1 Special Relativity

In order to understand the dynamics of electromagnetic particles and fields, we need to develop a fully relativistic viewpoint on the electric and magnetic fields. Before we start with this formalism, we will expand a bit more on the subject of special relativity itself.

1.1 Lorentz Transformations & Spacetime Intervals

Our starting point for the special relativity are of course equations known as Lorentz transformations. These mix together the temporal and space coordinates of a certain event, depending on some velocity $\vec{v}$. Their form is, for $\vec{v}$ parallel to the x axis

$$ct' = \gamma(ct - \beta x) \tag{1}$$

$$x = \gamma(x - \beta ct) \tag{2}$$

where c is the speed of light, t and x are the original coordinates of the event, t' and x' are the transformed coordinates of the event, and

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The other coordinates in this special case remain unchanged, i.e. $y' = y$ and $z' = z$.
These transformations have a particular property of not changing the d'Alembertian operator - the wave operator - for electromagnetic radiation, which moves at phase speed of c. The operator for such waves has form

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

where ∇^2 is the Laplacian operator. Using the chain rule, we can determine

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \gamma\frac{\partial}{\partial t'} - \gamma\beta c\frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = -\frac{\gamma\beta}{c}\frac{\partial}{\partial t'} + \gamma\frac{\partial}{\partial x'}$$

So

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{1}{c^2}\left(\gamma^2\frac{\partial^2}{\partial(t')^2} - 2\gamma\beta c\frac{\partial^2}{\partial x'\partial t'} + \gamma^2\beta^2c^2\frac{\partial^2}{\partial(x')^2}\right) = \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial(t')^2} - 2\gamma^2\frac{\beta}{c}\frac{\partial^2}{\partial x'\partial t'} + \gamma^2\beta^2\frac{\partial^2}{\partial(x')^2}$$

$$\frac{\partial^2}{\partial x^2} = \gamma^2\frac{\beta^2}{c^2}\frac{\partial^2}{\partial(t')^2} - 2\gamma^2\frac{\beta}{c}\frac{\partial^2}{\partial x'\partial t'} + \gamma^2\frac{\partial^2}{\partial(x')^2}$$

And therefore

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{c^2}\gamma^2\left(1-\beta^2\right)\frac{\partial^2}{\partial(t')^2} - \gamma^2(1-\beta^2)\frac{\partial^2}{\partial(x')^2}$$

But, given the definition of γ, $\gamma^2(1-\beta^2) = 1$, and therefore

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2}{\partial(t')^2} - \frac{\partial^2}{\partial(x')^2}$$

Since y and z remain unchanged, we can state that $\Box = \Box'$ - the d'Alembertian is invariant under this transformation. Normally, this would be but a specific mathematical property of specific operator. But, d'Alembertian has a very central role to a good proportion of classical physics. To show this, consider Maxwell equations in vacuum

$$\nabla\cdot\vec{E} = 0$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Taking curl of the third equation

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$

Substituting in from the first ($\nabla \cdot \vec{E} = 0$) and fourth equation ($\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$)

$$-\nabla^2 \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

So

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = \Box \vec{E} = 0$$

By starting with the curl of the fourth equation, we could also derive $\Box \vec{B} = 0$. But since the operator does not change under Lorentz transformations, it seems that these transformations do not change the physics of electromagnetism in vacuum.
The proper interpretation of the Lorentz transformations was later provided by Einstain, as transformations into a reference frame moving at velocity $\vec{v}$ relative to the starting frame.
Lorentz transformations can be also regarded in a somewhat different mathematical viewpoint if we notice that coefficients γ and β follow

$$\gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$$

$$1 - \frac{\gamma^2\beta^2}{\gamma^2} = 1 - \beta^2 = \frac{1}{\gamma^2}$$

This is the same behaviour that we could achieve by setting $\gamma = \cosh(\alpha)$ and $\gamma\beta = \sinh(\alpha)$, so that $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$ and $1 - \tanh^2(\alpha) = \frac{1}{\cosh^2(\alpha)}$. The Lorentz transformations then take form (again, for $\vec{v} \parallel x$)

