PX3A3 Electrodynamics

Warwick Physics Society

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Contents

Acknowledgements and Credits

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- Enoch Ko: Chapters on Relativity, the Dipole and Retarded Potentials
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These notes follow Prof David Leadley's 2023-24 lectures for PX3A3 Electrodynamics closely. We have tried our best to follow his notation and any differences will be made clear. Please use the current lecturer's notation wherever possible.
Other useful resources:

- Prof Sandra Chapman's book "Core Electrodynamics"
- Prof David Tong's lecture notes: http://www.damtp.cam.ac.uk/user/tong/teaching.html on *Vector Calculus* and *Electromagnetism*
- David Griffiths's *Introduction to Electrodynamics*.
- John David Jackson's *Classical Electrodynamics*

Notation and conventions

- Regular 3-vectors in $\mathbb{R}$ are denoted in **boldfont**, e,g, the position vector $(x_1, x_2, x_3) = \mathbf{x}$.
- 4-vectors and tensors will be denoted in index notation. For example, the 4-position $(x_0, x_1, x_2, x_3) = x^\mu$

All equations and content in boxes such as

Important box

$$\text{This is a math box} \tag{0.1}$$

or

$$\text{Another math box} \tag{0.2}$$

are denoted as **important information** and you should ensure the surrounding ideas relating to the content/equation are well understood to get high marks.

1 Tips for this module

In no particular order of importance

- Get used to index notation.
- Know how to do vector calculus and some vector identities.
- Practice problem sheets and past papers. Try some questions from textbooks.

You do not need to memorise most of the equations nor do you need to memorise integral results - integrals will be given in the exam and so will a bunch of tensors and equations. See past papers in PX3A3 and PX384 (old version of this module) as an indicative guide to what doesn't need memorising.

Things worth memorising however:

- Maxwell's equations in free space
- The vector and scalar potentials $\mathbf{A}, \phi$ respectively
- Lorentz transformations
- 4-vectors: 4-position, 4-momentum, 4-wavevector etc.

2 Electromagnetism and Special Relativity

God said, "Let there be light!"
And so, there was Maxwell and his equations.
Also, Tim Gershon was there. Somehow.

2.1 Revision: The Maxwell Equations & EM Waves

Recall from PX285 Statistical Mechanics, Electromagnetic Theory and Optics:

Maxwell's Equations (in free space)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{M1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{M4}$$

Equivalently, we can rewrite the first and fourth equation as $\nabla \cdot \mathbf{D} = \rho$ and $\nabla \times \mathbf{H} = \mathbf{J} + \partial_t \mathbf{D}$ respectively, where $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$ (in free space).

Note: We will not consider dielectrics in PX3A3, thus one can always use the above relations.

Using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we also obtained that in free space,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0} \approx 3 \times 10^8 \text{ ms}^{-1}$ is the speed of light in vacuum, showing that light is in fact electromagnetic waves(!).

2.2 Revision: Energy in EM Fields and Waves

Again recalling from PX263, conservation of energy gives

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_f = \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) = \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

where we can identify $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ as the *Poynting vector*, and in linear, isotropic materials ($\mathbf{H} = \mathbf{B}/\mu_0$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$), we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right) \implies u = \frac{1}{2} \left(\frac{B^2}{\mu_0} + \epsilon_0 E^2 \right)$$

For waves, we let $\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$ and $\mathbf{B} = \mathbf{B}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$ such that $\nabla \mapsto -i\mathbf{k}$ and $\partial_t \mapsto i\omega$. The Maxwell equations (in free space) then becomes

$$\mathbf{k} \cdot \mathbf{E} = 0 \tag{2.3}$$

$$\mathbf{k} \cdot \mathbf{B} = 0 \tag{2.4}$$

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \tag{2.5}$$

$$\mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2} \mathbf{E} \tag{2.6}$$

Thus the *power flow* (aka *energy flux*) is given by the time-averaged Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = EH\,\hat{\mathbf{k}} = \frac{E^2}{Z} \hat{\mathbf{k}} \implies \langle \mathbf{S} \rangle = \frac{1}{2} \frac{E_0^2}{Z} \hat{\mathbf{k}}$$

with *impedance* Z defined as

$$Z = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} = Z_0\sqrt{\frac{\mu_r}{\epsilon_r}}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\,\Omega$ is the *impedance of free space.*

2.3 EM in terms of Potentials

Looking at (M2), one might define a magnetic field via

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{2.7}$$

where $\mathbf{A}$ is some vector field known as the *magnetic vector potential.* Note that this automatically satisfies (M2) since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for all vector fields (in $\mathbb{R}^3$)[1].

Similarly, since

$$0 = \nabla \times \mathbf{E} + \frac{\partial(\nabla \times \mathbf{A})}{\partial t} = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)$$

we can define a *scalar potential*[2] via

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi$$

Rearranging, we have

$$\boxed{\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}} \tag{2.8}$$

This reduces the 6 components of $\mathbf{E}$ and $\mathbf{B}$ to the 4 components of $\mathbf{A}$ and ϕ!

Example 1.

Consider a uniform magnetic field

$$\mathbf{B} = (0, 0, B_0) = \nabla \times \mathbf{A} = (\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x)$$

Here, $\mathbf{A}$ can be $B_0(0, x, 0)$, $B_0(-y, 0, 0)$, or $\frac{B_0}{2}(-y, x, 0)$. In particular, there is a freedom to choose between different potentials for a given field (though the third one is usually preferred for its symmetry).

Now, note that for any scalar field ψ, $\nabla \times (\nabla\psi) = 0$. Therefore, for any magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\mathbf{B}' = \nabla \times (\mathbf{A} + \nabla\psi) = \mathbf{B}$$

i.e. we can arbitrarily choose ψ and have $\mathbf{B}$ expressed in terms of $\mathbf{A}' \equiv \mathbf{A} + \nabla\psi$.

However, from (2.8),

$$-\mathbf{E} = \nabla\phi' + \frac{\partial \mathbf{A}'}{\partial t} = \nabla\phi' + \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial(\nabla\psi)}{\partial t}$$

Therefore, for $\mathbf{E}$ to remain unchanged, we require

$$\nabla\phi' = \nabla\phi - \frac{\partial(\nabla\psi)}{\partial t} \implies \boxed{\phi' = \phi - \frac{\partial \psi}{\partial t}} \tag{2.9}$$

[1] In fact, for any "physical" divergence-free vector fields (smooth in $\mathbb{R}^3$ with compact support, in this case a magnetic field $\mathbf{B}$), *Helmholtz's theorem* guarantees that one can always define such a vector potential $\mathbf{A}$.

[2] The existence of ϕ is again guaranteed by *Helmholtz's theorem.*

The selection of such scalar field ψ is called "choosing the *gauge*", which is usually expressed in terms of $\nabla \cdot \mathbf{A}$. A useful choice for relativity is the *Lorenz gauge*[3][4]

Lorenz gauge

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{2.10}$$

Substituting (2.8) into (M1), we get

$$-\nabla^2 \phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \frac{\rho}{\epsilon_0} \tag{2.11}$$

Similarly, substituting (2.7) and (2.8) into (M4), we get

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c^2}\left[\frac{\partial(\nabla\phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right] = \mu_0 \mathbf{J} \tag{2.12}$$

Finally, substituting the Lorenz gauge (2.10) into (2.11) and (2.12) gives

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{2.13}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{2.14}$$

Defining the *d'Alembertian* by

$$\boxed{\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2} \tag{2.15}$$

we can rewrite the equations as

Maxwell's Equations in terms of potentials

$$\Box \phi = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \Box \mathbf{A} = \mu_0 \mathbf{J} \tag{2.16}$$

These two equations contain all the information we had in Maxwell's equations. In particular, in free space ($\rho = 0, \mathbf{J} = 0$), we recover wave equations for $\mathbf{A}$ and ϕ ($\Box\phi = 0$ and $\Box\mathbf{A} = 0$), which gives a description of electromagnetic waves (of speed c, as expected).

2.4 EM Fields in Materials

Note: This is mainly PX285 revision, and serves as a clarification between $\mathbf{E}$ and $\mathbf{D}$; it is not particularly related to the rest of this module as we will only consider EM fields in free space.

Recall that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla \cdot \mathbf{D} = \rho_{free}$$

where

$$\rho = \rho_{free} + \rho_p \implies \nabla \cdot (\epsilon_0 \mathbf{E}) = \nabla \cdot \mathbf{D} - \nabla \cdot \mathbf{P} \implies \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

Usually (for linear, isotropic materials), we have

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \implies \mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$$

[3] NOT Lorentz, typo in Chapman's *Core Electrodynamics*!

[4] In electrostatics this reduces to the *Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$ (useful for plane waves), which gives Poisson equations for both ψ and $\mathbf{A}$:

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

where $\epsilon_r = 1 + \chi$. For non-linear and/or anisotropic materials, this generalizes to the non-linear tensor relation (Einstein notation, summing over repeated indices)

$$P_i = \epsilon_0 \chi_{ij} E_j + \epsilon_0^2 \chi_{ijk}^{(2)} E_j E_k + \cdots$$

Again, we (fortunately) will not consider such complications in this module as we will only be looking at EM fields in free space.

2.5 Prelude to Special Relativity

In 1887, Michelson-Morley disproved the existence of the aether (medium for propagation of light waves) by improving upon the Michelson interferometer. In 1892, Lorentz first derived the *Lorentz Force Law*

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{2.17}$$

bridging Maxwell's EM fields to Newtonian mechanics. Later in the same year, as an attempt to reconcile the aether with Michelson-Morley, he proposed a coordinate transformation for a frame of reference moving at velocity v in the $+x$ direction:

$$\begin{aligned} t' &= \gamma(t - \frac{v}{c^2}x) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where he derived much later in 1903 that

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = (1 - \beta^2)^{-1/2}} \tag{2.18}$$

where $\beta := v/c$. However, as we know, it wasn't until later in 1905 that Einstein provided the correct physical interpretation for this transformation.

2.6 Special Relativity

In his paper *The Electrodynamics of Moving Bodies* (1905), Einstein reconciled Lorentz Transformation with mechanics by introducing a new understanding of time, under which Gallilean transformation and Newtonian mechanics work as approximations for the case $v/c \ll 1$.

Later in 1907, Minkowski[5] coined the term *spacetime* (more formally the *Minkowski space*[6]), a 4-dimensional object on which events are points with coordinates

$$\boxed{x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)}$$

with indices deliberately written as superscripts to denote *contravariant* rank-1 tensors (vectors); we shall introduce its counterpart (*covariant* vectors) later, where indices are written as subscripts (e.g. x_μ).

Note that we are free to choose the origin of each axis and the orientation of the spatial axes via a rotation matrix, typically (rotating by θ around the z-axis),

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[5] Einstein's PhD supervisor!

[6] Topologically $\mathbb{R}^4$, endowed with the *Minkowski inner product* $v \cdot w = \eta(v, w) := \eta_{\mu\nu} v^\mu w^\nu$.

We can also change our frame of reference via a *Lorentz Boost*

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

or equivalently,

$$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}}$$

where

$$\Lambda := \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\alpha := \tanh^{-1}\beta$ (such that Λ looks like a rotation matrix)[7].

Definition 2.1. $\alpha = \tanh^{-1}\beta$ *is known as the* ***rapidity***

We also define

Definition 2.2. *The* ***metric distance*** *(***Minkowski norm** squared*)*

$$s^2 := (ct)^2 - x^2 - y^2 - z^2$$

We often consider the difference between two events (*Minkowski inner product*[8], aka the *relativistic dot product*) given by

Definition 2.3.

$$ds^2 := (cdt)^2 - dx^2 - dy^2 - dz^2$$

This quantity ds^2 is usually referred to as the ***spacetime interval****, the difference between 2 spacetime events.*

This is again Lorentz invariant. Here, the opposite signature of time and spatial coordinates gives three cases (inside, on, and outside *light cone*):

$$ds^2 > 0\text{: } \textit{time-like} \text{ interval}$$
$$ds^2 = 0\text{: } \textit{light-like} \text{ interval}$$
$$ds^2 < 0\text{: } \textit{space-like} \text{ interval}$$

Lemma 2.1. *The spacetime interval and metric distance are invariant under Lorentz transformations (LT).*

Proof. Algebra problem, fairly long. Metric distance can be shown directly by plugging in the regular Lorentz transforms. For a proof of the invariance of the spacetime interval, see this pdf ☐

Definition 2.4. *If something happens in spacetime at a defined 4-position, then this is a* ***spacetime event****.*

[7] In fact, just like regular rotations (which form the *special orthogonal groups* $SO(n)$), the translations, rotations and Lorentz boosts altogether form what's known as the *Poincaré group*.

[8] Mathematical note for those who took Advanced Linear/Multilinear Algebra: More preciesly, the Minkowski inner product is a *pseudo-inner product*, which one does not require to be positive definite, i.e. we allow $\eta(u,v) < 0$. Furthermore, the product satisfies *linearity in first argument*, *symmetry*, and *non-degeneracy* (i.e. $\eta(u,v) = 0\ \forall v \in M \implies u = 0$); note that the first two properties automatically imply *bilinearity*, hence the "inner product" is well-defined via this (non-degenerate) symmetric bilinear form.

2.7 Introduction to 4-vectors

4-vectors represent physical quantities and respect the symmetry of spacetime, i.e. transform according to the Lorentz Boosts. They are usually of the form

$$x^\mu = (ct, \mathbf{r}) = (ct, x, y, z)$$

Often, we denote 4-vector components with Greek indices x^μ, where $\mu = 0, 1, 2, 3$, and reserve Italic indices x^i, $i = 1, 2, 3$ for ONLY the spatial components.

In particular, the *Minkowski distance* is

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

As hinted earlier, we can also define the *covariant vector* x_μ, where the lower indices are defined as $x_0 = x^0, x_i = -x^i$, i.e.

$$\boxed{x_\mu = (ct, -x, -y, -z)}$$

Therefore, we have

Spacetime interval

$$s^2 := x_\mu x^\mu = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 \tag{2.19}$$

where repeated indices are again summed over with the *Einstein summation convention.*

More generally, for any 4-vector v and w, we have the (Lorentz invariant) inner product $v \cdot w \equiv v_\mu w^\mu = v^\mu w_\mu$, and again all 4-vectors a transform according to LT via

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

2.7.1 4-vector Examples

There are various useful 4-vectors one can define. For relativistic mechanics, we have

- *4-position:*
$$x^\mu = (ct, \mathbf{r}) = (ct, x, y, z)$$
- *4-velocity:*
$$u^\mu = \gamma(c, \mathbf{v})$$
where a factor of γ is added such that $u_\mu u^\mu = c^2$ is Lorentz invariant.
- *4-momentum:*
$$p^\mu = (E/c, \mathbf{p})$$

Lemma 2.2.
$$p_\mu p^\mu = E^2/c^2 - p^2 = m^2c^2 \tag{2.20}$$

Proof. **This is a useful quantity to remember**. Directly take 4-vector inner products and compare this to the usual form of the Einstein energy relation $E^2 - p^2c^2 = m^2c^4$ □

Since m is the rest mass and c is just the speed of light, we have shown another Lorentz-invariant quantity.

Note that this is consistent with the 4-velocity via

$$p^\mu = mu^\mu = (\gamma mc^2/c, \gamma m\mathbf{u}) = (E/c, \mathbf{p})$$

We shall derive the expressions for the 4-velocity and 4-momentum more carefully later.

Furthermore, for electromagnetism, we have

- *4-current (density):*
$$j^\mu = (\rho c, \mathbf{J})$$

- *4-potential:*
$$A^\mu = (\phi/c, \mathbf{A})$$

We shall also derive these expressions properly later. However, observe that since $\Box\phi = \frac{\rho}{\epsilon_0}$ and $\Box\mathbf{A} = \mu_0\mathbf{J}$, we have

Electrodynamics in one equation

$$\Box A^\mu = \mu_0 j^\mu \tag{2.21}$$

2.8 More on Tensors

This subsection provides a bit more on tensors needed for this module[9].

Tensors are usually classified in terms of their *rank*, which is roughly defined by the number of *independent* indices[10]. Some useful examples include

- Rank 0 (scalar): $ds^2 = dx_\mu dx^\mu$, $c^2 = u_\mu u^\mu$
- Rank 1: 4-vectors, e.g. $x^\mu, p^\mu, j^\mu, A^\mu, \ldots$
- Rank 2: *metric tensor* $g^{\mu\nu}$, *Faraday tensor* $F^{\mu\nu}$
- Rank 3: $\partial_\lambda F_{\mu\nu}$
- Rank 4: *antisymmetric tensor* $\epsilon_{\alpha\beta\gamma\delta}$, *Riemann curvature tensor* $R^\alpha_{\beta\gamma\delta}$

All tensors must transform according to LT, e.g. $F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$.

We shall now define the *metric tensor*, which is given by[11]

Metric Tensor (Special Relativity)

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.22}$$

In particular, this allows us to transform between covariant and contravariant vectors (called **raising and lowering indices**), i.e.

$$x^\mu = g^{\mu\nu} x_\nu \qquad \text{and} \qquad x_\mu = g_{\mu\nu} x^\nu$$

which lets us rewrite the *spacetime interval* as

$$ds^2 = g_{\mu\nu} x^\mu x^\nu$$

Note: In general, the metric tensor $g_{\mu\nu}$ is more complicated than this (more to come in GR!). To distinguish this "special" relativity case from the "general" case (pun intended), we often replace $g_{\mu\nu}$ by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}$$

using the metric tensor allows us to rigorously raise and lower indices. We've seen how it works for rank-1 tensors above. Now we can do it for tensors of rank-2 and higher. For some fully

[9] David Tong's brilliant lecture notes for a far more comprehensive introduction to tensors.

