Gamma |f(Re^{i\theta})|$ and thus

$$\left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| \le \frac{n! M}{2\pi R^n} \int_0^{2\pi} d\theta$$

$$\left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| \le \frac{n! M}{R^n}$$

Hence, if function $f(z)$ is bounded along the path Γ, so are all of its derivatives.

3.3.3 Liouville's Theorem

By taking a special case of function f analytic over entire complex plane (called an entire function) that is also bounded as we move $R \to \infty$, we have

$$\forall n > 0 : \left| \frac{d^n f}{dz^n}\Big|_{z_0} \right| \le 0$$

But this means that all the derivatives of f are zero, hence the only bounded entire function is a constant function.

3.4 Series Expansions of Complex Functions

We have seen that for closed contour integral of a function, only terms of form $\frac{1}{z-z_0}$ contribute to the value of the integral. It is therefore desirable to devise a way how to expand a complex function into series of terms of this form. Two ways of doing this are discussed - the geometrical series and Laurent series.

3.4.1 Geometrical Series

Geometrical series formula dictates that for $z : |z| < 1$

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$$

This enables are to determine the series expansion of any function of type $\frac{1}{1-z}$. But, the expansion is only valid for $|z| < 1$.
Consider, for example, a function $f(z) = \frac{1}{(z-2)(z-3i)}$ can be expanded as

$$f = \frac{1}{(z-2)(z-3i)} = \frac{\frac{1}{2-3i}}{z-2} - \frac{\frac{1}{2-3i}}{z-3i} = \frac{1}{2-3i}\left(\frac{1}{3i}\frac{1}{1-\frac{z}{3i}} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}\right)$$

For $|z| < 2$, $|z/2| < 1$ and $|z/(3i)| < 1$, hence both fractions can be expanded

$$f = \frac{1}{2-3i}\left(\frac{1}{3i}\sum_{j=0}^{\infty}\left(\frac{z}{3i}\right)^j - \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^i\right)$$

It should be noted that we can obtain the series expansion even outside the classical region of convergence. In this region, $|z| > 1$, but $1/|z| < 1$, hence we could write.

$$f = \frac{1}{1-z} = \frac{1}{z}\frac{1}{\frac{1}{z}-1} = \frac{-1}{z}\frac{1}{1-\frac{1}{z}} = \frac{-1}{z}\sum_{j=0}^{\infty}\frac{1}{z^j} = \frac{-1}{z}\sum_{j=0}^{\infty} z^{-j}$$

We now see that we are summing over negative powers of j. This concept can be generalized in new alternative to Taylor series for complex numbers, called Laurent series.

3.4.2 Laurent Series

We would like to capture the behaviour of a function in a series expansion similar to Taylor series. But, since the only values contributing to contour integrals are from the divergent/non-analytic parts of otherwise analytical functions, we would like to somehow preserve the information about the non-analycity in the series. The Laurent series are a natural way of doing this, by including the negative powers in Taylor expansion, so that a function f can be expanded around some z_0 as

$$f = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

To find out coefficients a_j, we must however use a slightly different approach than for Taylor series. In this case, we will only consider the case when the minimum value j can take is $-m$, where m is some positive integer. Then, we say that the function f has pole of order m at z_0. Consider then multiplying the function f by $(z-z_0)^m$.

$$f(z)(z-z_0)^m = \sum_{j=-m}^{\infty} a_j(z-z_0)^j(z-z_0)^m = \sum_{j=-m}^{\infty} a_j(z-z_0)^{j+m} = \sum_{j=0}^{\infty} a_{j-m}(z-z_0)^j$$

This is the classical description of Taylor series, and we can therefore use formula for Taylor series elements to obtain coefficients a_{j-m}

$$a_{j-m} = \lim_{z\to z_0}\frac{1}{j!}\frac{d^j}{dz^j}\left(f(z)(z-z_0)^m\right)\Big|_{z=0} \tag{14}$$

For this formula to work, it is important that m is truly the highest order of poles of f at z_0. As an example, consider expansion of $f(z) = \frac{1}{\sin(z)}$ around 0. To find the order of the pole of f at 0, try taking a limit of $f(z)(z-0)^j$ with progressively increasing j. For $j < m$, the limit will diverge (as there will be still some $\frac{1}{(z-0)^{m-j}}$ left in leading pole term). For $j = m$, there will be some finite value of the limit, equal to a_{-m}. For $j > m$, the limit will go to zero (as there will be some extra $(z-z_0)^j - m$). So, first we try $j = 1$.