$$ct' = \cosh(\alpha)ct - \sinh(\alpha)x$$

$$x' = \cosh(\alpha)x - \sinh(\alpha)ct$$

This looks somewhat like some kind of rotation of coordinates when we include time as one of the coordinates. Minkowski later comes with the idea of spacetime - including the time as a spatial coordinate in a mathematical space that describes the physical world with 4 different coordinates (4 dimensional space). If we are to declare this spacetime a proper space, we need to know its geometry, or at least some notion of distance in this space. This distance should be invariant under the rotations of the real space coordinates and under Lorentz transformations. We know that in Euclidian space (which for now will be the starting point, although not even the 3 dimensional space is in fact Euclidian) has notion of distance described by the sum of squares of the difference of spatial coordinates between two points - if we have coordinates for point 1 and 2 in two rotated systems of coordinets S and S', the expression

$$D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2$$

So, we expect that the expression for the distance of two points in our new spacetime should still have this part in and then some additional part, mixing together other coordinates. So, we expect that the distance s is something like

$$s^2 = F(ct_1, x_1, y_1, z_1, ct_2, x_2, y_2, z_2) + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

where F is some unknown function. It should follow that this d is invariant under Lorentz transformations. To satisfy this requirement, we can calculate how original distance D changes under Lorentz transformations. Since the Lorentz transformations are linear in the coordinates, we can see that the differences transform in the same way as coordinates themselves, i.e. that

$$(ct_1' - ct_2') = \gamma((ct_1 - ct_2) - \beta(x_1 - x_2))$$

$$(x_1' - x_2') = \gamma((x_1 - x_2) - \beta(ct_1 - ct_2))$$

$$(y_1' - y_2') = (y_1 - y_2)$$

$$(z_1' - z_2') = (z_1 - z_2)$$

Writing $\Delta x = x_1 - x_2$ and similarly for other coordinates, we then have

$$(D')^2 = \Delta(x')^2 + \Delta(y')^2 + \Delta(z')^2 = \gamma^2(\Delta x^2 - 2\beta c\Delta x\Delta t + \beta^2 c^2\Delta t^2) + \Delta y^2 + \Delta z^2$$

Then, we can notice that

$$c^2\Delta(t')^2 = \gamma^2(c^2\Delta t^2 - 2\beta c\Delta x\Delta t + \beta^2\Delta x^2)$$

And therefore that

$$c^2\Delta(t')^2 - (D')^2 = \gamma^2(1-\beta^2)c^2\Delta t^2 - \gamma^2(1-\beta^2)\Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t^2 - D^2$$

And therefore we can set the "distance" in the spacetime as

$$s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{3}$$

Commonly, we also speak of the interval with respect to the origin of the coordinate system, which is an event at the origin of spatial coordinate system happening at time $t = 0$, for which we simply have

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

It is important to notice that this interval is also invariant when considering it in infinitesimal distances between the points, i.e. we can set

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

This result leads to many interesting results, mainly, we can directly see that the time τ that passes in the frame of reference where certain particle is not moving compares to the time t passed for the particle in the frame in which it is moving at speed v is given (as $dx = dy = dz = 0$ in the so called rest frame of the particle) by

$$c^2(d\tau)^2 = c^2(dt)^2\sqrt{1 - \frac{(dx)^2 + (dy)^2 + (dz)^2}{c^2(dt)^2}} = c^2(dt)^2\sqrt{1 - \frac{v^2}{c^2}}$$

$$\tau = t\sqrt{1 - \frac{v^2}{c^2}} \tag{4}$$

This time is called the proper time and has the specific property of being independent of reference frame, as for particles which do have a rest frame, we are always able to determine their proper time.
Since we have a notion of distance in this spacetime and we have transformations that mix the coordinates of an event together and conserve this distance, we have essentially a similar thing to a vector in 3D - a set of coordinates which express something that is independent of our reference frame. Indeed, we call the set of these 4 numbers a fourvector (or 4-vector).