[10] Not everything with indices are tensors! Tensors have to transform properly under a change of coordinates. Even in GR, we deal with *Christoffel symbols* (more about them in PX436 General Relativity or MA3D9 Geometry of Curves and Surfaces), which usually look something like $\Gamma^i{}_{jk}$, but they do not transform like tensors under a change of coordinates, hence are not tensors.

Aside: More precisely, in differential geometry, given a smooth manifold, one may choose a coordinate basis for the tangent vector space at a point, and consider a tensor with components that are functions of points on the manifold. Then we can speak of the transformation law of tensors, which can be expressed explicitly in terms of partial derivatives of the chosen coordinate functions. In our case, the Λ's of LT are exactly the said laws:

$$\Lambda^\mu_\nu \equiv \frac{\partial x^\mu}{\partial x'^\nu} \implies a'^\mu = \frac{\partial x^\mu}{\partial x'^\nu} a^\nu = \Lambda^\mu_\nu a^\nu$$

More about this in MA3H5 Manifolds and PX453 Advanced Quantum Theory.

[11] Potential confusion: We use the $(+---)$ convention in this module (and for particle physics in general). However, in other fields (e.g. GR), the $(-+++)$ convention may be used instead, where $ds^2 \equiv -(ct)^2 + x^2 + y^2 + z^2$.

contravariant rank-2 tensor $A^{\mu\nu}$ we apply the *covariant* metric tensor once to lower the first index:

$$A_\mu{}^\nu = \eta_{\mu\lambda} A^{\lambda\nu}$$

which is defined as

$$A_\mu{}^\nu = \begin{cases} A^{\mu\nu} & \mu = 0 \\ -A^{\mu\nu} & \mu > 0 \end{cases}$$

It helps to imagine if rank-2 tensors are matrices and compute this on some random matrix.

We can apply $\eta_{\alpha\beta}$ again to get the fully covariant form $A_{\mu\nu}$:

$$A_{\mu\nu} = \eta_{\mu\beta}\eta_{\alpha\nu} A^{\alpha\beta}$$

defined as

$$A_{\mu\nu} = \begin{cases} A^{\mu\nu} & \mu = \nu = 0 \\ -A^{\mu\nu} & \mu > 0, \nu = 0 \quad \text{or} \quad \mu = 0, \nu \neq 0 \end{cases}$$

Note we can do the same to raise indices of fully covariant tensors back to fully contravariant tensors by applying $\eta^{\mu\nu}$ instead. This conveniently proves a lemma about **symmetric tensors**:

Lemma 2.3. *If $A^{\mu\nu}$ is symmetric then $A^\mu{}_\nu = A_\nu{}^\mu$ and $A^{\mu\nu} = A^{\nu\mu}$*

3 Introduction to Relativistic Electrodynamics

3.1 Relativistic Equations of Motion

A consistent set of relativistic eqautions of motion must

1. approach the Newtonian results in the classical limit $v/c \ll 1$; and
2. transform as 4-vectors via Lorentz transformations.

To formulate such equations, we shall first look at the *proper time* τ

Definition 3.1. *The* ***proper time*** τ *is the time in the* co-moving frame *of our particle in question, i.e. the time measured in a frame of reference where the particle is stationary.*

In this co-moving frame, the spacetime interval between two events is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 d\tau^2 - dx_\tau^2 - dy_\tau^2 - dz_\tau^2$$

In particular, we have $dx_\tau = dy_\tau = dz_\tau = 0$ for the particle in this frame, so, assuming the interval is timelike[12], i.e. $ds^2 > 0$, we have

$$ds^2 = c^2 d\tau^2 \implies ds = c d\tau$$

or equivalently,

$$\boxed{d\tau = \frac{ds}{c}} \tag{3.23}$$

By definition, the proper time is a scalar agreeable by all observers (Lorentz invariant).

Note that in a different frame of reference, we have

$$\begin{aligned} c^2 d\tau^2 &= ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ d\tau^2 &= dt^2 \left(1 - \frac{1}{c^2}\frac{dx^2}{dt^2} - \frac{1}{c^2}\frac{dy^2}{dt^2} - \frac{1}{c^2}\frac{dz^2}{dt^2}\right) \\ &= dt^2(1 - \beta^2) \end{aligned}$$

Therefore,

Proper time

$$d\tau = \frac{dt}{\gamma} \tag{3.24}$$

Now, we are ready to properly formulate the *4-velocity* by

$$\begin{aligned} u^\mu &:= \frac{dx^\mu}{d\tau} \\ &= \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \\ &= \gamma \left(\frac{dx^0}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) \end{aligned}$$

i.e.

[12] This is always true for a physical particle travelling inside the light cone (ignoring so-called “tachyons”, which are now commonly interpreted as instabilities in quantum fields rather than particles doomed to travel on space-like trajectories anyway).

4-Velocity

$$u^\mu = \gamma(c, \mathbf{v}) \tag{3.25}$$

where we differentiated position against τ to ensure $u_\mu u^\mu = c^2$ is observer-independent[13]. This is also a useful identity to remember!

Similarly, we define the *4-momentum*[14]by

$$p^\mu := mu^\mu = \gamma m(c, \mathbf{v}) = (\gamma mc^2/c, \gamma m\mathbf{v})$$

i.e.

4-Momentum

$$p^\mu = (E/c, \mathbf{p}) \tag{3.26}$$

We also have the analogue for acceleration. To do this we must differentiate the 4-velocity u^μ with respect to proper time along the particle's worldline:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(c, \mathbf{v}) = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\gamma(c, \mathbf{v})$$

If γ constant then $a^\mu = \gamma^2(0, \mathbf{a})$. However when accelerating $\mathbf{v}$ changes and hence, so does γ so must find the time derivative of γ:

$$\begin{aligned} a^0 = \gamma\frac{d}{dt}(\gamma c) = \frac{c}{2}\frac{d}{dt}(\gamma^2) = \frac{c}{2}\frac{d}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-1} &= \frac{c}{2}\cdot(-1)\cdot\left(1 - \frac{v^2}{c^2}\right)^{-2}\cdot\frac{-2v}{c}\frac{dv}{dt} \\ &= \frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4 \end{aligned}$$

The spatial components where $i = 1, 2, 3$ are given by

$$a^{i=1,2,3} = \gamma\gamma\frac{d}{dt}(\gamma v_i) = \gamma^2\frac{dv_i}{dt} + \frac{v_i}{2}\frac{d\gamma^2}{dt} = \gamma^2 a_i + \gamma^4\frac{v_i}{c^2}(\mathbf{v}\cdot\mathbf{a})$$

therefore we can gather our results to get the

4-acceleration

$$a^\mu = \left(\gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c}, \gamma^2\mathbf{a} + \gamma^4\frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\mathbf{a})\right) \tag{3.27}$$

To define the *4-force*

$$f^\mu := \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \gamma\left(\frac{\mathrm{d}p^0}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)$$

Note that

$$u_\mu f^\mu = u_\mu\frac{\mathrm{d}}{\mathrm{d}\tau}(mu^\mu) = mu_\mu\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\mu u^\mu) = 0$$

In particular,

$$0 = u^0 f^0 - (u^1 f^1 + u^2 f^2 + u^3 f^3) = \gamma c f^0 - \gamma\mathbf{v}\cdot\gamma\mathbf{F} \implies f^0 = \frac{\gamma}{c}(\mathbf{v}\cdot\mathbf{F})$$

Therefore,

[13]Derivation: $u_\mu u^\mu = \gamma^2(c^2 - v^2) = (1-\beta^2)^{-1}(1-\beta^2)c^2 = c^2$.

[14]Similarly, this is defined such that $p_\mu p^\mu = \gamma^2 m^2(c^2 - v^2) = m^2c^2$ is observer-independent.

4-Force

$$f^\mu = \gamma \left(\frac{\mathbf{v} \cdot \mathbf{F}}{c}, \mathbf{F} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \mathbf{F} \right) \tag{3.28}$$

Looking at the Newtonian work done relationship $dE/dt = \mathbf{F} \cdot \mathbf{v}$, the 4-force we obtained from the 4-momentum does somehow resemble what's expected classically.

Remark. *Observe that the spatial components of 4-acceleration and 4-force are* ***no longer parallel to their regular 3-vectors*** *unlike in the 4-velocity and 4-momentum.*

This means talking about relativity in terms of forces is ***hard****.*

3.2 Relativistic EM Forces and the Faraday Tensor

To use our new 4-vectors, we will need a relativistic form of the Lorentz Force Law. Substituting (2.17) into (3.28), and using (3.25), we get[15]

$$
\begin{aligned}
f^0 &= \frac{\gamma}{c}(\mathbf{v}\cdot\mathbf{F}) = \frac{q\gamma}{c}(\mathbf{v}\cdot\mathbf{E} + \mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})) = \frac{q}{c}(U^1E_x + U^2E_y + U^3E_z) \\
f^1 &= \gamma F_x = \gamma q(E_x + v_yB_z - v_zB_y) = q\left(u^0\frac{E_x}{c} + u^2B_z - u^3B_y\right) \\
f^2 &= \gamma F_x = \gamma q(E_y + v_zB_x - v_xB_z) = q\left(u^0\frac{E_y}{c} + u^3B_x - u^1B_z\right) \\
f^3 &= \gamma F_x = \gamma q(E_z + v_xB_y - v_yB_x) = q\left(u^0\frac{E_z}{c} + u^1B_y - u^2B_x\right)
\end{aligned}
$$

Putting all together, we have

$$\boxed{f^\mu = qF^\mu{}_\nu u^\nu} \tag{3.29}$$

where $F^\mu{}_\nu$ is the *Faraday Tensor* (aka *Electromagnetic Tensor*), which is an *antisymmetric* rank-2 tensor

$$
F^\mu{}_\nu := \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}
$$

with μ and ν indexing the rows and columns respectively (REMEMBER THE INDICES STILL START FROM ZERO), e.g. $F^1{}_2 = B_z$.

Usually, one uses either the (fully) contravariant form or the (fully) covariant form, i.e.

Faraday/Electromagnetic Tensor

$$
F^{\mu\nu} = \eta^{\nu\beta}F^\mu{}_\beta = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{3.30}
$$

$$
F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{3.31}
$$

In particular, note that $F^{\mu\nu}$ and $F_{\mu\nu}$ are fully antisymmetric, i.e. $F^{\mu\nu} = -F^{\nu\mu}$ and $F_{\mu\nu} = -F_{\nu\mu}$. Note also that the matrices all consist of a time-like block (0th row and column) of electric field entries, and a space-like block (bottom-right 3×3 matrix) of magnetic field entries.

With these new forms, one writes the *relativistic Lorentz Force Law* as

Relativistic Lorentz Force Law

$$
f^\mu = qF^{\mu\nu}u_\nu \qquad \text{or} \qquad f_\mu = qF_{\mu\nu}u^\nu \tag{3.32}
$$

[15] We also used the vector identity $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$

3.3 Motion in Simple EM Fields

Now, using Eq. (3.29), we have

$$f^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \implies \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{m}F^\mu{}_\nu u^\nu \tag{3.33}$$

This is an eigenvalue problem of the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = M\mathbf{u}$$

which has general solution

$$\mathbf{u} = \sum_\lambda A_\lambda e^{\lambda\tau}\mathbf{e}_\lambda$$

where λ and $\mathbf{e}_\lambda$ are the eigenvalues and corresponding eigenvectors of M respectively. We shall now try and solve Eq. (3.33) under different (boundary) conditions.

Case 1: Uniform E-field; $\mathbf{E} = (E, 0, 0)$, $\mathbf{B} = \mathbf{0}$

The Faraday Tensor is

$$F^\mu_\nu = \eta_{\nu\beta}F^{\mu\beta} = \begin{pmatrix} 0 & E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has eigenvalues $\lambda = 0$ (repeated) and $\pm E/c$, with eigenvectors $(0, 0, 0, 1), (0, 0, 1, 0)$, $(1, 1, 0, 0)$, and $(1, -1, 0, 0)$ respectively. In particular, our general solution is

$$U^\mu = Ae^{a\tau}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Be^{-a\tau}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + C\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + D\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where $a = qE/mc$.

Assuming[16] $v_x(\tau = 0) = 0$, we have $A = B$. Furthermore, from $U^0(\tau = 0) = \gamma_0 c = A + B$, we have $A = B = \gamma_0 c/2$. Therefore,

$$\begin{pmatrix} \gamma c \\ \gamma v_x \end{pmatrix} = \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \gamma_0 c \begin{pmatrix} \cosh\left(\frac{qE}{mc}\tau\right) \\ \sinh\left(\frac{qE}{mc}\tau\right) \end{pmatrix}$$

In particular, γ changes with time via

$$\gamma(\tau) = \gamma_0 \cosh\left(\frac{qE}{mc}\tau\right)$$

and so

$$v_x(\tau) = \frac{U^1}{\gamma(\tau)} = c\tanh\left(\frac{qE}{mc}\tau\right)$$

This should make sense since

$$\lim_{\tau\to\infty}|v_x(\tau)| = c \cdot \lim_{x\to\pm\infty}|\tanh(x)| = c$$

as is expected for a relativistic accelerator.

[16] We can always do this by changing reference frames. In particular, we shall later see that Lorentz Transformations leave the parallel components $\mathbf{E}_\parallel$ and $\mathbf{B}_\parallel$ unchanged, thereby justifying our (Lorentz boosted) calculations.

Using this, one can also find the coordinate time t via

$$dt = \gamma(\tau)d\tau \implies t = \int_0^t \gamma(\tau)d\tau \propto \sinh\left(\frac{qE}{mc}\tau\right)$$

Now, for the transverse velocities v_y and v_z, note that

$$u^2 = \gamma(\tau)v_y(\tau) = C \implies v_y(\tau) = \frac{C}{\gamma(\tau)}$$

and similarly,

$$u^3 = \gamma(\tau)v_z(\tau) = D \implies v_z(\tau) = \frac{D}{\gamma(\tau)}$$

Therefore,

The transverse velocities *decrease* as the particle accelerates!

This is expected as

$$p^2 = \gamma m v_y = \frac{\gamma m C}{\gamma} = mC \qquad \text{and} \qquad p^3 = \gamma m v_z = \frac{\gamma m D}{\gamma} = mD$$

i.e. the transverse momenta are conserved (C and D are just constants).

Case 2: Uniform B-field; $\mathbf{E} = \mathbf{0}, \mathbf{B} = (B, 0, 0)$

The Faraday Tensor is

$$F^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}$$

So, u^0 and u^1 are constant. Also,

- $u^0 = \gamma c \implies \gamma = \text{constant} \implies |\mathbf{v}| = \text{constant} \implies$ Energy is conserved; and
- $u^1 = \gamma v_x \implies$ velocity along the direction of $\mathbf{B}$, v_x, remains constant.

This time, $F^\mu{}_\nu$ has eigenvalues 0 (repeated) and $\pm iB$, with eigenvectors $(1, 0, 0, 0), (0, 1, 0, 0)$, $(0, 0, 1, i)$ and $(0, 0, 1, -i)$ respectively. This has general solution

$$u^\mu = A' \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + B' \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Ce^{i\omega\tau} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} + De^{-i\omega\tau} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}$$

where $\omega = qB/m$ is the **Larmor frequency**. In particular, assuming[17] $v_z(\tau = 0) = 0$, we have $C = D$, and we can write

$$\begin{pmatrix} U^2 \\ U^3 \end{pmatrix} = A \begin{pmatrix} \cos(\omega\tau) \\ -\sin(\omega\tau) \end{pmatrix} = A \begin{pmatrix} \sin(\omega\tau + \phi) \\ \cos(\omega\tau + \phi) \end{pmatrix}$$

where $A = C + D$, and a phase ϕ is added for generality.

In particular, this results in *helical motion* around the x-axis, with $\omega(t) = \omega(\tau)/\gamma$.

Case 3: Crossed $\mathbf{E}$ and $\mathbf{B}$ fields; $\mathbf{E} = (0, E_y, 0), \mathbf{B} = (0, 0, B_z)$

[17] We can always do this by choosing the $+z$ direction to be perpendicular to $\mathbf{v}(\tau = 0)$.

The Faraday Tensor is

$$F^{\mu}_{\ \nu} = \begin{pmatrix} 0 & 0 & E_y/c & 0 \\ 0 & 0 & B_z & 0 \\ E_y/c & -B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, $u^\mu = (c, E_y/B_z, 0, 0)$ is an eigenvector with eigenvalue 0, and is therefore a steady-state solution for 3.33. In particular, we have

$$v_x = \frac{E_y}{B_z}$$

with overall motion perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. This is known as the *Hall drift*.
Aside: More generally, Hall drift occurs whenever $\mathbf{E} \cdot \mathbf{B} = 0$, with the general velocity being

Hall drift velocity

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2} \tag{3.34}$$

3.4 Lorentz Transformation of Tensors

Rank-0 tensors (scalars):

Scalars are trivially invariant, i.e. they are the same in all reference frames.