$$\lim_{z\to 0} \frac{1}{\sin z}(z-0) = \lim_{z\to 0} \frac{z}{\sin z} = \lim z \to 0 \frac{1}{\cos z} = 1$$

where we used L'Hopitals rule (if both numerator and denominator approach zero in the limit, the value of the limit is equal to the limit of the ratio of the derivatives of numerator and denominator, respectively), which was introduced in the first year. Hence we have a finite value, suggesting that the pole at 0 will be of order 1. We can quickly check that second order term goes to zero, i.e.

$$\lim_{z\to 0} \frac{1}{\sin z}(z-0)^2 = \lim_{z\to 0} \frac{z^2}{\sin z} = \lim_{z\to 0} \frac{2z}{\cos z} = 0$$

Hence, we can write that around 0

$$\frac{1}{\sin z} = \sum_{j=-1}^{\infty} a_j (z-0)^j = \sum_{j=0} (z)^{j-1} \lim_{z\to z_0} \frac{1}{j!} \frac{d^j}{dz^j} \left(\frac{z}{\sin z}\right)\Big|_{z=0}$$

Inversly to the poles, we could say that a function $G(z)$ has a zero of order m at z_0, if all derivatives up to m-th derivative at z_0 are zero. In this case

$$\frac{1}{G(z)} = \frac{1}{\sum_{j=0}^{m} a_j (z-z_0)^j}$$

has pole of order m.

3.4.3 Convergence of Laurent Series

For Laurent series to converge successfully, we need that z_0 is an isolated singularity, i.e. we are able to draw a circle of some finite, even though perhaps small radius around the singularity such that all points on the circle are points where f does not diverge.

3.5 Calculus of Residues

Consider now a function $f(z)$ expandable as a Laurent series that has a pole of order m around z_0, so that

$$f(z) = \sum_{j=0}^{\infty} a_{j-m} (z-z_0)^{j-m}$$

Consider now taking a closed contour integral around circle centered on z_0. The value of the integral is (using Cauchy's integral formula)

$$\oint_\Gamma f(z)dz = \oint_\Gamma \left(\sum_{j=0}^{\infty} a_{j-m}(z-z_0)^{j-m}\right) dz = \sum_{j=0}^{\infty} \oint_\Gamma \frac{a_{j-m}}{(z-z_0)^{m-j}} dz =$$

$$= \sum_{j=0}^{m} \frac{2\pi i}{(m-j)!} \frac{d^{m-j-1}}{dz^{m-j-1}} a_{j-m}$$

where terms for $j > m$ dissappear as $m - j < 0$ and the integrals become integrals of polynomials, which always go to zero, as polynomials are entire functions. Furthermore, since a_{j-m} coefficients are constants, their derivatives are zero as well. Hence, we are left with only one term, when we are not taking the derivative of the coefficients, which corresponds to $m - j - 1 = 0$ or $j = m - 1$. Hence

$$\oint_\Gamma f(z)dz = \frac{2\pi i}{(m-(m-1))!} a_{m-1-m} = 2\pi i a_{-1} \tag{15}$$

Hence, the whole complex integral can be in fact solved by looking at a single component of the Laurent series expansion of the function $f(z)$. This single component a_{-1} is given a specific name to highlight its importance - its called the residue of function $f(z)$ at z_0, and we write

$$\text{Res}[f(z), z_0] = a_{-1}$$

For functions with finite pole order, the complex integration then becomes exercise in complex differentiation, which is generally much easier to acomplish.
One more generalization is possible by considering a case when several distinct isolated poles are present inside the area enclosed by Γ. Then, we can use similar approach as illustrated in Fig. 3 to isolate each singularity into a small circle around it. Let there be n of these circles, indexed by j so that we can call these circles Γ_j and the poles z_j. Then, the integral over Γ of $f(z)$ becomes

$$\oint_\Gamma f(z)dz = \sum_{j=1}^{n} \oint_{\Gamma_j} f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}[f(z), z_j]$$

Therefore, we have Cauchy's residue theorem

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}[f(z), z_j] \tag{16}$$

This is pretty much the intellectual peak of the course. In the remaining sections, we discuss some specific applications of the Cauchy's residue theorem.
Remember that for this to work as written, the contour integration has to be carried out in the positive (anti-clockwise) direction, otherwise the expression on the right has the opposite sign.