1.2 Fourvectors

Fourvectors is essentially any set of 4 variables which satisfies the following conditions

1. The four variables have the same dimensions
2. The four variables transform in the same way as (ct, x, y, z), both in Lorentz transformations and in purely spatial transformations such as rotation

Clearly, (ct, x, y, z) is a fourvector.
In relativistic calculations, fourvectors tend to have a special notation, where we assign a letter to each fourvector and access its components by different indices. For the interval, we assign it usually a letter x and assign indices as $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. To define the Lorentz invariant,

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

we introduce a notational trick by setting a different type of index, defined as $x_0 = x^0$, $x_1 = -x_1$, $x_2 = -x_2$ and $x_3 = -x_3$, so that

$$s^2 = \sum_{\mu=0}^{3} x^\mu x_\mu$$

Usually, we further condense the notation by application of so called Einstain summation convention. This convention states that if we see a product of two numbers given by the same indices, we instead see it as a sum of products over the different possible values of the indices, ranging from 0 to 3, i.e.

$$x_\mu x^\mu \leftrightarrow \sum_{\mu=0}^{3} x_\mu x^\mu$$

Fourvectors also commonly relate somehow to the 3D spatial vectors. Sometimes, it is therefore useful to write fourvectors as

$$a^\mu = (a^0, \vec{b})$$

where $\vec{b}$ is some 3D vector.

1.2.1 Finding fourvectors

Other ways how to find fourvectors is usually by derivation from other fourvectors. For example, if we have a scalar - a single Lorentz invariant variable - we can multiply some fourvector by this variable to receive a new fourvector. Similarly, we could do other operations that involve this scalar, for example differentiation with respect to this scalar.
One other way to show that certain set of 4 numbers b^μ is a fourvector is by showing that the alternative of scalar product with a fourvector a^μ produces a Lorentz invariant. The scalar product can be taken as $a^\mu b_\mu$, which can be expressed as

$$a^\mu b_\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = I$$

If we know for sure that I is invariant, it means that in different reference frame

$$a^{0'} b^{0'} - a^{1'} b^{1'} - a^{2'} b^{2'} - a^{3'} b^{3'} = I$$

Notice that I am writing the indices as primed instead of fourvectors as primed - this is to emphasize the fact that the fourvector is something more than just a set of numbers, as it represents some Lorentz invariant. Comparing these two expressions and substituting transformations of a^μ, which we know is a fourvector, leads to

$$a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = \gamma(a^0 - \beta a^1) b^{0'} - \gamma(a^1 - \beta a^0) b^{1'} - a^2 b^{2'} - a^3 b^{3'}$$

Moving all the expressions to the left hand side, we are left with

$$a^0(b^0 - \gamma(b^{0'} + \beta b^{1'})) - a^1(b^1 - \gamma(b^{1'} + \beta b^{0'})) - a^2(b^2 - b^{2'}) - a^3(b^3 - b^{3'}) = 0$$

Which definitely applies when

$$\begin{aligned} b^0 &= \gamma(b^{0'} + \beta b^{1'}) \\ b^1 &= \gamma(b^{1'} + \beta b^{0'}) \\ b^2 &= b^{2'} \\ b^3 &= b^{3'} \end{aligned}$$

which is the inverse of Fourier transforms. Since we can choose any initial reference frame for a^μ, these equations are in fact required, as different components of a^μ can be different numbers. This means that b^μ is also a fourvector.

1.2.2 Fourtensors

Fourtensors are linear maps that map fourvectors and other fourtensors onto each other. Usually, they are designated by two or more indices, such as $F^{\mu\nu}$ or T^μ_ν. In Einstain summation convention, we tend to sum out indices as follows

$$T^{\mu\nu} x_\nu = T^{\mu 0} x_0 + T^{\mu 1} x_1 + T^{\mu 2} x_2 + T^{\mu 3} x_3 = y^\mu$$

but we only do this if the indices are on opposite positions - up and down, also called contravariant (up) and covariant (down) indices. This is because only this kind of operation usually leads to a meaningful fourvector, scalar or fourtensor as a result.