Rank-1 tensors (vectors):

In this module, we consider 4-vectors which transform according to LT via

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

with the Lorentz boost in the $+x$-direction given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rank-2 tensors (matrices):[18]

The Lorentz Transformation generalizes easily to higher-rank tensors. For example, we looked at the Faraday Tensor, which transforms according to LT via

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^\alpha{}_\beta$$

More explicitly, suppose we want to find E'_x for a particle travelling along the x-direction. Then

$$\begin{aligned} \frac{E'_x}{c} &= F'^{10} = \Lambda^1_{\ \mu}\Lambda^0_{\ \nu}F^{\mu\nu} = \Lambda^1_{\ 0}(\Lambda^0_{\ 0}F^{00} + \Lambda^0_{\ 1}F^{01}) + \Lambda^1_{\ 1}(\Lambda^0_{\ 0}F^{10} + \Lambda^0_{\ 1}F^{11}) \\ &= -\beta\gamma\left(0 + \beta\gamma\frac{E_x}{c}\right) + \gamma\left(\gamma\frac{E_x}{c} + 0\right) = (1-\beta^2)\gamma^2 \cdot \frac{E_x}{c} = \frac{E_x}{c} \end{aligned}$$

So, $E'_x = E_x$, i.e. the parallel component of $\mathbf{E}$ remains unchanged.

Similarly, we have

$$\frac{E'_y}{c} = F'^{20} = \Lambda^2_{\ \mu}\Lambda^0_{\ \nu}F^{\mu\nu} = \Lambda^2_{\ 2}(\Lambda^0_{\ 0}F^{20} + \Lambda^0_{\ 1}F^{21}) = \frac{\gamma E_y}{c} - \beta\gamma B_z$$

[18] Note: Matrices are fundamentally different from rank-2 tensors, they are merely a convenient representation we use in this module (and in most other circumstances)!

and

$$\frac{E'_z}{c} = F'^{30} = \Lambda^3{}_\mu \Lambda^0{}_\nu F^{\mu\nu} = \Lambda^3{}_3(\Lambda^0{}_0 F^{30} + \Lambda^0{}_1 F^{31}) = \frac{\gamma E_z}{c} + \beta\gamma B_y$$

In particular, observe that

$$\begin{aligned} E'_y &= \gamma(E_y - v_x B_z) = \gamma[\mathbf{E} + \mathbf{v} \times \mathbf{B}]_y \\ E'_z &= \gamma(E_z + v_x B_y) = \gamma[\mathbf{E} + \mathbf{v} \times \mathbf{B}]_z \end{aligned}$$

Now, for magnetic fields, we have

$$B'_x = F'^{32} = \Lambda^3{}_\mu \Lambda^2{}_\nu F^{\mu\nu} = \Lambda^3{}_3 \Lambda^2{}_2 F^{32} = B_x$$

So, the parallel component of $\mathbf{B}$ remains unchanged.

Similarly,

$$B'_y = \Lambda^1{}_\mu \Lambda^3{}_\nu F^{\mu\nu} = \Lambda^1{}_0 F^{03} + \Lambda^1{}_1 F^{13} = \frac{\beta\gamma E_z}{c} + \gamma B_y$$

and

$$B'_z = \Lambda^2{}_\mu \Lambda^1{}_\nu F^{\mu\nu} = \Lambda^1{}_0 F^{20} + \Lambda^1{}_1 F^{21} = -\frac{\beta\gamma E_y}{c} + \gamma B_z$$

Again, we can rewrite these expressions as

$$B'_y = \gamma\left[\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right]_y \qquad \text{and} \qquad B'_z = \gamma\left[\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right]_z$$

To summarize, we have

Lorentz Transformation of E and B

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel \tag{3.35}$$

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel \tag{3.36}$$

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}\right) \tag{3.37}$$

$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right) \tag{3.38}$$

An important consequence of this is quantities that are *frame invariant* (aka *Lorentz invariants*).

Theorem 1. *The dot product $\boldsymbol{E}\cdot\boldsymbol{B}$ is frame invariant.*

Proof. WLOG, assume $\mathbf{v} = (v, 0, 0)$, which we are allowed to as spatial rotations always leave dot products unchanged (this has nothing to do with Lorentz Transformations).

$$\begin{aligned} \mathbf{E}'\cdot\mathbf{B}' &= E'_x B'_x + E'_y B'_y + E'_z B'_z \\ &= E_x B_x + \gamma^2(E_y - vB_z)\left(B_y + \frac{vE_z}{c^2}\right) + \gamma^2(E_z + vB_y)\left(B_z - \frac{vE_y}{c^2}\right) \\ &= E_x B_x + E_y B_y \gamma^2(1-\beta^2) + E_z B_z \gamma^2(1-\beta^2) = \mathbf{E}\cdot\mathbf{B} \end{aligned}$$

Alternatively, we can also consider the determinant

$$\left|F^{\mu'\nu'}\right| = \left|\Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu F^{\mu\nu}\right| = \left|\Lambda^{\mu'}{}_\mu\right| \left|\Lambda^{\nu'}{}_\nu\right| \left|F^{\mu\nu}\right|$$

Since $\left|\Lambda^{\mu'}{}_\mu\right| = \left|\Lambda^{\nu'}{}_\nu\right| = (1-\beta^2)\gamma^2 = 1$, we have $\left|F^{\mu'\nu'}\right| = |F^{\mu\nu}|$. Therefore, the determinant of $F^{\mu\nu}$ is Lorentz invariant. One can then show that

$$\det(F^{\mu\nu}) = \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{B})^2$$

i.e. $\mathbf{E}\cdot\mathbf{B}$ is Lorentz invariant. $\square$

As we can see, this is tedious work. We had to somehow know to calculate $\mathbf{E}\cdot\mathbf{B}$ at the first place, then hope that things work out as above, or grind through the determinant of a 4×4 matrix. A more useful way would be to find invariants (scalars) directly from the electromagnetic tensor, which we shall demonstrate below.

Theorem 2. *$B^2 - E^2/c^2$ is frame invariant.*

Proof. We simply use the fact that tensor products with all indices contracted (i.e. scalars) are frame invariant. In particular,

$$F^{\mu\nu}F_{\mu\nu} = 2(B_x^2 + B_y^2 + B_z^2) - 2\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

□

3.4.1 Aside: The Dual Electromagnetic Tensor

The same idea can be used to prove the invariance of $\mathbf{E}\cdot\mathbf{B}$, but requires the introduction of the *dual electromagnetic tensor*[19], which is defined as[20]

Aside: The dual electromagnetic tensor $\tilde{F}^{\mu\nu}$

$$\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \tag{3.39}$$

Note that $\tilde{F}^{\mu\nu}$ arises from $F^{\mu\nu}$ by the substitution $\mathbf{E}\mapsto c\mathbf{B}$ and $\mathbf{B}\mapsto -\mathbf{E}/c$. Furthermore, the fact that $\tilde{F}^{\mu\nu}$ is a *tensor* (and not just a matrix) means that it also transforms nicely under Lorentz Transformations, with

$$\tilde{F'}^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \tilde{F}^{\rho\sigma}$$

Taking the obvious square of $\tilde{F}$ gives[21]

$$\begin{aligned}
\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}F_{\rho\sigma} \\
&= \frac{1}{4}\left(-2!\,\delta^{\rho\sigma}{}_{\alpha\beta}\right)F^{\alpha\beta}F_{\rho\sigma} \\
&= -\frac{1}{2}\left(\delta^\rho{}_\alpha\delta^\sigma{}_\beta - \delta^\rho{}_\beta\delta^\sigma{}_\alpha\right)F^{\alpha\beta}F_{\rho\sigma} \\
&= -\frac{1}{2}\left(F^{\alpha\beta}F_{\alpha\beta} - F^{\alpha\beta}F_{\beta\alpha}\right) \\
&= -F^{\alpha\beta}F_{\alpha\beta}
\end{aligned}$$

So, we get nothing new. Of course, the next most natural thing to do[22] is to contract $\tilde{F}$ with our original F, giving

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = -\frac{4}{c}(\mathbf{E}\cdot\mathbf{B})$$

---

[19] See p.112 of David Tong's lecture notes on electromagnetism.

[20] $\tilde{F}^{\mu\nu}$ is sometimes also written as $\star F^{\mu\nu}$. This alternative (in fact more common) notation comes from the fact that $\star F^{\mu\nu}$ is the so-called *Hodge dual* of the Faraday tensor $F^{\mu\nu}$ (whatever this means).

[21] We can derive $\tilde{F}_{\mu\nu}$ with the usual raising and lowering of indices (try this yourself!):

$$\tilde{F}_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}\tilde{F}^{\alpha\beta} = \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\epsilon^{\alpha\beta\gamma\tau}(\eta_{\gamma\rho}\eta_{\tau\sigma}F^{\rho\sigma}) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

[22] If we look closer, the fact that $\tilde{F}^{\mu\nu}$ has the word *dual* in its name does suggest us to perform this operation. For the mathematicians among us physics peasants, see this Wikipedia article on how *tensor contraction* actually arises from the natural pairing of a vector space and its dual.

and voilá! We have shown for the third time that $\mathbf{E} \cdot \mathbf{B}$ is Lorentz invariant, this time with a much more elegant and powerful tool, i.e. by contracting tensors into scalars.

3.5 Four-gradients

For a general scalar field $\psi(ct, x, y, z)$, its differential is given by

$$d\psi = \frac{\partial \psi}{\partial(ct)}d(ct) + \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dz = \sum_{\mu=0}^{4} \frac{\partial \psi}{\partial x^\mu}dx^\mu$$

With $d\psi$ on the LHS being a scalar (since it has no indices), we must have $\frac{\partial \psi}{\partial X^\mu}$ as a covariant 4-vector such that all indices are contracted with dX^μ. In particular, we define the *covariant 4-derivative* by

Covariant 4-gradient

$$\partial_\mu := \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{3.40}$$

As usual, we can then raise the indices to get the *contravariant 4-derivative*

Contravariant 4-gradient

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \tag{3.41}$$

Notice here that the contravariant 4-gradient has negative spatial components, as opposed to its covariant counterpart (as is the case for other 4-vectors, e.g. $x_\mu = (ct, -\mathbf{r})$).

Furthermore, the d'Alembertian operator is now automatically frame invariant, since

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$$

With these 4-gradients, we can simplify much of the equations we have seen earlier using 4-vectors. For example, we have

Euler's (charge continuity) equation

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{3.42}$$

Consider now the inhomogeneous Maxwell's equations. We can rewrite (M1) as

$$\begin{aligned}\nabla \cdot \left(\frac{\mathbf{E}}{c}\right) &= \frac{\rho/\epsilon_0}{c} \\ \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} &= \mu_0(\rho c) \\ \partial_\mu F^{\mu 0} &= \mu_0 j^0\end{aligned}$$

and looking at (M4):

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \mathbf{J}$$

we can rewrite the x-, y-, and z-components of the equation as

$$\begin{aligned}\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} + \frac{\partial(E_x/c)}{\partial t} &= \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} + \frac{\partial F^{01}}{\partial x^0} = \mu_0 j^1 \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} + \frac{\partial(E_y/c)}{\partial t} &= \frac{\partial F^{32}}{\partial x^3} + \frac{\partial F^{12}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^0} = \mu_0 j^2 \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + \frac{\partial(E_x/c)}{\partial t} &= \frac{\partial F^{13}}{\partial x^1} - \frac{\partial F^{23}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^0} = \mu_0 j^3\end{aligned}$$

Combining everyting, we get

Inhomogeneous Maxwell's equation

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{3.43}$$

Similarly for the homogeneous Maxwell equations, from (M2), we have

$$0 = -\nabla \cdot \mathbf{B} = \frac{\partial(-B_x)}{\partial x} + \frac{\partial(-B_y)}{\partial y} + \frac{\partial(-B_z)}{\partial z} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}$$

and from (Eq. (M3)):

$$\frac{1}{c}\left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) = 0$$

we can again rewrite the x-, y-, and z-components separately as

$$\begin{aligned}
\frac{\partial(E_z/c)}{\partial y} - \frac{\partial(E_y/c)}{\partial z} + \frac{\partial B_x}{\partial(ct)} &= \partial_2 F_{03} + \partial_3 F_{20} + \partial_0 F_{32} = 0 \\
\frac{\partial(E_x/c)}{\partial z} - \frac{\partial(E_z/c)}{\partial x} + \frac{\partial B_y}{\partial(ct)} &= \partial_3 F_{01} + \partial_1 F_{30} + \partial_0 F_{31} = 0 \\
\frac{\partial(E_y/c)}{\partial x} - \frac{\partial(E_x/c)}{\partial y} + \frac{\partial B_z}{\partial(ct)} &= \partial_1 F_{02} + \partial_2 F_{10} + \partial_0 F_{21} = 0
\end{aligned}$$

Summarizing these, we have the so-called *Bianchi Identity*

Bianchi Identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{3.44}$$

or equivalently as

$$\partial_{[\alpha} F_{\beta\gamma]} = 0$$

where [...] stands for anti-symmetrization of any groups of indicies, i.e. for a rank-n tensor T,

$$T_{[12\cdots n]} := \frac{1}{n!}\epsilon^{i_1 i_2 \cdots i_n} T_{i_1 i_2 \cdots i_n}$$

In our case of three indices, it reads

$$\partial_{[\alpha} F_{\beta\gamma]} \equiv \frac{1}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma} = \frac{1}{6}\left(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} - \partial_\alpha F_{\gamma\beta} - \partial_\beta F_{\alpha\gamma} - \partial_\gamma F_{\beta\alpha}\right)$$

and we recover (3.44) using the fact that F is antisymmetric, i.e. $F_{\mu\nu} = -F_{\nu\mu}$.

In fact, we can define the Faraday tensor alternatively as follows[23]:

Faraday tensor in terms of the 4-potential

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.45}$$

where A_μ is of course the usual (covariant) 4-potential $A_\mu = (\phi/c, -\mathbf{A})$. Notice that $F_{\mu\nu}$ (and therefore $F^{\mu\nu}$) is then antisymmetric by definition.

[23] Mathematically, the 4-potential, A, is known as a *differential 1-form*. F is then defined as the *exterior derivative* of A, i.e. $F := dA$, or in terms of local coordinates, $F_{\mu\nu} = (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (as defined above), which is therefore a *differential 2-form*. Again, more about this in MA3H5 Manifolds.

One can easily check that this definition of $F_{\mu\nu}$ is indeed consistent with our previous matrix definitions via direct calculation. For example,

$$F_{12} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} = \frac{\partial(-A_y)}{\partial x} - \frac{\partial(-A_x)}{\partial y} = -[\nabla \times \mathbf{A}]_z = -B_z$$

Bianchi's identity (and hence the homogeneous Maxwell Equations) now follows automatically:

$$\begin{aligned}
&\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \\
&= (\partial_\alpha\partial_\beta A_\gamma - \partial_\alpha\partial_\gamma A_\beta) + (\partial_\beta\partial_\gamma A_\alpha - \partial_\beta\partial_\alpha A_\gamma) + (\partial_\gamma\partial_\alpha A_\beta - \partial_\gamma\partial_\beta A_\alpha) \\
&= \partial_\alpha\partial_\beta A_\gamma - \partial_\alpha\partial_\gamma A_\beta + \partial_\beta\partial_\gamma A_\alpha - \partial_\alpha\partial_\beta A_\gamma + \partial_\alpha\partial_\gamma A_\beta - \partial_\beta\partial_\gamma A_\alpha \\
&= 0
\end{aligned}$$

So far, we have been rewriting Maxwell's equations using the 4-gradient and the Faraday tensor, which are all gauge-independent. In particular, substituting our new definition (3.45) into the inhomogeneous Maxwell's equation (3.43), we get

Gauge independent Maxwell's equations

$$\mu_0 j^\nu = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) \tag{3.46}$$

However, notice that if we choose the Lorenz gauge, i.e.

$$\partial_\mu A^\mu \equiv \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Maxwell's equations can be elegantly summarized as

Maxwell's equations in the Lorenz gauge

$$\Box A^\mu = \mu_0 j^\mu \tag{3.47}$$

Noice!

4 Narnia™: The Dipole, the Antenna & the Retarded Potential

"All shall be done, but it may be harder than you think."
C.S. Lewis, *Narnia: The Lion, the Witch and the Wardrobe*;
also me on *How to Pass the PX3A3 Exam.*

4.1 Moving Charges and the Retarded Potential

We first consider the electrostatic setting of a point charge q located at $\mathbf{r}_0$. Recall from PX120 that the charge generates an electric field given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}_0\right|^2} \frac{\mathbf{r} - \mathbf{r}_0}{\left|\mathbf{r} - \mathbf{r}_0\right|}$$

or equivalently, a scalar electrostatic potential

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}_0\right|}$$

For multiple charges, we simply sum over all constituents, giving

$$V(\mathbf{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}_i\right|}$$

We can also integrate over a continuous region of charges with

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r}' \, \frac{\rho(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

where $\rho(\mathbf{r}')$ is the charge density at $\mathbf{r}$'.

Now, this is all well and good, but what if the charges are moving instead? We might reasonably expect V to have the form

$$V(t, \mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r}' \, \frac{\rho(t, \mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

i.e. the potential now depends on time due to the time-dependence of ρ.