3.5.1 Simple pole functions

Consider a function of type $f(z) = \frac{P(z)}{Q(z)}$, where $Q(z)$ has a first order zero at z_0 (a simple zero) and $P(z_0) \neq 0$, so that $f(z)$ has a first order pole (a simple pole) at z_0. The residue of the function at z_0 is

$$\text{Res}\left[\frac{P(z)}{Q(z)}, z_0\right] = \frac{1}{0!} \lim_{z\to z_0} \frac{P(z)}{Q(z)}(z - z_0)$$

where I used (14) for a_{-1} with $m = 1$ (a simple pole) and $j = 0$. Since $Q(z_0) = 0$ by our definition, we can subtract it in the denominator, which leads to

$$\text{Res}\left[\frac{P(z)}{Q(z)}, z_0\right] = \lim_{z\to z_0} \frac{P(z)}{Q(z) - Q(z_0)}(z - z_0) = \lim_{z\to z_0} \frac{P(z)}{\frac{Q(z)-Q(z_0)}{z-z_0}} = \frac{P(z_0)}{Q'(z_0)}$$

where $Q'(z)$ is the first derivative of Q with respect to z. This is a useful simplification for some residue calculations.

3.5.2 Harmonic Integrals

Consider a rational function of sines and cosines $R(\cos\theta, \sin\theta)$ in a real integral

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$$

We can view this integral as a contour integral in the complex plane along the unit circle. Parametrizing $z = e^{i\theta}$, so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(\frac{z}{-}\frac{1}{z}\right)$ and $d\theta = -i\frac{1}{z}dz$. Hence, the integral becomes

$$I = -i \oint_{\text{unit circle}} \frac{R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(\frac{z}{-}\frac{1}{z}\right)\right)}{z} dz$$

The integrand is a analytical function everywhere except at $z = 0$ and at the poles of $R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(\frac{z}{-}\frac{1}{z}\right)\right)$. Therefore, we can apply Cauchy's residue theorem to calculate this integral easily.

3.5.3 Inverse square integrals

Consider integral of form

$$I = \int_{-\infty}^{\infty} \frac{f(x)}{x^2 + a^2} dx$$

where $f(x)$ is some function which grows as it approaches infinity slower than $\frac{1}{x}$ and a is some real number. This integral can be interpreted as

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{f(x)}{x^2+a^2} dx$$

We can try to convert the inner integral into the closed contour integral by closing the contour by a half-circle in one of the half planes of complex plane. This half circle will have radius R and will go from $z = R$ to $z = -R$ through, for the sake of correct orientation of the contour, $z = iR$. This will however introduce extra part to the integral, the part when we travel along the half circle Γ_c. Hence, we can write that for the whole closed contour Γ

$$\int_\Gamma \frac{f(z)}{z^2+a^2} dz = \int_{-R}^{R} \frac{f(z)}{z^2+a^2} dz + \int_{\Gamma_c} \frac{f(z)}{z^2+a^2} dz$$

But, we can estimate the value of the second integral as

$$\left| \int_{\Gamma_c} \frac{f(z)}{z^2+a^2} dz \right| \leq \int_{\Gamma_c} \frac{|f(z)||dz|}{|z^2+a^2|} = \int_0^\pi \frac{|f(Re^{i\theta})|Rd\theta}{|R^2e^{2i\theta}+a^2|} =$$

$$\int_0^\pi \frac{|f(Re^{i\theta})|Rd\theta}{|(Re^{i\theta}+ia)(Re^{i\theta}-ia)|}$$

In the limit of $R \to \infty$, the integrand tends to $\frac{f(Re^{i\theta})}{R}$, which, if f grows slower than z at all points as z goes to infinity, tends to zero. Therefore, this part of the integral as a whole tends to zero, and we have

$$I = \int_\Gamma \frac{f(z)}{z^2+a^2} dz$$

and we can use Cauchy's residue theorem and count the residues in the upper plane, which there is one of (either at ia or at $-ia$).
I will end the discussion of the applied integrals here. Of course, there are many types of integrals we could furher discover to be solvable by calculus of residues, but I refer the readers to the examples problems, where there were many problems of this nature solved.