In fact, we can clasify the fourtensors into orders by the number of indices they have. Zeroth order fourtensors are then scalars, first order fourtensors are fourvectors etc. Importantly, we can describe second order fourtensors as matrices, and expressions such as $T^{\mu\nu}x_\nu$ as matrix multiplication, although we have to be careful. Matrix multiplication is defined as $\mathbf{AB} = \sum_j A_{ij}B_{jk}$ where i indexes rows in $\mathbf{A}$, j columns in $\mathbf{A}$ and rows in $\mathbf{B}$ and k indexes columns in $\mathbf{B}$. So, if we have fourtensor Λ^μ_ν and fourtensor $T^{\alpha\nu}$, expressed in matrix form as $\mathbf{\Lambda}$ with μ indexing rows and ν columns, and $\mathbf{T}$ with α indexing rows and ν columns, the summation convention can be rewritten in matrix form only with use of transposes as

$$\Lambda^\mu_\nu T^{\alpha\nu} = \mathbf{T}(\mathbf{\Lambda})^T$$

Sometimes, the fourtensors have some symmetry, allowing for easy calculation of the transposes, other times it is not a useful way how to make the problems easier for calculation. Notice also that the order in matrix multiplication matters, and can be different from the order in summation convention, which is essentially irrelevant.

1.2.3 Metric Tensor

Metric tensor is a special tensor in the relativity which describes how distances or magnitudes are derived in this formalism. We defined the distance as $s^2 = x^\mu x_\mu$ where x^μ is the interval fourvector (I will now start calling it fourposition, with name interval reserved exclusively for differences of two fourpositions). We can then define the metric tensor $g_{\mu\nu}$ as linear map of the fourposition on scalars as

$$s^2 = x^\mu g_{\mu\nu} x^\nu$$

which leads to $g_{\mu\nu}x^\nu = x_\mu$. Here, we can see other typical property of fourtensors - they transform contravariant components onto covariant and vice-versa. To have an intuitive definition, we also require that

$$s^2 = x_\mu g^{\mu\nu} x_\nu$$

so that $g^{\mu\nu}x_\nu = x^\mu$. In our case, for special relativity, we can describe $g^{\mu\nu} = \eta^{\mu\nu}$ as

$$\eta^{\mu\nu} = \delta_{0\mu}\delta_{0\nu} - \delta_{1\mu}\delta_{1\nu} - \delta_{2\mu}\delta_{2\nu} - \delta_{3\mu}\delta_{3\nu}$$

where δ_{ij} is the Kronecker delta. This special metric tensor $\eta^{\mu\nu}$ is called the Minkowski metric. This means that $\eta^{\mu\nu}$ is non-zero only on the diagonal, and is equal to 1 for $\mu = \nu = 0$ and -1 for other components on the diagonal.
This also directly means that $\eta^{\mu\nu}$ is symmetrical - if we represent it by a matrix η

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where μ indexes rows and ν columns, $\eta^T = \eta$. This can be also understood as $\eta^{\mu\nu} = \eta^{\nu\mu}$. Also, we can see that from the definition of covariant and contravariant components, the effect of $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ is exactly the same, so

$$\eta_{\mu\nu} = \eta^{\mu\nu}$$

Finally, we provide some usefull identities:

$$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha$$

where δ^γ_α is the identity operation (matrix), which ofcourse satisfies $\delta^\gamma_\alpha = \delta_{\gamma\alpha} = \delta^{\gamma\alpha} = 1$. Also

$$\eta_{\alpha\beta}T^{\mu\beta} = T^\mu_\alpha$$

1.2.4 Lorentz Transformation Tensor

Lorentz transformation tensor $\Lambda^{\mu'}_\mu$ that transforms from reference frame S to reference frame S' as

$$x^{\mu'} = \Lambda^{\mu'}_\mu x^\mu$$

In the matrix form, we can express

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where μ indexes rows and ν columns and we have $\vec{v} \parallel x$. We can also transform fourtensors, but we need to transpose both components of the fourtensor as

$$T^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} T^{\mu\nu}$$

This leads to notion that even though Lorentz transformation tensor is very important, it is not a fourtensor - there is no meaning in transforming Lorentz transform.
The important properties of this tensor is notibly its symmetry - $\Lambda^{\mu}_{\nu} = \Lambda^{\nu}_{\mu}$. Lastly, we can use the Lorentz transformation to show that the metric tensor is independent of frame of reference, as

$$\eta^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \eta^{\mu\nu} = \Lambda^{\mu'}_{\mu} \eta^{\mu\nu} \Lambda^{\nu'}_{\nu} =$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & 0 & 0 & 0 \\ 0 & \beta^2\gamma^2 - \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}$$

Hence we have shown that $\eta^{\mu'\nu'} = \eta^{\mu\nu}$ - metric tensor is frame independent.