But what should the time t here be in this case? Naturally, we want t to be the time at which we, the observer makes a measurement in our own frame of reference. However, since radiation[24] from the electric field travels at a finite speed c, the speed of light (in vacuum), the charges we measure were actually generated by the source at time $t' = t - \left|\mathbf{r} - \mathbf{r}'\right|/c$.

We define this as the *retarded time*

Retarded time

$$t_R := t - \frac{\left|\mathbf{r} - \mathbf{r}'\right|}{c} \tag{4.48}$$

In particular, ρ depends on t_R, the retarded time, NOT t, the time of measurement. That is, for moving charges, the generated electric potential is given by

$$V(t, \mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r}' \, \frac{\rho(t_R, \mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} \tag{4.49}$$

[24] Some justification for why radiation comes up here: Recall that radiation is merely some fluctuation in the electric and magnetic fields (aka EM waves). With time-varying charges/currents, we have a time-varying **E**-field, and can reasonably expect the generation of a **B**-field (e.g. by Lenz's Law). In other words, we have radiation, which must travel (in vacuum) with speed c due to Special Relativity. More on this in the next few sections.

In fact, solving Maxwell's equations[25] in the Lorenz gauge $\Box A^\mu = \mu_0 j^\mu$, or separately as

$$\Box\phi = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \Box\mathbf{A} = \mu_0\mathbf{J}$$

we get the very similar looking retarded potentials

$$\phi(t,\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\mathbf{r}'\,\frac{\rho(t_R,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.50}$$

$$\mathbf{A}(t,\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{r}'\,\frac{\mathbf{J}(t_R,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.51}$$

which combines into a 4-potential as

Retarded potential

$$A^\mu = \frac{\mu_0}{4\pi}\int d\mathbf{r}'\,\frac{[j^\mu]}{|\mathbf{r}-\mathbf{r}'|} \tag{4.52}$$

where the square brackets mean that we evaluate j^μ using **r**' and the retarded time t_R, i.e.

$$[j^\mu] = j^\mu(ct_R,\mathbf{r}')$$

instead of the regular 4-position $X^\mu = (ct,\mathbf{r})$ (as is the case for A^μ). Intuitively, this potential should make sense if we compare it to the form of $V(t,\mathbf{r})$ as derived in (4.49).

4.2 4-potential from a Hertzian Dipole

As hinted above, one important consequence of the 4-potential retardation is the emission of radiation from time-dependent charges, as we shall see below.

Consider a pair of opposite oscillating point charges $\pm q(t)$ situated with distance b apart, or more precisely, with a *displacement vector* $\mathbf{b} = b\hat{\mathbf{z}}$, i.e. we align the dipole along the $+z$-axis. This is commonly set up in, say, radio antennae using some AC current $I(t)$.

The dipole moment $\mathbf{p}$ (with origin set at the midpoint of the two charges) is given by

$$\mathbf{p}(t) = q(t)\mathbf{b}$$

which varies in time as

$$\dot{\mathbf{p}} = \dot{q}\mathbf{b} = I(t)\mathbf{b}$$

Now, if we assume $|\mathbf{r}| \gg \lambda \gg b$, where λ is the wavelength of radiation, we then have $|\mathbf{r}| \gg |\mathbf{r}'|$, and can therefore approximate **r**' and hence, t_R to be uniform across the dipole. Dipoles satisfying this assumption are known as *Hertzian dipoles* (sometimes also *short dipoles*).

Therefore, the vector potential can be approximated as

$$\mathbf{A}(t,\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{r}'\,\frac{\mathbf{J}(t_R,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{\mu_0}{4\pi r}\int d\mathbf{r}'\,[\mathbf{J}] = \frac{\mu_0\hat{\mathbf{z}}}{4\pi r}\int_{-b/2}^{b/2} dz\,[I] = \frac{\mu_0[I]b}{4\pi r}\hat{\mathbf{z}} = \frac{\mu_0[\dot{p}]}{4\pi r}\hat{\mathbf{z}}$$

where $r := |\mathbf{r}|$. This is called the *electric dipole approximation*.

As for the scalar potential, we have

$$\phi(t,\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\mathbf{r}'\,\frac{\rho(t_R,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\left[\frac{q(t_R,\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|} + \frac{q(t_R,\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|}\right]$$

[25] This can be done by introducing appropriate Green functions. For more details, see J.D.Jackson *Classical Electrodynamics*, Second Edition, Section 6.6, p.223 (please don't, it really is just 4 pages of painful maths).

Labelling $R_+ := |\mathbf{r} - \mathbf{r}_1|$ and $R_- := |\mathbf{r} - \mathbf{r}_2|$ as the distances from the respective charges to point $\mathbf{r}$, this becomes

$$\phi(t,\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q\left(t - \frac{R_+}{c}\right)}{R_+} + \frac{-q\left(t - \frac{R_-}{c}\right)}{R_-}\right]$$

If we denote the angle between $\mathbf{r}$ and $\mathbf{b}$ as θ (such that $\mathbf{r}\cdot\mathbf{b} = \cos\theta$), then

$$R_\pm = \left|\mathbf{r} \mp \frac{\mathbf{b}}{2}\right| = \left(r^2 \mp \mathbf{r}\cdot\mathbf{b} + \frac{b^2}{4}\right)^{1/2} = r \pm \frac{\mathbf{r}\cdot\mathbf{b}}{2} + \mathcal{O}(b^2) \approx r \mp \frac{b}{2}\cos\theta$$

where we assumed $b \ll r$ as before. Therefore,

$$q\left(t - \frac{R_\pm}{c}\right) = q\left(t - \frac{r}{c} \pm \frac{b\cos\theta}{2c}\right) \approx q\left(t - \frac{r}{c}\right) \pm \frac{b\cos\theta}{2c}\dot{q}\left(t - \frac{r}{c}\right)$$

Hence, we can write

$$\begin{aligned}\phi(t,\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q + \frac{b\cos\theta}{2c}\dot{q}}{r - \frac{b}{2}\cos\theta} - \frac{q - \frac{b\cos\theta}{2c}\dot{q}}{r + \frac{b}{2}\cos\theta}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{\frac{b\cos\theta}{c}q + \frac{rb\cos\theta}{c}\dot{q}}{r^2 - \frac{b^2}{4}\cos^2\theta}\right) \\ &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{\frac{b\cos\theta}{c}q + \frac{rb\cos\theta}{c}\dot{q}}{r^2}\right) = \frac{1}{4\pi\epsilon_0 r}\left(\frac{qb}{r} + \frac{\dot{q}b}{c}\right)\cos\theta = \frac{1}{4\pi\epsilon_0 r}\left(\frac{p}{r} + \frac{\dot{p}}{c}\right)\cos\theta\end{aligned}$$

Now, with the dipole oscillating with some frequency ω, such that

$$p = p_0 e^{i\omega t_R}$$

We have (only considering the real parts of p and $\dot{p}$)

$$\dot{p} = \omega p \implies \frac{\dot{p}}{c} = \frac{\omega}{c}p = \frac{2\pi}{\lambda}p = 2\pi\left(\frac{r}{\lambda}\right)\cdot\frac{p}{r}$$

With the *far field approximation*, i.e. $r \gg \lambda$, this implies

$$\frac{\dot{p}}{r} \gg \frac{p}{r}$$

Therefore,

$$\phi(t,\mathbf{r}) = \frac{[\dot{p}]\cos\theta}{4\pi\epsilon_0 rc} \implies \frac{\phi(t,\mathbf{r})}{c} = \frac{\mu_0[\dot{p}]}{4\pi r}\cos\theta$$

Finally, combining everything into a 4-potential, we have

4-potential from a Hertzian dipole

$$A^\mu = \frac{\mu_0[\dot{p}]}{4\pi r}(\cos\theta, \hat{\mathbf{z}}) \tag{4.53}$$

where as before, $[\dot{p}] := \dot{p}(t - r/c)$.

Remark 1. Note that since $[\dot{p}] = \omega[p] \propto e^{i\omega t_R}$, we have $A^\mu \propto e^{i\omega(t-r/c)}/r$, which geometrically are just outgoing spherical waves with a decaying amplitude $\propto 1/r$.

Remark 2. As shown in the derivation, this 4-potential ONLY holds for *Hertzian dipoles* in the *far field approximation*, i.e. we require the assumption $r \gg \lambda \gg b$.

4.3 Radiation from a Hertzian Dipole

Now that we have the 4-potential, the corresponding EM fields can be found easily using (3.45). In spherical coordinates, defining $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{b}} = \cos\theta$ as before, we have $\hat{\mathbf{z}} = (\cos\theta, -\sin\theta, 0)$, and the 4-gradient becomes

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial r}, -\frac{1}{r}\frac{\partial}{\partial \theta}, -\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)$$

Therefore, noting that $[\ddot{p}] = \omega[\dot{p}]$ and $\partial[\dot{p}]/\partial r = -\omega[\dot{p}]/c$, we have

$$\begin{aligned}
\partial^0 A^\nu &= \frac{1}{c}\frac{\mu_0\omega[\dot{p}]}{4\pi r}(\cos\theta, \cos\theta, -\sin\theta, 0) \\
\partial^1 A^\nu &= \frac{\mu_0}{4\pi}\left(\frac{[\dot{p}]}{r^2} + \frac{\omega[\dot{p}]}{cr}\right)(\cos\theta, \cos\theta, -\sin\theta, 0) \\
\partial^2 A^\mu &= \frac{\mu_0[\dot{p}]}{4\pi r^2}(\sin\theta, \sin\theta, \cos\theta, 0) \\
\partial^3 A^\mu &= 0
\end{aligned}$$

or equivalently,

$$\partial^\mu A^\nu = \frac{\mu_0[\dot{p}]}{4\pi r}\left[\frac{\omega}{c}\begin{pmatrix} \cos\theta & \cos\theta & -\sin\theta & 0 \\ \cos\theta & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \cos\theta & \cos\theta & -\sin\theta & 0 \\ \sin\theta & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]$$

Now, since $\lambda \ll r$, we have

$$\frac{\omega}{c} = \frac{2\pi}{\lambda} \gg \frac{1}{r}$$

so the first matrix dominates in the far field approximation. Hence,

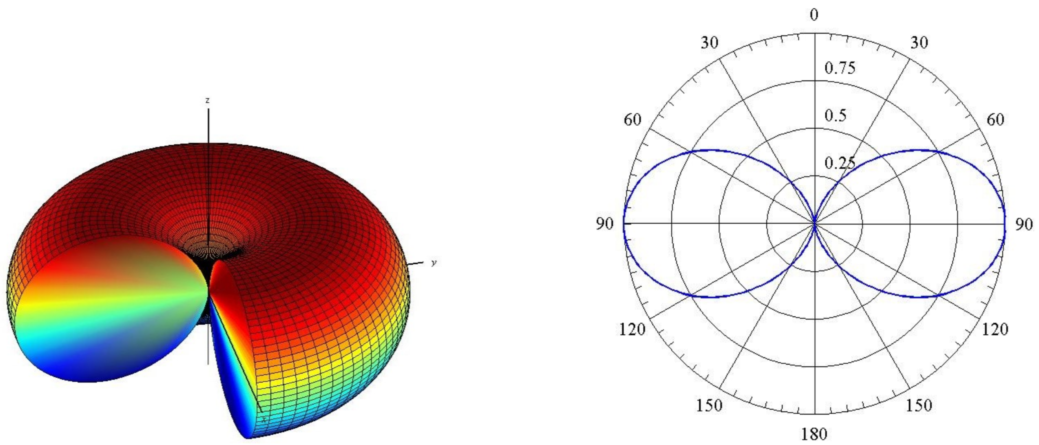
$$\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
&= \frac{\mu_0\omega[\dot{p}]}{4\pi rc}\left[\begin{pmatrix} \cos\theta & \cos\theta & -\sin\theta & 0 \\ \cos\theta & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \cos\theta & \cos\theta & 0 & 0 \\ \cos\theta & \cos\theta & 0 & 0 \\ -\sin\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\
&= \frac{\mu_0[\ddot{p}]}{4\pi rc}\begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & -\sin\theta & 0 \\ \sin\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Identifying this matrix with

$$\begin{pmatrix} 0 & -E_r/c & -E_\theta/c & -E_\phi/c \\ E_r/c & 0 & B_\phi & B_\theta \\ E_\theta/c & B_\phi & 0 & -B_r \\ E_\phi/c & -B_\theta & B_r & 0 \end{pmatrix} \equiv F^{\mu\nu} = \frac{\mu_0[\ddot{p}]}{4\pi rc}\begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & -\sin\theta & 0 \\ \sin\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we see that in spherical coordinates, a Hertzian dipole generates the following EM fields[26]:

[26] More precisely, whenever we speak of *"Radiation Fields"*, we are referring to fields in the limit $r \gg \lambda$ (alongside the usual $\lambda \gg b$ for Hertzian dipoles), as shown here.



(a) 3D distribution of dipole radiation

(b) Radiation pattern in the y-z plane

Figure 1: Radiation pattern of a Hertzian dipole[27]

Radiation from a Hertzian dipole

$$E_\theta = \frac{\mu_0[\ddot{p}]}{4\pi r}\sin\theta \tag{4.54}$$

$$B_\phi = \frac{\mu_0[\ddot{p}]}{4\pi rc}\sin\theta \tag{4.55}$$

$$E_r = E_\phi = 0 \tag{4.56}$$

$$B_r = B_\theta = 0 \tag{4.57}$$

Notice that the fields are mutually orthogonal with $E/B = c$, so we indeed have radiation as expected. Furthermore, recall that the *Poynting vector* for electromagnetic radiation is defined as

$$\mathbf{S} := \mathbf{E}\times\mathbf{H} = \mathbf{E}\times\frac{\mathbf{B}}{\mu_0} \tag{4.58}$$

Therefore, we have that for radiation from a Hertzian dipole,

$$\mathbf{S} = \frac{1}{\mu_0}\frac{\mu_0^2[\ddot{p}]^2}{(4\pi r)^2 c}\sin^2\theta\,(\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}) = \frac{\mu_0[\ddot{p}]^2}{(4\pi r)^2 c}\sin^2\theta\,\hat{\mathbf{r}}$$

Observe the angle dependence of $\mathbf{S}$: It has maximum magnitude when $\theta = \pi/2$, and minimum magnitude when $\theta = 0$ or π (with $R_{\max}/2$ at $\theta = \pi/4$). In particular, since $\mathbf{S}$ represents the directional energy flux (aka *power flow*) of an electromagnetic field, the radiation is concentrated around the (x-y) plane *perpendicular* to the dipole moment $\mathbf{p} \equiv p\hat{\mathbf{z}}$ (see Figure 1).

To quantify this angular dependence of radiated power, we define the *directivity* D as

Directivity

$$D := \frac{\text{Maximum radiated power}}{\text{Average radiated power}} \tag{4.59}$$

[27] Images taken from Photonics 101 (this website also comes with a rather handy and comprehensive introduction to electrodynamics).

We can first compute the total power radiated by simply integrating over some spherical surface of radius r, i.e.

$$
\begin{aligned}
P_{\text{total}}(t) &= \int_{\text{sphere}} \mathbf{S} \cdot \mathbf{dA} \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\mu_0 [\ddot{p}]^2}{(4\pi r)^2 c} \sin^2\theta\, \hat{\mathbf{r}} \cdot (r^2 \hat{\mathbf{r}} \sin\theta d\theta d\phi) \\
&= \frac{\mu_0 [\ddot{p}]^2}{(4\pi)^2 c} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} d\theta\ \sin^3\theta \\
&= \frac{\mu_0 [\ddot{p}]^2}{(4\pi)^2 c} \cdot 2\pi \cdot \left[\frac{1}{3}\cos^3\theta - \cos\theta\right]_0^{\pi} \\
&= \frac{\mu_0 [\ddot{p}]^2}{6\pi c}
\end{aligned}
$$

Then, the average power will be given by the total power divided by $4\pi r^2$. In particular, we have the directivity

$$
\boxed{D = \frac{R_{\max}}{R_{\text{avg}}} = \frac{\mu_0 [\ddot{p}]^2}{(4\pi r)^2 c} \left[\frac{1}{4\pi r^2}\left(\frac{\mu_0 [\ddot{p}]^2}{6\pi c}\right)\right]^{-1} = \frac{3}{2}}
$$

It is often useful to also include the angular dependence of $\mathbf{S}$ explicitly in the so-called *angular gain*. In this case, the angular gain is

$$
G(\theta, \phi) = \frac{3}{2}\sin^2\theta
$$

Furthermore, assuming $\mathbf{p}$ to be sinusoidal (e.g. if it is of the form $p = p_0 e^{i\omega t}$ as before), then

$$
\langle \ddot{p} \rangle = \omega^2 \langle p \rangle = \omega^2 p_0 / \sqrt{2}
$$

Hence, the total time-averaged power is

Total time-averaged power from dipole radiation

$$
\langle P_{\text{total}} \rangle = \frac{\mu_0 \langle [\ddot{p}]^2 \rangle}{6\pi c} = \frac{\mu_0 \omega^4 p_0^2}{12\pi c} \propto \frac{1}{\lambda^4} \tag{4.60}
$$

The key takeaway here is that $\langle P_{\text{total}} \rangle \propto \omega^4 \propto \lambda^{-4}$. In particular, the radiation is skewed heavily towards shorter wavelengths. This leads to the infamous *Rayleigh scattering*!