1.3 Acceleration and Forces

In order to study relativistic dynamics, we would like to rewrite something like Newton's second law in relativistic approach. We start by defining velocity fourvector u^{μ}

$$u^{\mu} = \frac{dx^{\mu}}{d\tau}$$

where x^{μ} is the fourposition and τ is the proper time. We know that (see (4))

$$\tau = t\sqrt{1 - \frac{v^2}{c^2}} = t\sqrt{1 - \beta^2} = \frac{t}{\gamma}$$

hence

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma \frac{d}{dt}$$

So

$$u^{\mu} = \gamma \left(\frac{d(ct)}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \gamma \left(c, \vec{v} \right)$$

where $\vec{v}$ is the 3D velocity. Fourvelocity is a fourvector as we derived it from the fourposition using only scalar operations. We can check that its magnitude is Lorentz invariant

$$u^{\mu}u_{\mu} = \gamma^2(c^2 - v^2) = c^2\gamma^2 \left(1 - \frac{v^2}{c^2}\right) = c^2\gamma^2(1 - \beta^2) = c^2$$

where $v = |\vec{v}|$. Speed of light is indeed a Lorentz invariant.

Having determined the fourvelocity, we can move onto the fourmomentum, which we simply take to be $p^\mu = mu^\mu$, where m is the rest mass of the object - the mass in its rest frame. I will not be repeating the rest mass calculations here, but I will simply state that it is a Lorentz invariant which can be determined as $m^2c^4 = E^2 - p^2c^2$, where E is the total energy of the particle and p is the 3-momentum of the particle in a certain frame of reference. We can also derive that

$$p^\mu p_\mu = m^2 u^\mu u_\mu = m^2c^2 = \frac{E^2}{c^2} - p^2$$

If we notice that

$$p^\mu = mu^\mu = (\gamma mc, \gamma m\vec{v}) = (\gamma mc, \vec{p})$$

where $\vec{p}$ is the 3D relativistic momentum (derived in first year), we clearly see that $\gamma mc = \frac{E}{c}$, which is what we expect for the total energy. Hence, we can also state that four momentum is

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

Now, we can move forward to describe the force. Again, via correspondence principle, we define fourforce as

$$f^\mu = \frac{dp^\mu}{d\tau}$$

Using the definition of p^μ and τ

$$f^\mu = \gamma\frac{dp^\mu}{dt} = \gamma m\left(c\frac{d\gamma}{dt}, \frac{d(\gamma\vec{v})}{dt}\right)$$

Here

$$\frac{d\gamma}{dt} = \frac{1}{(1-\beta^2)^{\frac{3}{2}}} \times \left(\frac{-1}{2}\right)(-2\beta)\frac{d\beta}{dt} = \gamma^3\beta\frac{a}{c} = \gamma^3\frac{\vec{v}\cdot\vec{a}}{c^2}$$

where $\vec{a}$ is the classical 3D acceleration, $\vec{a} = \frac{d\vec{v}}{dt}$. Also

$$\frac{d(\gamma\vec{v})}{dt} = \frac{d\gamma}{dt}\vec{v} + \gamma\frac{d\vec{v}}{dt} = \gamma^3\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + \gamma\vec{a}$$

Hence

$$f^\mu = \left(mc\gamma^4\frac{\vec{v}\cdot\vec{a}}{c^2}, m\gamma^4\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + m\gamma^2\vec{a}\right) \tag{5}$$

To check whether this is truly a fourvector, consider taking a scalar product with fourvelocity

$$f^\mu u_\mu = m\frac{du^\mu}{d\tau}u_\mu = m\frac{du^\mu}{d\tau}\eta_{\mu\nu}u^\nu$$

As $\eta_{\mu\nu}$ is time independent, we can write

$$f^\mu u_\mu = m\eta_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu$$

But, consider that

$$\eta_{\mu\nu}\frac{d(u^\mu u^\nu)}{d\tau} = \eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} + \eta_{\mu\nu}u^\nu\frac{du^\mu}{d\tau} = u_\nu\frac{du^\nu}{d\tau} + u_\mu\frac{du^\mu}{d\tau} = 2u_\mu\frac{du^\mu}{d\tau} = 2\eta_{\mu\nu}u^\nu\frac{du^\mu}{d\tau}$$