4.4 Rayleigh Scattering: Blue Skies and Red Sunsets

Due to incoming solar radiation, diatomic molecules in the atmosphere (e.g. N_2, O_2) experience an oscillatory electric field, which induces molecular dipoles of the form

$$
\mathbf{p}(t) = \alpha \mathbf{E}(t)
$$

for some constant $\alpha \in \mathbb{R}$. From above, we know that the molecules in turn give out radiation with a time-averaged power $\langle P \rangle \propto \lambda^{-4}$. That is, incoming sunlight of all wavelengths are scattered, but shorter wavelengths are scattered more strongly, hence Mr. Blue Sky™![28][29][30]

[28] Absolute banger from the Electric Light Orchestra (ELO), a band quite fittingly named for this module :)

[29] But why isn't the sky violet? The Sun, like every other star, has its own radiation spectrum, in this case the intensity peaks at around 500nm (green) and falls off in the violet region (as seen from, e.g. *Wien's displacement law*). Additionally, oxygen in the Earth's atmosphere absorbs photons of near-ultraviolet wavelengths. The resulting colour, which appears pale blue, is therefore a mixture of all the scattered colours (mainly blue and green).

[30] A fun fact: In locations with little light pollution, the moonlit night sky is also blue since moonlight is just reflected sunlight. The moonlit sky is not perceived as blue, however, because at low light levels, human vision comes mainly from rod cells which do not produce any colour perception (this is knows as the *Purkinje effect*).

Figure 2: Rayleigh scattering in opalescent glass[32]

During sunsets, due to the oblique angle, we see the sun through a much larger and denser proportion of the atmosphere near the Earth's surface, in which Rayleigh scattering removes a significant proportion of the shorter wavelength (blue/green) light from the direct path to the observer. The remaining unscattered light is therefore mostly of longer wavelengths, i.e. we get red sunsets. Note that this is also why we can safely look at sunsets directly in the first place — the sun's intense radiation is much attenuated by scattering along the way.

Aside: Interestingly, Rayleigh-type $\propto \lambda^{-4}$ scattering can also be demonstrated using nanoporous materials[31], e.g. scattered light in a piece of opalescent glass makes the glass appear blue from the side, while longer-wavelength orange light shines through (see Figure 2).

4.5 The Short Dipole Antenna

As mentioned before, electric dipoles are commonly found in antennae, and are easily set up by passing through an AC current $I(t)$ from the centre of the dipole to two nearby points (forming the two opposite "point charges").

More specifically, suppose we have a current given by

$$I(t_R) = I_0 \sin(\omega t_R)$$

Then, with dipole moment $\mathbf{p} = q(t_R)b\,\hat{\mathbf{z}} \implies \dot{\mathbf{p}} = I(t_R)b\,\hat{\mathbf{z}}$, we have

$$\ddot{\mathbf{p}}(t_R) = \frac{dI}{dt_R}b\,\hat{\mathbf{z}} = \omega I_0 b\cos(\omega t_R)\,\hat{\mathbf{z}} = 2\pi c\left(\frac{b}{\lambda}\right)I_0\cos(\omega t_R)\,\hat{\mathbf{z}}$$

where we note that $b \ll \lambda \implies b/\lambda \ll 1$ for Hertzian dipoles.

From above, we know that this antenna generates radiation with an average power of

$$\langle P_{\text{rad}}\rangle = \frac{\mu_0\langle[\ddot{p}]^2\rangle}{6\pi c} = \mu_0\frac{4\pi^2c^2}{6\pi c}\left(\frac{b}{\lambda}\right)^2\langle I_0^2\cos^2(\omega t_R)\rangle = \left[\frac{2\pi}{3}\left(\frac{b}{\lambda}\right)^2 Z_0\right]I_{\text{rms}}^2$$

where $I_{\text{rms}}^2 \equiv \langle I_0^2\sin^2(\omega t_R)\rangle = \langle I_0^2\cos^2(\omega t_R)\rangle$, and $Z_0 = \mu_0 c = \sqrt{\mu_0/\epsilon_0} \approx 377\Omega$ is the impedance of free space.

[31] In this case, the strong scattering is due instead to the large difference in refractive index between pores and solid parts within the material.

[32] Image (along with a long discussion on why the sky is blue) found on https://www.flickr.com/photos/optick/112909824/

Relating this to the usual power of an electrical circuit $P = I^2R$, we define a new quantity known as the *radiation resistance*, given by

$$R_{\text{rad}} := \frac{\langle P_{\text{rad}} \rangle}{\langle I^2 \rangle}$$

This is a so-called *effective* resistance. Unlike conventional (Ohmic) resistance, radiation resistance is not due to the resistivity of the imperfect conducting materials the antenna is made of, but rather due to the power carried from the antenna as radiation (usually as radio waves).

For dipole antennae, we have

Radiation resistance of a dipole antenna

$$R_{\text{rad}} = \frac{\langle P_{\text{rad}} \rangle}{\langle I^2 \rangle} = \frac{2\pi}{3}\left(\frac{b}{\lambda}\right)^2 Z_0 \tag{4.61}$$

We can also speak of an antenna's *radiation efficiency*

$$\eta := \frac{P_{\text{rad}}}{P_{\text{in}}} = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{Ohm}}}$$

where R_{Ohm} is the Ohmic resistance of the antenna. So, for an efficient (high η) antenna, we want $R_{\text{rad}} \gg R_{\text{Ohm}}$.

Example 2.

For a short dipole with $b = 1\,\text{cm}$, $\lambda = 1\,\text{m}$ ($\sim 300\,\text{MHz}$), we have

$$R_{\text{rad}} = \frac{2\pi}{3}\left(\frac{1}{100}\right)^2 (377) \approx 80\,\text{m}\Omega$$

To radiate 1W of power, we require a driving current of

$$I = \frac{P}{R_{\text{rad}}^2} = \frac{1}{0.08^2} \approx 156\,\text{A}$$

Notice here how the low R_{rad} of Hertzian dipoles requires antennae to draw impractically high currents. In practice, we use longer antennae to increase R_{rad}, though this does mean that our original $b \ll \lambda$ assumption will no longer hold.

4.6 A Longer Antenna: The Half-Wave Dipole Antenna/Aerial

Suppose now we have a longer dipole, i.e. we remove the assumption that $b \ll \lambda$ (while keeping the far field approximation $r \gg \lambda$). In this case, the retarded time t_R is no longer uniform across the dipole, but rather given by

$$t_R(z) = t - \frac{r - z\cos\theta}{c}$$

where z is the relative height of a point on the antenna from the centre of the dipole. Furthermore, by requiring that $I \to 0$ at $z = \pm b/2$ (due to the physical configuration of the wire), the current varies along the antenna (along $\hat{\mathbf{z}}$) as

$$I(t_R, z) = I_0 \cos\left(\frac{\pi z}{b}\right) e^{i\omega t_R} = I_0 \cos\left(\frac{\pi z}{b}\right) \exp\left[i\omega\left(t - \frac{r - z\cos\theta}{c}\right)\right]$$

for some maximum current I_0.

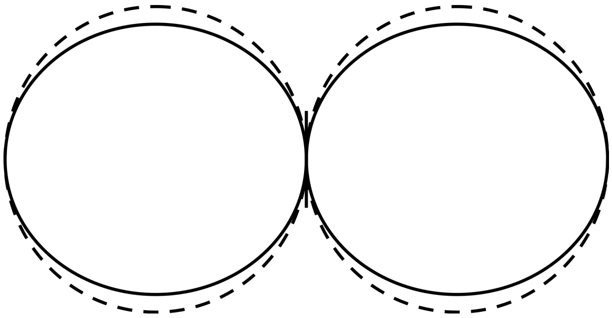


Figure 3: Radiation pattern of a half-wave dipole (solid line) vs a Hertzian dipole (dashed line)[33]

The retarded potential (with $r \gg b$) is then given by

$$\begin{aligned} A^\mu(t,r) &= \frac{\mu_0}{4\pi} \int_{-b/2}^{+b/2} dz \, \frac{I(t_R,z)}{|r - z\cos\theta|} \\ &= \frac{\mu_0 I_0}{4\pi r} \int_{-b/2}^{+b/2} dz \, \cos\left(\frac{\pi z}{b}\right) \exp\left[i\omega\left(t - \frac{r - z\cos\theta}{c}\right)\right] \\ &= \frac{\mu_0 I_0}{4\pi r} \exp\left[i\omega\left(t - \frac{r}{c}\right)\right] F(\theta) \end{aligned}$$

where

$$F(\theta) := \int_{-b/2}^{+b/2} dz \, \cos\left(\frac{\pi z}{b}\right) \exp\left(i\omega \frac{z\cos\theta}{c}\right) = \frac{\cos\left(\frac{\pi b}{\lambda}\cos\theta\right) - \cos\left(\frac{\pi b}{\lambda}\right)}{\sin^2\theta}$$

For the *half-wave dipole* (HWD), we choose $b = \lambda/2$ such that

$$F(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Substituting this gives a radiation field of

$$E_\theta = cA\sin\theta = c\,\frac{\mu_0 I_0}{4\pi r}\,\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \exp\left[i\omega\left(t - \frac{r}{c}\right)\right]$$

Hence, we get a radiated power of

$$P \propto E^2 \propto \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right]$$

We can also compute the directivity of a half-wave dipole via

$$D_{\text{HWD}} = \frac{R_{\text{max}}}{R_{\text{avg}}} = \frac{4\pi}{\iint d\theta d\phi \, \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin\theta} \approx 1.66 > 1.5 = D_{\text{Hertzian}}$$

Plotting out the radiation pattern of both the HWD and the Hertzian dipole (see Figure 3) confirms that radiation from the HWD is indeed more focused radially (along the x-y plane).

[33] Image taken from https://commons.wikimedia.org/wiki/File:L-over2-rad-pat.svg

4.7 Antenna reciprocity

Reciprocity states that the receiving and transmitting properties of an antenna are identical. The emitting antenna has Poynting vector

$$N_+ = \frac{N_0}{r^2} G_1(\theta, \phi)$$

and the receiving antenna has Poynting vector

$$N_- = \frac{N_0}{r^2} G_2(\theta, \phi)$$

where G_1 and G_2 are the gain of the antennas. Hence, the power emitted is

$$P_{R_1} = N_- A_1(\theta, \phi)$$

and the power received is

$$P_{R_2} = N_+ A_2(\theta', \phi').$$

Reciprocity tells us that $P_{R_1} = P_{R_2}$, hence

$$\begin{aligned}
N_- A_1(\theta, \phi) &= N_+ A_2(\theta', \phi') \\
\frac{N_0}{r^2} G_2(\theta', \phi') A_1(\theta, \phi) &= \frac{N_0}{r^2} G_1(\theta, \phi) A_2(\theta', \phi') \\
\frac{A_1(\theta, \phi)}{G_1(\theta, \phi)} &= \frac{A_2(\theta', \phi')}{G_2(\theta', \phi')} = \text{ constant} \\
&= \frac{\lambda^2}{4\pi}.
\end{aligned}$$

Hence,

$$A_1(\theta, \phi) = \frac{\lambda^2}{4\pi} G_1(\theta, \phi). \tag{4.62}$$

4.8 Multi-element antennae

If we have two short dipoles separated by a distance d, then the power received at (r, θ, ϕ) is proportional to

$$\frac{\sin^2\theta}{r^2}\left(1 + e^{i\alpha}\epsilon^{i\frac{2\pi}{\lambda}d\sin\theta}\right)^2 = \frac{\sin^2\theta}{r^2}\cos^2\left(\alpha + \frac{2\pi d}{\lambda}\sin\phi\right).$$

Setting $d = \frac{\lambda}{4}$, we get

$$P \propto \frac{\sin^2\theta}{r^2}\cos^2\left(\alpha + \frac{\pi}{2}\sin\theta\right).$$

For a linear array of N dipoles, separated by a distance d, we have

$$E(r, \theta, \phi) \propto \frac{\sin\theta}{r}\sum_{m=0}^{N} \epsilon^{im\alpha}\epsilon^{i\frac{2\pi}{\lambda}d\sin\phi}.$$

So,

$$\begin{aligned}
P &\propto \frac{\sin^2\theta}{r^2}\left[\frac{1 - \epsilon^{i\left(\alpha + \frac{2\pi}{\lambda}d\sin\phi\right)N}}{1 - \epsilon^{i\left(\alpha + \frac{2\pi}{\lambda}d\sin\phi\right)}}\right]^2 \\
&\propto \frac{\sin^2\theta}{r^2}\left[\frac{\sin^2\left(\alpha + N\frac{\pi d}{\lambda}\sin\phi\right)}{\sin^2\left(\alpha + \frac{\pi d}{\lambda}\sin\phi\right)}\right].
\end{aligned}$$

4.9 Radio power transmission

Suppose we have a transmitter with an oscillating current $I = I_0 \epsilon^{i2\pi ft}$, then it will radiate a power $P_S = I_S^2 R_{\text{Rad}}$. The power emitted at a distance r is

$$P_R = \langle N \rangle 4\pi r^2 = \frac{N_{S_0}}{D_S} 4\pi r^2,$$

where N_{S_0} is the peak Poynting vector and D_S is the directivity of the transmitter. Hence, the power received is

$$P_R = \frac{\lambda^2}{4\pi} D_R N_{S_0} = \frac{\lambda^2}{4\pi r^2} D_R D_S P_S.$$

4.10 Parabolic antennae

The electric field at an angle θ from the axis of a parabolic antenna of diameter D (not to be confused with the directivity) is

$$E(\theta) = E_0 \left[\frac{J_1\left(\frac{\pi D}{\lambda} \sin\theta\right)}{\frac{\pi D}{\lambda} \sin\theta} \right],$$

where J_1 is a first-order Bessel function. Determining the first nulls of the radiation pattern gives the beam-width θ_0. The term $J_1(x) = 0$ whenever $x = 3.83$. Thus,

$$\frac{\pi D}{\lambda} \sin\theta_0 = 3.83 \quad \Longrightarrow \quad \theta_0 = 1.22 \frac{\lambda}{D},$$

using the small angle approximation.

For a distant source N is approximately constant across the dish, so

$$P_R = \int N \cdot dS = N\pi \left(\frac{D}{2}\right)^2.$$

We also know that

$$P_R = \frac{\lambda^2}{4\pi r^2} G(\theta, \phi) N,$$

hence

$$G(\theta, \phi) = \frac{4\pi}{\lambda^2} \left(\frac{D}{2}\right)^2 = \left(\frac{\pi D}{\lambda}\right)^2.$$

The directivity is equal to $\frac{4\pi}{\Omega}$, where Ω is the solid angle subtended by the antenna.

5 Waveguides

Consider a rectangular waveguide with cross-sectional dimensions a and b along the x and y axes respectively. Along the walls of the waveguide, the electric field is zero, so the boundary conditions are $E_x = 0$ (for $y = 0$ or $y = b$) and $E_y = 0$ (for $x = 0$ or $x = a$). Elsewhere in the waveguide, the electric field satisfies

$$E_x = \sin\left(\frac{n\pi x}{b}\right) f(x) \qquad\qquad E_y = \sin\left(\frac{m\pi y}{a}\right) g(y),$$

where $f(x)$ and $g(y)$ are unknown functions, and n and m are integers.

We look for a transverse electric (TE) mode, so $E_z = 0$. Maxwell's equations (M1) give

$$\nabla \cdot \mathbf{E} = 0 \quad \Longrightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0.$$

By inspection, we find that the electric field is

$$\mathbf{E} = \begin{pmatrix} -\frac{a}{m} E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ \frac{b}{n} E_0 \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \\ 0 \end{pmatrix} e^{i(\omega t - k_z z)}.$$

To find the magnetic field, we use (M3), that is $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$. Setting $n = 0$[34], we get

$$\mathbf{E} = E_0 \sin\left(\frac{m\pi x}{a}\right) e^{i(\omega t - k_z z)} \hat{y}.$$

Then

$$\frac{\partial \mathbf{B}}{\partial t} = \begin{pmatrix} ikE_0 \sin\left(\frac{m\pi x}{a}\right) \\ 0 \\ E_0 \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \end{pmatrix} e^{i(\omega t - k_z z)},$$

and

$$\mathbf{B} = \begin{pmatrix} \frac{k_z}{\omega} E_0 \sin\left(\frac{m\pi x}{a}\right) \\ 0 \\ \frac{1}{i\omega} E_0 \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \end{pmatrix} e^{i(\omega t - k_z z)}.$$

Since, $\mathbf{B}$ has a component in the direction of propagation, $\hat{z}$, this is not a transverse magnetic field. If we set $m = 1$ then we have the TE_{10} mode.

Let

$$k_x = \frac{m\pi}{a} \quad \text{and} \quad k_y = \frac{n\pi}{b},$$

and recall that $k_z = \frac{2\pi}{\lambda_g}$, where λ_g is the guide wavelength. The contravariant wave vector is $k^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$, and $k^\mu k_\mu = 0$ in all frames. Hence,

$$\frac{\omega^2}{c^2} - k_x^2 - k_y^2 - k_z^2 = 0 \quad \Longrightarrow \quad k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2. \tag{5.63}$$

In free space $\lambda_0 = \frac{2\pi c}{\omega}$. For propagation, we require k_z to be real, so

$$\frac{\omega^2}{c^2} > \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \quad \Longrightarrow \quad \left(\frac{2\pi}{\lambda_g}\right)^2 > \left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{2\pi}{\lambda_c}\right)^2,$$

[34] As you can see, we obviously don't set $n = 0$ so we divide by zero. What we really mean is take the limit as $n \to 0$ and use the famous limit that

$$\lim_{n \to 0} \frac{\cos(nx)}{nx} = 1.$$

Here, $x = \frac{\pi y}{b}$ and to get the expression in the right form, we just absorb these constants into E_0 after taking the limit.

where λ_c is the **critical wavelength** defined by

Critical Wavelength

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}.$$

The critical wavelength is the longest wavelength that can propagate in the waveguide.

5.1 Waveguide dispersion

Substituting $k_c = \frac{2\pi}{\lambda_c}$ into Eq. (5.63), we get

$$k_z^2 = \frac{\omega^2}{c^2} - k_c^2.$$

The phase velocity is

$$v_\phi = \frac{\omega}{k_z} = \frac{c}{\sqrt{1 - \left(\frac{\lambda_c}{\lambda_0}\right)^2}}.$$

The group velocity is

$$v_g = \frac{d\omega}{dk_z} = c\sqrt{1 - \left(\frac{\lambda_c}{\lambda_0}\right)^2}.$$

The geometric mean of the phase and group velocities is given by $\sqrt{v_\phi v_g} = c$.

We would like to see which modes can propagate in a square waveguide, that is $a = b$. The critical wavelength is

$$\lambda_c = \frac{2a}{\sqrt{m^2 + n^2}}.$$

If $a = 1\mathrm{cm}$, we find that

- $\lambda_{10} < 2\,\mathrm{cm}$ or $f_{10} > 15\,\mathrm{GHz}$.
- $\lambda_{11} < 1.4\,\mathrm{cm}$ or $f_{11} > 21\,\mathrm{GHz}$.
- $\lambda_{20} < 1\,\mathrm{cm}$ or $f_{20} > 30\,\mathrm{GHz}$.

If $f < 15\,\mathrm{GHz}$, then

$$k_z^2 = \left(\frac{2\pi f}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 < 0,$$

which implies that k_z is imaginary, and the wave decays exponentially.

5.2 Power flow in waveguides

Recall that the time-averaged Poynting vector (power flow) is

$$\langle \mathbf{N} \rangle = \frac{1}{2}\,\mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right) \frac{E_0^2}{2\mu_0 c^2} = \frac{1}{2}\varepsilon_0 c E_0^2 = \frac{1}{2}\frac{E_0^2}{Z_0}.$$

Consider a the TE_{m0} mode in a rectangular waveguide, where

$$\mathbf{E} = \begin{pmatrix} 0 \\ E_0 \sin\left(\frac{m\pi x}{a}\right) \\ 0 \end{pmatrix} e^{i(\omega t - k_z z)} \quad \text{and} \quad \mathbf{B} = \frac{E_0}{\omega}\begin{pmatrix} k_z \sin\left(\frac{m\pi x}{a}\right) \\ 0 \\ i\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right) \end{pmatrix} e^{i(\omega t - k_z z)}.$$

Hence, the power flow is

$$\begin{aligned}\langle \mathbf{N} \rangle &= \frac{E_0^2}{2\omega} \operatorname{Re} \begin{bmatrix} -i\left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \\ 0 \\ k_z \sin^2\left(\frac{m\pi x}{a}\right) \end{bmatrix} \\ &= \frac{E_0^2}{2\omega} k_z \sin^2\left(\frac{m\pi x}{a}\right) \hat{z} \\ &= \frac{1}{2}\varepsilon_0 E_0^2 v_g \sin^2\left(\frac{m\pi x}{a}\right) \hat{z}.\end{aligned}$$

5.3 Total power in waveguides

The total power in a waveguide is

$$\begin{aligned}P &= \int \langle \mathbf{N} \rangle \cdot \mathbf{dS} \\ &= \int_{y=0}^{b} \int_{x=0}^{a} \langle \mathbf{N} \rangle \, dx dy \\ &= \frac{1}{2}\varepsilon_0 E_0^2 v_g \left(\frac{ab}{2}\right).\end{aligned}$$

The power does not depend on m, so all modes carry the same power.

5.4 TM modes

For a TM mode, we have $B_z = 0$. Since $\nabla \cdot \mathbf{B} = 0$ (M2), $\mathbf{B}$ must be in loops. We find

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i(\omega t - k_z z)}.$$

The lowest TM mode is the TM_{11} mode, as we can't have $m = 0$ or $n = 0$.

5.5 Circular waveguides

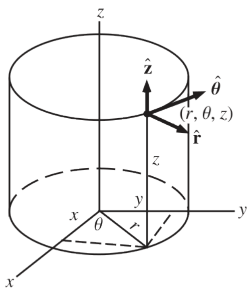


Figure 4: Cylindrical coordinates

Let our circular waveguide have a radius of R. For a TE mode, we have $E_z = 0$. The boundary conditions are $E_\theta = 0$ (for $r = R$). The TE_{01} mode, i.e. $r = 0$ and $\theta = 1$, is the low-loss mode, as $\mathbf{E}$ does not touch the walls of the waveguide. The lowest energy mode is the TE_{11} mode, which dominates. The TE_{01} mode is preferred for low-noise applications, while the TE_{11} mode is preferred for high-power applications.

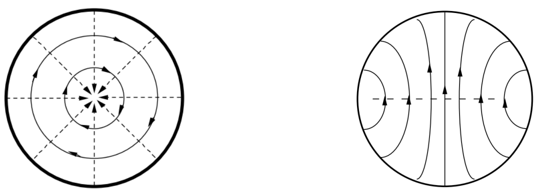


Figure 5: TE_{01} and TE_{11} modes

5.6 Optical fibres

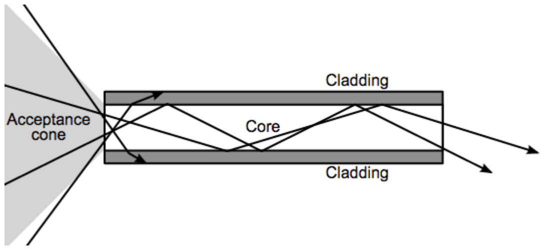


Figure 6: Optical fibre

The core has a radius a and refractive index n_1, the cladding has a radius b and refractive index n_2, and the air has a refractive index n_0. Total internal reflection occurs when the angle of incidence is greater than the critical angle, θ_c. The critical angle is given by

$$\sin\theta_c = \frac{n_2}{n_1}.$$

At the entrance,

$$\sin\theta_c = \frac{n_1}{n_0}\cos\theta_1 = \frac{n_1}{n_0}\sqrt{1-\left(\frac{n_2}{n_1}\right)^2},$$

where θ_1 is the angle of incidence.

6 Potentials for moving charges

Recall the Liénard-Wiechert potential of a single charge

$$A^{\beta}(x^{\mu}) = \frac{\mu_0}{4\pi}\int \frac{[j^{\beta}]}{|\mathbf{r}-\tilde{\mathbf{r}}|} d^4\tilde{x}^{\mu},$$

where $[j^{\beta}] = j^{\beta}(ct_R, \tilde{\mathbf{r}})$ is the 4-current of the source at the retarded time $t_R = t - \frac{|\mathbf{r}-\tilde{\mathbf{r}}|}{c}$ (also denoted $\tilde{t}$). We can split the variables of integration $d^4\tilde{x}^{\mu} = d\left(c\tilde{t}\right) d^3\tilde{\mathbf{r}}$ into time and space components. Integrating $\left[j^{\beta}\right]$ over the space components gives

$$\begin{aligned}\int j^{\beta}(c\tilde{t}, \tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}} &= (cq, \tilde{\mathbf{v}}q) \\ &= q\left(c, \tilde{\mathbf{v}}\right) \\ &= \frac{q}{\gamma}\left[\tilde{u}^{\beta}\right],\end{aligned}$$

where $\tilde{u}^{\beta} = (c, \tilde{\mathbf{v}})$ is the four-velocity of the source. Notice that A^{β} relates to $\tilde{u}^{\beta}$, both of which are 4-vectors. Hence, the quantity that relates them must be a scalar, which is frame invariant. Evaluating the potential

$$A^{\beta} = \frac{\mu_0 q}{4\pi}\left[\tilde{u}^{\beta}\right]\int \frac{1}{\gamma}\frac{\delta(t_R)}{|\mathbf{r}-\tilde{\mathbf{r}}|} d(c\tilde{t}),$$

in the rest frame of the source, we have $\gamma = 1$ and the integral $\int \ldots d\left(c\tilde{t}\right) = \frac{1}{R_s}$, where $R_s = |\mathbf{r}-\tilde{\mathbf{r}}|$ is the distance between the source and the observer in the rest frame of the source at the retarded time. Hence,

Liénard-Wiechert potential

$$y = 6A^{\beta} = \frac{\mu_0 q}{4\pi}\frac{\left[\tilde{u}^{\beta}\right]}{R_s}. \tag{6.64}$$

Consider the 4-separation between the source and the observer $X^{\beta} = x^{\beta} - \tilde{x}^{\beta}$. We create an invariant scalar

$$X^{\beta}\tilde{u}_{\beta} = \left(c\left(t-\tilde{t}\right), \mathbf{r}-\tilde{\mathbf{r}}\right)\cdot\gamma\left(c, -\tilde{\mathbf{v}}\right).$$

In the source frame of reference

$$X^{\beta}\tilde{u}_{\beta} = (R_s, \mathbf{R}_s)\cdot(c, 0) = cR_s,$$

which implies that

$$R_s = \frac{X^{\beta}\tilde{u}_{\beta}}{c}.$$

6.1 Fields of moving charges

First let's rewrite the Liénard-Wiechert potential explicitly

$$A_{\nu}\left(x^{\mu}, \tilde{\tau}(\tilde{x}^{\mu})\right) = \frac{\mu_0 q}{4\pi}\frac{\tilde{u}_{\nu}(\tilde{\tau})}{R_s(x^{\mu}, \tilde{\tau})}.$$

To find the electric field, we need to calculate

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}.$$

Recall that $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$. Hence,

$$\partial_{\mu}A_{\nu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} + \frac{\partial A_{\nu}}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial x^{\mu}}.$$

First term

We note that the only term in A_ν dependent on x^μ is R_s. Hence,

$$\frac{\partial A_\nu}{\partial x^\mu} = \frac{\mu_0 q}{4\pi}\frac{\tilde{u}_\nu}{R_s^2}\frac{\partial R_s}{\partial x^\mu}.$$

We compute

$$\frac{\partial R_s}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\left(\frac{X^\mu \tilde{u}_\mu}{c}\right) = \frac{\partial}{\partial x^\mu}\left(\frac{(x^\mu - \tilde{x}^\mu)\tilde{u}_\mu}{c}\right) = \frac{1}{c}\tilde{u}_\mu,$$

to find that

$$\frac{\partial A_\nu}{\partial x^\mu} = \frac{\mu_0 q}{4\pi}\frac{\tilde{u}_\nu \tilde{u}_\mu}{cR_s^2} \quad\Longrightarrow\quad \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = 0.$$

Second term

By the product rule we compute

$$\frac{\partial A_\nu}{\partial \tilde{\tau}} = \frac{\mu_0 q}{4\pi}\left[\frac{1}{R_s}\frac{\partial \tilde{u}_\nu}{\partial \tilde{\tau}} - \frac{\tilde{u}_\nu}{R_s^2}\frac{\partial R_s}{\partial \tilde{\tau}}\right] = \frac{\mu_0 q}{4\pi}\left[\frac{\tilde{a}_\nu}{R_s} - \frac{\tilde{u}_\nu}{R_s^2 c}\left(X^\beta \tilde{a}_\beta - c^2\right)\right],$$

where we computed

$$\begin{aligned}
\frac{\partial R_s}{\partial \tilde{\tau}} &= \frac{\partial}{\partial \tilde{\tau}}\left(\frac{X^\beta \tilde{u}_\beta}{c}\right) \\
&= \frac{1}{c}\left(X^\beta \frac{\partial \tilde{u}_\beta}{\partial \tilde{\tau}} - \tilde{u}_\beta \frac{\partial X^\beta}{\partial \tilde{\tau}}\right) \\
&= \frac{1}{c}\left(X^\beta \tilde{a}_\beta - \tilde{u}_\beta \frac{\partial X^\beta}{\partial \tilde{x}^\beta}\frac{\partial \tilde{x}^\beta}{\partial \tilde{\tau}}\right) \\
&= \frac{1}{c}\left(X^\beta \tilde{a}_\beta - \tilde{u}_\beta \tilde{u}^\beta\right) \\
&= \frac{1}{c}\left(X^\beta \tilde{a}_\beta - c^2\right).
\end{aligned}$$

Third term

We have that

$$\frac{\partial \tilde{\tau}}{\partial x^\mu} = \frac{\partial \tilde{\tau}}{\partial \tilde{x}^\mu}\frac{\partial \tilde{x}^\mu}{\partial x^\mu} = \frac{1}{\tilde{u}^\mu} = \frac{X_\mu}{cR_s},$$

where we used the fact that the frame invariant separation is light-like, i.e.

$$\begin{aligned}
&X_\mu X^\mu = 0 \\
&(x_\mu - \tilde{x}_\mu)(x^\mu - \tilde{x}^\mu) = 0 \quad\Longrightarrow\quad \tilde{x}^\mu = x^\mu \quad\Longrightarrow\quad \frac{\partial \tilde{x}^\mu}{\partial x^\mu} = 1.
\end{aligned}$$

Putting it all together

Multiplying the second and third terms, we get

$$\begin{aligned}
\frac{\partial A_\nu}{\partial \tilde{\tau}}\frac{\partial \tilde{\tau}}{\partial x^\mu} &= \frac{\mu_0 q}{4\pi}\left[\frac{\tilde{a}_\nu}{R_s} - \frac{\tilde{u}_\nu}{R_s^2 c}\left(X^\beta \tilde{a}_\beta - c^2\right)\right]\frac{X_\mu}{cR_s} \\
&= \frac{\mu_0 q}{4\pi}\left[\frac{\tilde{a}_\nu X_\mu}{cR_s^2} + \frac{\tilde{u}_\nu X_\mu}{R_s^3}\left(1 - \frac{X^\beta \tilde{a}_\beta}{c^2}\right)\right].
\end{aligned}$$

Hence,

Faraday Tensor for moving charges

$$F_{\mu\nu} = \frac{\mu_0 q}{4\pi}\left[\frac{\tilde{a}_\nu X_\mu - \tilde{a}_\mu X_\nu}{cR_s^2} + \frac{\tilde{u}_\nu X_\mu - \tilde{u}_\mu X_\nu}{R_s^3}\left(1 - \frac{X^\beta \tilde{a}_\beta}{c^2}\right)\right]. \tag{6.65}$$

The energy flows in the direction of the Poynting vector $\mathbf{N} \propto E^2$. For a particle at constant velocity, we deduce that $F_{\mu\nu} \propto \frac{1}{r^2}$ and $\mathbf{N} \propto \frac{1}{r^4}$. If we consider the power radiated $P = \int_{\text{sphere}} \mathbf{N} \cdot d\mathbf{S} \propto \frac{1}{r^2}$, which tends to 0 as $r \to \infty$.

For an accelerating charge, we have $F_{\mu\nu} \propto \frac{1}{r}$ and $\mathbf{N} \propto \frac{1}{r^2}$. Hence, $P \propto r^0$, i.e. some finite amount of power is radiated at large distances. Its dependence on the acceleration means that it's highly directional.