Hence

$$f^\mu u_\mu = \frac{m}{2}\eta_{\mu\nu}\frac{d(u^\mu u^\nu)}{d\tau} = \frac{m}{2}\frac{d(u^\mu\eta_{\mu\nu}u^\nu)}{d\tau} = \frac{m}{2}\frac{d(u^\mu u_\mu)}{d\tau}$$

And since $u^\mu u_\mu = c^2$, which is time independent, we have

$$f^\mu u_\mu = 0 \tag{6}$$

Besides being a Lorentz invariant, this result should be somewhat reassuring. Classicaly, if we found that the magnitude of velocity does not change by an action of a force, we would expect the force to act perpendicularly to the velocity vector. This would cause the cartesian scalar product of the force and the velocity to go to zero. As the magnitude of the fourvelocity $u^\mu u_\mu$ does not change, it is a nice result that the scalar product in the fourvector notation with the fourforce also results in zero.

Finally, (5) gives a kinematical definition of fourforce - it uses velocities and accelerations. We are, however, also interested in the dynamical description of the fourforce with relation to 3D dynamical variables. To discover these, consider the expression for fourmomentum

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

Then

$$f^\mu = \frac{dp^\mu}{d\tau} = \gamma \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right) = \gamma \left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

Taking scalar product with u^μ then leads to

$$f^\mu u_\mu = 0 = \gamma(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}) \cdot (\gamma c, -\gamma \vec{v})$$

where $\cdot$ symbolizes simple scalar product (without swapping signs for spatial parts). Therefore, we have

$$\gamma^2 \left(\frac{dE}{dt} - \vec{v} \cdot \frac{d\vec{p}}{dt}\right) = 0$$

This is satisfied when

$$\frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt} = \vec{v} \cdot \vec{F}$$

where $\vec{F}$ is the 3D (but relativistic) force. This is a known relation for the power acting on a system, so again, we have good consistency with classical mechanics.
Therefore, we can condense the dynamic description of fourforce as

$$f^\mu = \left(\frac{\gamma}{c}\vec{v} \cdot \vec{F}, \gamma \vec{F}\right) \tag{7}$$

2 Relativistic Particle Motion

We will now develop an aparatus to describe a relativistic particle motion due to electromagnetic forces, while assuming that the form of these forces does not change relativisticly. The starting point is the Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Therefore, the relativistic force caused by this 3D force is

$$f^\mu = (\frac{\gamma}{c}\vec{F} \cdot \vec{v}, \gamma \vec{F})$$

Therefore, we can write

$$f^0 = \frac{q\gamma}{c}\vec{v} \cdot (\vec{E} + \vec{v} \times \vec{B}) = \frac{q\gamma}{c}\vec{v} \cdot \vec{E}$$

as $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$. So, in components of $\vec{v}$ and $\vec{E}$

$$f^0 = q(\gamma v_x)\frac{E_x}{c} + q(\gamma v_y)\frac{E_y}{c} + q(\gamma v_z)\frac{E_z}{c}$$

The spatial components of the force are, in components

$$f^1 = \gamma q E_x + \gamma q v_y B_z - \gamma q v_z B_y = q(\gamma c)\frac{E_x}{c} + q(\gamma v_y)B_z + q(\gamma v_z)(-B_y)$$

$$f^2 = q\gamma E_y + q\gamma v_z B_x - q\gamma v_x B_z = q(\gamma c)\frac{E_y}{c} + q(\gamma v_x)(-B_z) + q(\gamma v_z)B_x$$

$$f^3 = q\gamma E_z + q\gamma v_x B_y - q\gamma v_y B_x = q(\gamma c)\frac{E_z}{c} + q(\gamma v_x)B_y + q(\gamma v_y)(-B_x)$$