6.2 Fields of uniformly moving charges

We set all the acceleration terms to zero, such that

$$F_{\mu\nu} = \frac{\mu_0 q}{4\pi}\frac{\tilde{u}_\nu X_\mu - \tilde{u}_\mu X_\nu}{R_s^3}.$$

Source Frame	**Laboratory Frame**
$\tilde{u}_\nu = (c, 0)$	$u_\nu = \gamma(c, -\tilde{\mathbf{v}})$
$X_\mu = (R_s, -\mathbf{R}_s)$	$X_\mu = (x_\mu - \tilde{x}_\mu) = \left(c(t-\tilde{t}), -(\mathbf{r} - \tilde{\mathbf{r}})\right) = (R, -\mathbf{R})$
$R_s = \frac{X^\beta \tilde{u}_\beta}{c}$	$R_s = \gamma(R - \frac{\tilde{\mathbf{v}}\cdot\mathbf{R}}{c})$

Let's consider the case where the source is moving in the x-direction with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{x}}$. In the laboratory frame, we have

$$F_{0\nu} = \frac{q\mu_0}{4\pi}\frac{\gamma}{R_s^3}\begin{pmatrix} 0 \\ -R\tilde{v}_x + cx \\ cy \\ cz \end{pmatrix}.$$

Hence, we can find the electric field by identifying $\mathbf{E} = cF_{0i}$ where $i = 1, 2, 3$ denotes the spatial components. We find

$$\mathbf{E} = \frac{q\mu_0}{4\pi}\frac{\gamma\left(\mathbf{R} - \frac{R}{c}\mathbf{v}\right)}{R_s^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\left(\mathbf{R} - \frac{R}{c}\mathbf{v}\right)}{\left(R - \frac{\mathbf{R}\cdot\mathbf{v}}{c}\right)^3}.$$

If we write $\mathbf{r} = \mathbf{R} - R\frac{\mathbf{v}}{c}$, which denotes the position of the charge at the retarded time, where $\mathbf{R}$ is the position of the observer, and let

$$x = vt_R + r_\parallel = \frac{v}{c}R + r_\parallel$$
$$y = r_\perp$$
$$z = 0.$$

Then, we have

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\mathbf{r}}{R^3} = E_0\gamma\frac{\mathbf{r}}{R},$$

where $E_0 = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}$ is electrostatic case. Hence, the electric field for a moving charge is enhanced by a factor of γ in the direction pointing from the extrapolated position of the charge to the observer. In fact, it can be shown that

$$\mathbf{E} = E_0\gamma \, \frac{\mathbf{r}}{\sqrt{{r_\perp}^2 + \gamma^2 {r_\parallel}^2}}.$$

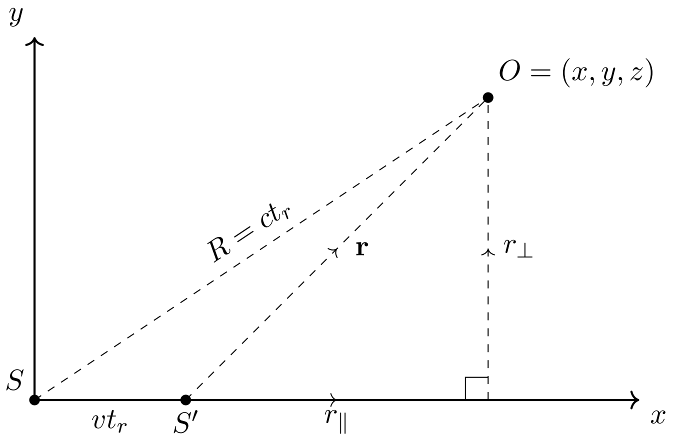


Figure 7: Diagram showing the positions of the source, its projected position, and the observer in the lab frame.

Note that for $r_\perp = 0$, we have $\mathbf{E} = E_0\hat{\mathbf{x}}$, which is the same as the electrostatic case. For $r_\parallel = 0$, we have $\mathbf{E} = E_0\gamma\hat{\mathbf{y}}$, so the electric field is enhanced by a factor of γ.

By identifying terms of the Faraday tensor, we find that the magnetic field is

$$\mathbf{B} = \begin{pmatrix} F_{32} \\ F_{13} \\ F_{21} \end{pmatrix} = \frac{q}{4\pi\varepsilon_0}\frac{1}{c^2R^3}\begin{pmatrix} -z\cdot 0 - (-y)\cdot 0 \\ -x\cdot 0 - (-z)\cdot(-\gamma v) \\ -y\cdot(-\gamma v) - (-z)\cdot 0 \end{pmatrix} = \frac{E_0}{c}\frac{\gamma v}{R^3}\begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}.$$

As expected, the magnetic field is perpendicular to the electric field and the direction of motion. The magnetic field is also enhanced by a factor of γ.

6.3 Fields of accelerating charge

Suppose we are starting off with an accelerating charge in the low-velocity limit, i.e. $\gamma \sim 1$ so our source 4-velocity $\tilde{u}^\mu \sim (c, \mathbf{v})$ and our source 4-acceleration is $\tilde{a}^\mu = (0, \mathbf{a})$. Recalling that $X^\mu = (R, \mathbf{R})$ is our 4-separation, then $X^\mu a_\mu = -\mathbf{a}\cdot\mathbf{R}$ and $R_s \sim R$.

Let us find $E_x = cF^{10} = cF_{01}$ which we will get from Eq. (6.65):

$$E_x = \frac{q}{4\pi\epsilon_0}\left[\frac{\tilde{a}_1X_0 - \tilde{a}_0X_1}{cR_s^2} + \frac{\tilde{u}_1X_0 - \tilde{u}_0X_1}{R_s^3}\left(1 - \frac{X^\beta\tilde{a}_\beta}{c^2}\right)\right].$$

Substituting our approximations in gives

$$E_x = \frac{q}{4\pi\epsilon_0}\left[-\frac{Ra_x}{c^2R^2} + \frac{xc - Rv_x}{cR^3}\left(1 + \frac{\mathbf{a}\cdot\mathbf{R}}{c^2}\right)\right].$$

This can be rewritten as

$$E_x = \frac{q}{4\pi\epsilon_0}\left[\frac{x}{R^3} - \frac{R^2a_x}{c^2R^3} + \frac{(\mathbf{a}\cdot\mathbf{R})x}{c^2R^3}\right].$$

where we eliminate the v_x terms in the small-velocity limit. Now we can simplify this greatly. Observe that if we do this for every component and turn it into a vector, $x \to \mathbf{R}$, $R^2 a_x \to (\mathbf{R}\cdot\mathbf{R})\mathbf{a}$ we see that

Electric field of accelerating charge

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{R}}{R^3} + \frac{(\mathbf{R}\times(\mathbf{R}\times\mathbf{a}))}{c^2R^3}\right] = \frac{q}{4\pi\epsilon_0}\left[\mathbf{E}_{\text{static}} + \mathbf{E}_{\text{acc}}\right] \tag{6.66}$$

Notice the acceleration term in addition to what you expect at electrostatics (when $\mathbf{a} = \mathbf{0}$). We can write this using the fact that $\mathbf{R}$ is an angle θ away from $\mathbf{a}$ and $\mathbf{R} = R\hat{\mathbf{e}}_\theta$ we get

$$\mathbf{E}_{\text{acc}} = \frac{q\,|\mathbf{a}|\sin\theta}{4\pi\epsilon_0 c^2 R}\hat{\mathbf{e}}_\theta$$

Notice however that $q\,|\mathbf{a}|\,\hat{\mathbf{e}}_\theta = \ddot{\mathbf{p}}$. This is something like an oscillating dipole!

6.3.1 Fields when $\tilde{\mathbf{a}} \parallel \tilde{\mathbf{v}}$

We will work in **spherical coordinates** centred on the charge with φ the azimuth angle, θ the zenith and r the radial distance and that the 3-acceleration and 3-velocity are both oriented in $\hat{\mathbf{e}}_z$. Then

- 4- separation $X^\mu = (R, R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$
- 4 - velocity $\tilde{u}^\mu = \gamma(c, 0, 0, v)$
- 4 -acceleration $\tilde{a}^\mu = \gamma^2\left(\gamma^2\frac{va}{c}, 0, 0, a\left(1 + \gamma^2\frac{v^2}{c^2}\right)\right)$

$$1 + \gamma^2\frac{v^2}{c^2} = 1 + \frac{1}{\left(1 + \frac{v^2}{c^2}\right)}\cdot\frac{v^2}{c^2} = \frac{1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} = \frac{1}{1 - \frac{v^2}{c^2}} = \gamma$$
$$\therefore \quad \tilde{a}^\mu = \gamma^4\left(\frac{v}{c}a, 0, 0, a\right)$$

Remark. *It is at this point in the lectures where he generally drops the $\sim$ symbol above the acceleration and velocity 4-vectors (for the charge) since only one spatial component has a non-zero value, but he still refers to the 4-vectors themselves. We've re-added back the $\sim$ symbols for 4-vectors and dropped them only when we are referring to the value of the element of the 4-vector.*

We need $X^\alpha\tilde{a}_\alpha = X^0\tilde{a}^0 - X^3\tilde{a}^3$ (since $\tilde{\alpha}^1 = \tilde{a}^2 = 1$) and $cR_s = X^\beta\tilde{u}_\beta$

$$\begin{aligned} &= X^0\tilde{u}^0 - X^3\tilde{u}^3 \qquad \text{since } \tilde{u}^1 = \tilde{u}^2 = 0 \\ &= R\cdot\gamma_c - R\cos\theta\gamma v = \gamma Rc\left(1 - \frac{v}{c}\cos\theta\right) \end{aligned}$$

Now, we want to find E-field from Faraday tensor in Eq. (6.65). We first consider the terms that involve the acceleration $\tilde{a}_\alpha$ which we denote by ${F^{\mu 0}}_{\text{acc}}$:

$${F^{\mu 0}}_{\text{acc}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{c^4R_s^3}\Big[X^0X^0\left(\tilde{u}^\mu\tilde{a}^0 - \tilde{u}^0\tilde{a}^\mu\right) + X^0X^3\left(\tilde{u}^3\tilde{a}^\mu - \tilde{u}^\mu\tilde{a}^3\right) + X^\mu X^3\left(\tilde{u}^0\tilde{a}^3 - \tilde{u}^3\tilde{a}^0\right)\Big] \tag{6.67}$$

<ins>Finding E_x:</ins>
For $E_x, \mu = 1$. $u^1 = a^1 = 0 \implies$ first 2 terms are 0. This reduces the accelerating term of the

electric field component to

$$\begin{aligned}
E_x &= \frac{q}{4\pi\varepsilon_0}\frac{1}{c^3R_s^3}\left[X^1X^3\left(\tilde{u}^0\tilde{a}^3-\tilde{u}^3\tilde{a}^0\right)\right]\\
&= \frac{q}{4\pi\varepsilon_0 c^3}\frac{1}{R_s^3}R^2\sin\theta\cos\theta\cos\varphi\left(\gamma c\cdot\gamma^4 a-\gamma\frac{v}{c}\cdot\gamma^4 a\right)\\
&= \frac{qca}{4\pi\varepsilon_0 c^3}\frac{\gamma^5R^2\sin\theta\cos\theta\cos}{\gamma^3R^3\left(1-\frac{v}{c}\cos\theta\right)^3}\left(1-\frac{v^2}{c^2}\right)\\
E_x &= \frac{qa}{4\pi\varepsilon_0c^2}\cdot\frac{1}{R}\cdot\frac{\sin\theta\cos\theta\cos\varphi}{\left(1-\frac{v}{c}\cos\theta\right)^3}
\end{aligned}$$

Finding E_y:
We get the result

$$E_y = \frac{qa}{4\pi\varepsilon_0c^2}\cdot\frac{1}{R}\cdot\frac{\sin\theta\cos\theta\sin\varphi}{\left(1-\frac{r}{c}\cos\theta\right)^3} \tag{6.68}$$

Finding E_z:
For $\mu=3$, the middle term in Eq. (6.67) is lost as $\tilde{u}^3\tilde{a}^3-\tilde{u}^3\tilde{a}^3=0$. Also note that by assumption, there is only acceleration and velocity in the $\hat{\mathbf{e}}_z$ direction so we lose those terms as well. This gives:

$$\begin{aligned}
E_z &= \frac{q}{4\pi\varepsilon_0}\frac{1}{c^3R_s^3}\left[X^0X^0\left(\tilde{u}^3\tilde{a}^0-\tilde{u}^0\tilde{a}^3\right)+X^3X^3\left(\tilde{u}^0\tilde{a}^3-\tilde{u}^3\tilde{a}^0\right)\right]\\
&= \frac{q}{4\pi\varepsilon_0c^3}\frac{1}{R_s^3}\cdot\left(\tilde{u}^0\tilde{a}^3-\tilde{u}^3\tilde{a}^0\right)\left(X^3X^3-X^0X^0\right)\\
&= \frac{q}{4\pi\varepsilon_0c^3}\frac{1}{R_s^3}\left(\gamma c\cdot\gamma^4a-\gamma v\cdot\gamma^4\frac{v}{c}a\right)\left(\cos^2\theta-1\right)R^2\\
&= \frac{qa}{4\pi\varepsilon_0c^2}\frac{R^2}{R_s^3}\gamma^5\left(1-\frac{v^2}{c^2}\right)\left(\cos^2\theta-1\right)\\
E_z &= \frac{-qa}{4\pi\varepsilon_0c^2}\cdot\frac{1}{R}\frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^3}
\end{aligned}$$

where again we use $R_s\sim R$. Putting all components together

$$\boxed{\mathbf{E}=\frac{qa}{4\pi\varepsilon_0c^2R}\frac{1}{\left(1-\frac{v}{c}\cos\theta\right)^3}\begin{pmatrix}\sin\theta\cos\theta\cos\varphi\\ \sin\theta\cos\theta\sin\varphi\\ -\sin^2\theta\end{pmatrix}}$$

$$\boxed{|\mathbf{E}|=\frac{qa}{4\pi\varepsilon_0c^2R}\frac{\sin\theta}{\left(1-\frac{v}{c}\cos\theta\right)^3}}$$

6.3.2 Magnetic field when $\tilde{\mathbf{a}}\parallel\tilde{\mathbf{v}}$

From the definition of the Faraday tensor, we need to find

$$B_x=F^{32};B_y=F^{13};B_z=F^{21}$$

To find B_x, note that $\tilde{a}^2=\tilde{u}^2=0$

$$\therefore \quad F^{32} = \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\left[-X^2\tilde{u}^3\left(X^3\tilde{a}^3 - X^0\tilde{a}^0\right) - X^2\tilde{a}^3\left(X^0\tilde{u}^0 - X^3\tilde{u}^3\right)\right]$$
$$= \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\left[X^0X^2\left(\tilde{u}^3\tilde{a}^0 - \tilde{u}^0\tilde{a}^3\right)\right]$$
$$B_x = \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\frac{a}{\gamma^2}\sin\theta\sin\varphi$$

For $B_y, \tilde{a}^1 = \tilde{u}^1 = 0$ and we get

$$F^{13} = \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\left[X^1\tilde{u}^3\left(X^3\tilde{a}^3 - X^0\tilde{a}^0\right) + \tilde{a}^3X^1\left(X^0\tilde{u}^0 - X^0\tilde{u}^3\right)\right]$$
$$= \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\left[X^0X^1\left(u^0a^3 - u^3a^0\right)\right]$$
$$B_y = \left(\frac{q}{4\pi\epsilon_0 c^2 R_s^3}\right)\frac{a}{\gamma^2}\sin\theta\cos\varphi$$

For $B_z, \tilde{u}^1 = \tilde{u}^2 = 0$ and $\tilde{a}^1 = \tilde{a}^2 = 0$. This eliminates most of the terms:

$$F^{21} \propto \left[\left(X^2\tilde{u}^1 - X^1\tilde{u}^2\right)\left(X^3\tilde{a}^3 - X^0\tilde{a}^0\right) + \left(X^2\tilde{a}^1 - X^1\tilde{a}^2\right)\left(X^0u^0 - X^3\tilde{u}^3\right.\right.$$
$$\therefore B_z = 0$$

Putting everything together, we get the magnetic field is

$$\boxed{\mathbf{B} = \frac{q}{4\pi\varepsilon_0 c^3 R}\frac{1}{\left(1 - \frac{v}{c}\cos\theta\right)^3}\begin{pmatrix}-\sin\theta\sin\varphi\\ \sin\theta\cos\varphi\\ 0\end{pmatrix}} \tag{6.69}$$

As expected, there is 0 magnetic field in the direction of motion

6.3.3 Power radiated from a moving charge

We find the power emitted by a moving charge using the Poynting vector in Eq. (4.58). Since we are again assuming free space, the magnitude of power emitted is

$$|\mathbf{S}| \coloneqq N(\theta) = \varepsilon_0 c E^2$$
$$= \varepsilon_0 c\left(E_x^2 + E_y^2 + E_z^2\right)$$
$$= \frac{\varepsilon_0 c \cdot (qa)^2}{\left(4\pi\varepsilon_0\right)^2 c^4 R^2}\frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^6}$$

We see in Figure 8 that as $\beta \to 1$, the power becomes more focused in the direction of motion. As we get more relativistic then, and thinking about what this distribution looks like, the power distribution tends towards a **cone of power** in the direction of motion. We can examine this by taking limits:

If $\frac{v}{c} \sim 1$ i.e. γ is large then θ can be thought of as small, so $\sin\theta \sim \theta$ and $\cos\theta \sim 1 - \frac{\theta^2}{2}$ (by Taylor expansion).

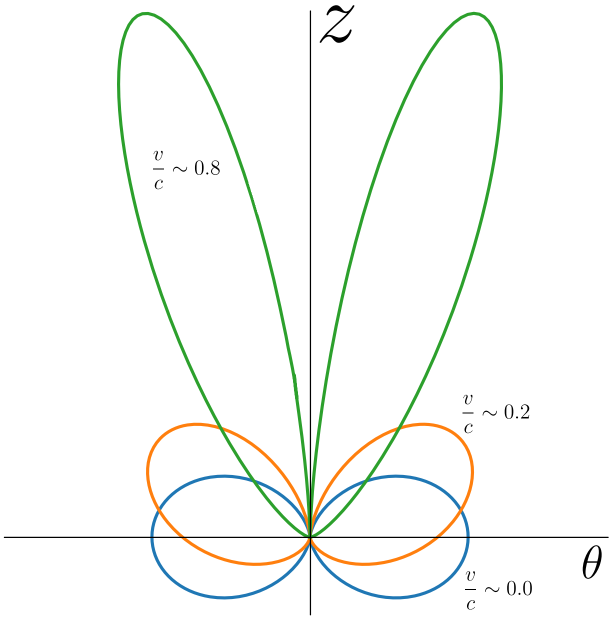


Figure 8: The power distribution emitted from a moving charge at different ratios of $\beta = v/c$. The axes are centred on the charge such that it is at the origin.

$$
\begin{aligned}
1 - \frac{v}{c}\cos\theta &\approx 1 - \frac{v}{c}\left(1 - \frac{\theta^2}{2}\right) \\
&\approx \left(1 - \frac{v}{c}\right) + \frac{v}{c}\frac{\theta^2}{2}
\end{aligned}
$$

Consider:

$$
\begin{aligned}
\left(1 - \left(\frac{v}{c}\right)^2\right) = \frac{1}{\gamma^2} &= \left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right) \\
\therefore \left(1 - \frac{v}{c}\right) &\simeq \frac{1}{2\gamma^2}
\end{aligned}
$$

Therefore

$$
1 - \frac{v}{c}\cos\theta \approx \frac{1 + \gamma^2\theta^2}{2\gamma^2} \tag{6.70}
$$

and

$$
N(\theta) = N_0 \frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^6} = N_0 \frac{\theta^2\left(2\gamma^2\right)^6}{\left(1 + \gamma^2\theta^2\right)^6}
$$

where N_0 is the value of N when $\theta = \pi/2$.

To find the *maximum* power at $\theta = \theta_M$, we must find extrema:

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} = 0 \quad \text{when } \left(1 + \gamma^2\theta_M^2\right) &= 6\theta_M^2 \\
\therefore \quad \theta_M &= \pm\frac{1}{\sqrt{5}\gamma}
\end{aligned}
$$

If we set $\gamma = 1000; \theta_M = \frac{1}{\sqrt{5}}$mrad then

$$
\begin{aligned}
N\left(\theta_M\right) &= N_0 \frac{\left(2\gamma^2\right)^6 \cdot \frac{1}{5\gamma^2}}{\left(1 + \frac{1}{5}\right)^6} \\
&= N_0 4\gamma^{10}
\end{aligned}
$$

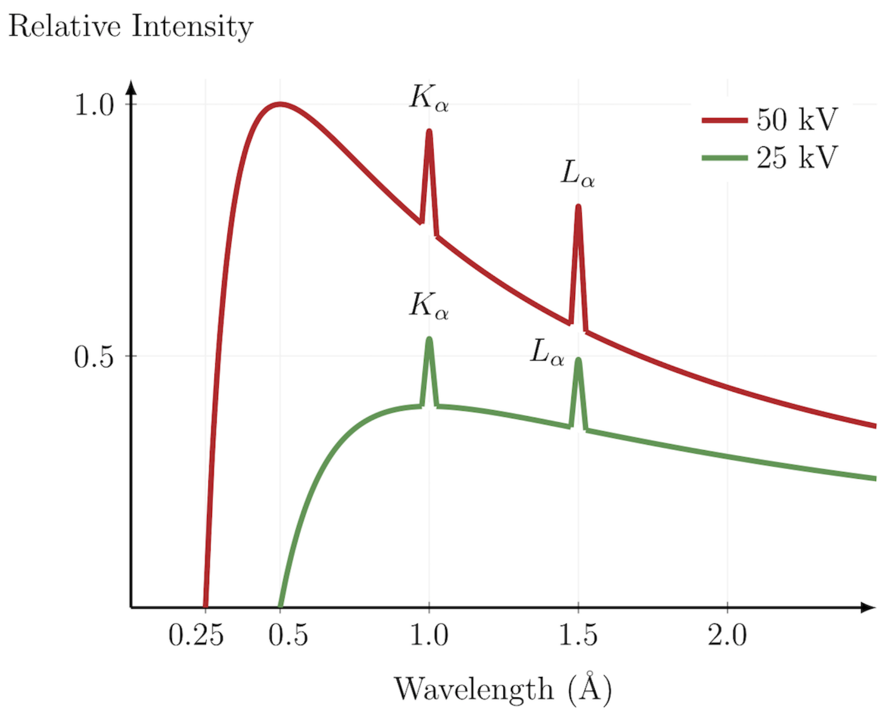


Figure 9: The spectrum shows the variation of X-rays intensity with wavelength. It consists of bremsstrahlung (**continuous spectrum**) and peaks (**characteristic spectrum**). Note the variation of cut-off wavelength with accelerating potential.[35]

i.e., we have extreme dependence on the value of γ. Note that in reality, γ is a function of speed v, and if we are accelerating a charged particle towards c, we get a huge amount of power emitted.

Definition 6.1. ***Bremsstrahlung*** *radiation are X-rays produced by decelerating electrons.*

We also get energy produced from *deceleration*, since the change in energy from a high to low intensity will be emitted as photons. A typical measurement of Bremsstrahlung is shown in Figure 9. The characteristic spectrum are indicated by the sharp peaks denoted K_α and L_α. These correspond to electron transitions to the K-shell (quantum number $n = 1$) and the L-shell ($n = 2$) respectively, and these carry on towards the right as high energy shells are closer together, so transitions will emit/absorb less energetic photons.

The α indicates that the photons came from the adjacent energy levels, e.g. K_α transitions involve $n = 2 \to 1$ (and of course $n = 0$ doesn't exist for atoms). L_α transitions are $n = 3 \to 2$ transitions. There also exist correspond beta-modes denoted K_β, L_β which are the same thing but coming from further energy levels, e.g. $n = 3 \to 1$ is a K_β transition and would show up to the left of the K_α peak because jumping multiple energy levels requires more energy which indicates a shorter wavelength photon.

Using these facts, we can simply plug in values into de Broglie's equation

Example 3.

Find the energy of an emitted photon if the minimum wavelength of a K_α transition is $\lambda_{\text{min}} = 0.02\,\text{nm}$

$$eV = \frac{hc}{\lambda_{\text{min}}} = 9.939 \times 10^{-15}\text{J} \sim 62\,\text{eV}$$

[35] Source: Dr. H. C. Verma

6.3.4 Acceleration perpendicular to velocity

Recall our 4-acceleration:

$$\tilde{a}^\mu = \gamma^2\left(\gamma^2\frac{\mathbf{v}\cdot\mathbf{a}}{c}, \gamma^2\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{a})}{c^2} + \mathbf{a}\right)$$

If $\mathbf{v} \perp \mathbf{a} \implies \mathbf{v}\cdot\mathbf{a} = 0 \implies \tilde{a}^\mu = \gamma^2(0, \mathbf{a})$ This means for a uniform $\mathbf{B}$, we get circular motion.

Relativistically, the 4-force f^μ on a particle of charge q obeys Eq. (3.29):

$$f^\alpha = qF^{\alpha\beta}u_\beta$$

If $\mathbf{v}$ in x-direction, so $\mathbf{v} = v\hat{\mathbf{e}}_x$ and $\mathbf{B} = B\hat{\mathbf{e}}_z$ then the 4-force becomes

$$f^\alpha = q\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ -\gamma v \\ 0 \\ 0 \end{pmatrix}$$

$$f^2 = -qB\gamma v = ma^2$$

$$\tilde{a}^2 = -\frac{\gamma B v q}{m} = \gamma^2 a_y$$

$$\therefore a_y = -\frac{qBV}{\gamma m}$$

where γm is the **relativistic mass**.

This physics is used for example, in particle experiments at CERN or for Diamond Imaging at the EJRF. Both use a **synchrotron** which is a metal torus with regularly-placed **bending magnets** around the ring to accelerate the particle around the torus.

> Example 4.
>
> Finding γ from energy of electrons. Suppose the energy of an electron is $3\,\text{GeV} = \gamma mc^2$ The rest mass of an electron is $m_e = 0.511\,\text{MeV}/c^2$.
>
> $$\gamma = \frac{3\times 10^9}{0.5\times 10^6} = 6000. \implies v \sim c.$$

Now consider an electron in a synchrotron as so

The 4-velocity is $\tilde{u}^\mu = \gamma(c, 0, 0, v)$, the 4-acceleration is $\tilde{a}^\mu = \gamma^2(0, a, 0, 0)$ with $a = qB/\gamma m$ and the 4-separation is $X^\mu = (R, 0, 0, R\cos\theta)$. You may remark that in a real torus, there should be other components in $\cos\theta$ and $\sin\varphi$, but the electrons are generally kept as in-plane as possible to minimise the necessity of corrections so we assume these terms are sufficiently small to be neglected.

Then $X^\mu\tilde{a}_\mu = 0$ and

$$\begin{aligned} R_s &= \frac{1}{c}X^\mu\tilde{a}_\mu \\ &= \frac{1}{c}\left(\gamma cR - rv\cos\theta\right) \\ &= \gamma R\left(1 - \frac{v}{c}\cos\theta\right) \\ &\approx \frac{R}{2\gamma}(1+\gamma^2\theta^2) \end{aligned}$$

where in the last line we used 2 approximations:

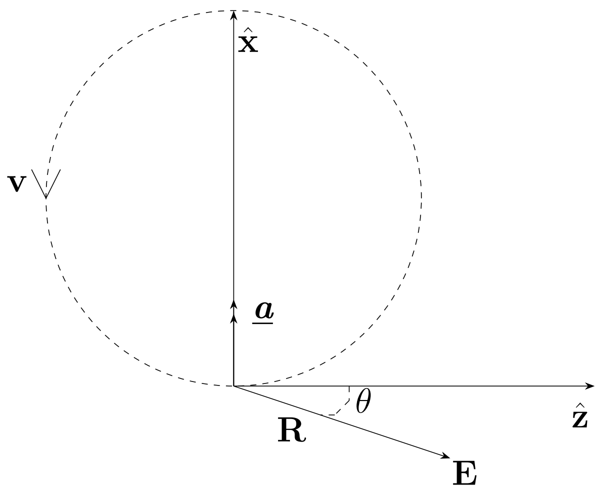


Figure 10: Electron in a synchrotron travelling at 3-velocity $\mathbf{v}$ in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_y$

- Large v
- Small θ (which we mentioned earlier about keeping electrons in the same plane.)

We will use the Faraday tensor to again find the fields with

$$F^{i0} = \frac{\mu_0 q}{4\pi}\left[\frac{X^i\tilde{a}^0 - X^0\tilde{a}^i}{cR_s^2}\right]$$

and remember that $E_x = cF^{10}$ which

$$\begin{aligned} E_x &= \frac{\mu_0 cq}{4\pi}\cdot\frac{R\gamma^2\frac{qB}{\gamma m}4\gamma^2}{cR^2(1+\gamma^2\theta^2)} \\ &= \frac{\mu_0 q^2 B}{\pi m}\frac{\gamma^3}{R}\cdot\frac{1}{(1+\gamma^2\theta^2)^2} \\ E_y &= E_z = 0 \end{aligned}$$

Then

$$\boxed{N(\theta) \propto \frac{E^2}{R^2} \propto \frac{1}{R^2}\frac{\gamma^6}{(1+\gamma^2\theta^2)^4}} \tag{6.71}$$

is the power radiated per unit area. The power distribution is shown in Figure 11.

Definition 6.2. *The equation in Eq.* (6.71) *is the **synchrotron radiation**.*

This is the emitted energy which is detected in experiments, and is a consequence of charged particles travelling at relativistic speeds.

Though synchrotron radiation might seem like a loss of energy for the particle, which is bad for trying to max out energy for collisions, there are uses:

- The high intensity of synchrotron light allows for the study of disease mechanisms, high-resolution imaging, and advances in microbiology and cancer radiation therapy.
- for high precision and time-dependent measurements that would be impossible under other circumstances (e.g. in materials research).

The FWHM for this distribution scales as $1/\gamma$. Visually, you can see this as synchrotron radiation again being emitted in a narrow cone around the particle for large γ, i.e. high intensity light is *tangential* to the ring. We also remark that there will be a large frequency of x-rays emitted (which will be a function of the power).

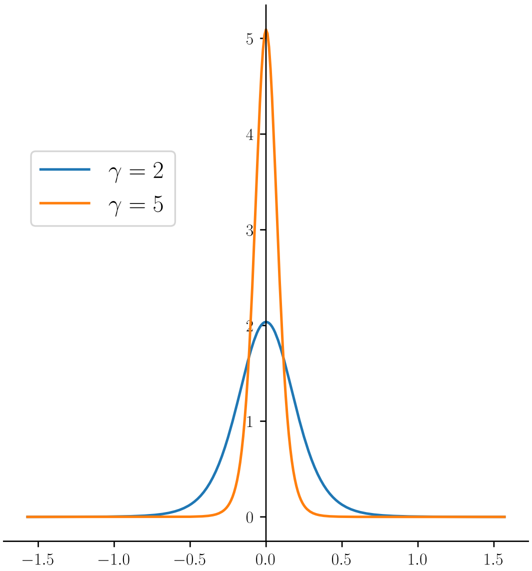


Figure 11: A normalised plot of power radiated against θ. We see more power is emitted in the direction of motion (when θ is small) and that as γ increases, the power is more concentrated in the direction of motion.

The **total radiated power** is then an integral over all angles,

$$P = \int N(\theta) d\theta = \frac{(qa)^2\gamma^4}{6\pi\epsilon_0 c^3} \tag{6.72}$$

Actually doing this integral is a total mess, and is non-examinable. What is important to know is that the total power $P \sim \gamma^4$.

The radius of curvature of a bending magnet, denoted ρ satisfies $a = v^2/\rho$ to maintain perfect circular motion. Therefore the power radiated is

$$\boxed{P = \frac{q^2}{6\pi\epsilon_0 c^3}\frac{v^4}{\rho^2}\gamma^4} \tag{6.73}$$

When the electron does one revolution, it traversers an arc of length $2\pi\rho$ at velocity v. Hence $\tau := 2\pi\rho/v$ is the time for one revolution. Now, light particles are easily accelerated to relativistic speeds, so assume $v \sim c$, then

$$P = \frac{q^2\gamma^4}{3\epsilon_0\rho}$$

is the power lost per revolution. If you suppose you are accelerating an electron with charge e, it can eb thought of as a current I and so the power lost is

$$P = \frac{e^2\gamma^4}{3\epsilon_0\rho} I$$

Example 5.

Suppose $\rho = 6\,\text{m}, I = 0.3\,\text{A}$ and the energy $E = 3\,\text{GeV}$ for the electron. Substituting above, we get the power lost is avout $390\,\text{kW}$.

Some more exotic sources of synchrotron radiation is

- Sunspots

- Jupiter radiation zones (for electron, $E \sim 5\,\mathrm{GeV} \implies \gamma \sim 10$)
- Nebulae, e.g. Crab nebula

Synchrotron rings which are designed to produce synchrotron radiation have regularly placed **inversion devices** where the magnetic field alternates between the 2 perpendicular directions. This causes the electron to oscillate up and down, i.e. be constantly accelerating.

Now, 2 things can happen to the electron (with arguably funny names). Suppose $\Delta\theta$ is the critical angle which differentiates between an electron escaping the inversion zones or remaining in it. Suppose electron is at angle ψ just before this point.

Wiggler[36]: this is when $\psi \gg \Delta\theta$. Here, the electron just escapes the inversion zone and leaves a forward narrow cone of radiation with apex angle $\Delta\theta$.

- The spectrum of this radiation is **broad**
- The light is **incoherent**
- The intensity is N times greater than what is emitted at a regular bending magnet, when you have N sections of alternating fields (beware clash of notation of N here from power earlier).

The other case is the
Undulator: this occurs when $\psi \ll \Delta\theta$. The electron does not escape the alternating fields.

- The light is **monochromatic** due to phase-matched sections (as it can continue traversing the fields)
- The light is **coherent**

Return ourselves to the **lab frame**. In the undulator, define the wavelength λ_0 as the distance between field reversals. The magnetic field is a function of position, so the electron acceleration is a function of time. We impose initial conditions $\omega_0 = 0$ (electron has no angular frequency before it enters the undulator) and $k_0 = 2\pi/\lambda_0$.

The 4-wavevector is

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right) \tag{6.74}$$

and in the **electron frame of reference**, its 4-wavevector satisfies

$$k^{\mu'} = \Lambda^{\mu'}{}_{\mu} k^\mu \tag{6.75}$$

$$\implies \omega' = \gamma c k_0 \tag{6.76}$$

so in its own frame, the electron oscillates at angular frequency $\omega' = \gamma c k_0$ and so it emits radiation of this same frequency.

The question is therefore, *what do we observe*? We will LT the angular frequency back to the lab frame:

$$\begin{aligned}
\omega_{\mathrm{obs}} &= \gamma(\omega' + vk') \\
&= \gamma\left(\omega' + \frac{v}{c}\omega'\right) \\
&= 2\gamma\omega' \\
&= 2\gamma^2 c k_0 = 2\gamma^2 c \frac{2\pi}{\lambda_0}
\end{aligned}$$

Namely, we see the observed radiation wavelength is

$$\lambda_{\mathrm{obs}} = \frac{\lambda_0}{2\gamma^2} \tag{6.77}$$

This can be useful for probing atoms.

Example 6.

Suppose $\lambda_0 = 1\,\mathrm{cm}, \gamma = 5000$. Then

$$\lambda_{\mathrm{obs}} = \frac{10^{-2}}{2 \times 5000^2} = 2 \times 10^{-10}\mathrm{m} \tag{6.78}$$

The atomic radius of hydrogen is about 0.5 Å whereas for much large atoms it can be a few angstroms, so we can use this radiation to probe atomic/molecular spacing.

The last lecture for this module was on the energy-momentum tensor, which is **non-examinable**.