In all of these components, we can see some scalar charge q, some components of the fourvelocity $u^\mu = (\gamma c, \gamma \vec{v})$ and some coefficients defined by electromagnetic fields. Since all expressions are linear in all these terms, we suspect that we might be able to describe the force via some fourtensor as

$$f^\mu = qF^\mu_\nu v^\nu$$

The ν is intentionally indented to indicate that μ indexes rows and ν columns in the matrix form (this is unneccessary for $\Lambda^{\mu}_{\ \nu}$ or $\eta^{\mu}_{\ \nu}$, as these matrices are symmetric, so switching rows and columns has no effect, while for general matrix, it is important to distinguish these). In this form, we would have

$$f^0 = qF^0_{\ 0}\gamma c + qF^0_{\ 1}\gamma v_x + qF^0_{\ 2}\gamma v_y + qF^0_{\ 3}\gamma v_z$$

By comparison with expression we derived from the Lorentz force, we can see that $F^0_{\ 0} = 0$, $F^0_{\ 1} = \frac{E_x}{c}$, $F^0_{\ 2} = \frac{E_y}{c}$ and $F^0_{\ 3} = \frac{E_z}{c}$. Similarly, we can see that

$$f^1 = qF^1_{\ 0}\gamma c + qF^1_{\ 1}\gamma v_x + qF^1_{\ 2}\gamma v_y + qF^1_{\ 3}\gamma v_z$$

$$F^1_{\ 0} = \frac{E_x}{c}\ ,\ F^1_{\ 1} = 0\ ,\ F^1_{\ 2}B_z\ ,\ F^1_{\ 3} = -B_y$$

$$f^2 = qF^2_{\ 0}\gamma c + qF^2_{\ 1}\gamma v_x + qF^2_{\ 2}\gamma v_y + qF^2_{\ 3}\gamma v_z$$

$$F^2_{\ 0} = \frac{E_y}{c}\ ,\ F^2_{\ 1} = -B_z\ ,\ F^2_{\ 2} = 0\ ,\ F^2_{\ 3} = B_x$$

$$f^3 = qF^3_{\ 0}\gamma c + qF^3_{\ 1}\gamma v_x + qF^3_{\ 2}\gamma v_y + qF^3_{\ 3}\gamma v_z$$

$$F^3_{\ 0} = \frac{E_z}{c}\ ,\ F^3_{\ 1} = B_y\ ,\ F^3_{\ 2} = -B_x\ ,\ F^3_{\ 3} = 0$$

Therefore, we can write the fourtensor as

$$F^{\mu}_{\ \nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

This tensor is the called the Faraday tensor, and it describes the dynamics of a charged particle in an electromagnetic field. We can therefore write the relativistic formulation of Lorentz force as

$$f^{\mu} = qF^{\mu}_{\ \nu}u^{\nu}$$

Preferrably, we would like to have $F^{\mu\nu}$, i.e. the form when both indices are contravariant. To achieve this, consider the unit operation $1 = \eta^{\nu\alpha}\eta_{\alpha\nu}$

$$f^{\mu} = qF^{\mu}_{\ \nu}\eta^{\nu\alpha}\eta_{\alpha\nu}u^{\nu} = q\eta^{\alpha\nu}F^{\mu}_{\ \nu}\eta_{\alpha\nu}u^{\nu} = qF^{\mu\alpha}u_{\alpha}$$

Therefore, we can write

$$f^{\mu} = qF^{\mu\nu}u_{\nu} \tag{8}$$

with

$$F^{\mu\nu} = \eta^{\nu\alpha}F^{\mu}_{\ \alpha} = F^{\mu}_{\ \alpha}\eta^{\alpha\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{9}$$

Before moving on, we should note that the Faraday Tensor is a legitimate fourtensor, as it maps a fourvector on a fourvector and will be Lorentz transformable. Furhtermore, it is a anti-symmetric tensor, so $F^{\mu\nu} = -F^{\nu\mu}$, which of course requires $F^{\mu\mu} = 0$, and in this case this applies without summation convention (i.e. for any μ rather than for the sum $F^{00} + F^{11} + ...$).

2.1 Static Fields

For static fields, the Lorentz force equation can be expressed as a static matrix differential equation.

$$f^\mu = m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu$$

$$\frac{du^\mu}{d\tau} = \frac{q}{m}F^{\mu\nu}u_\nu = \frac{q}{m}F^\mu_{\ \nu}u^\nu$$

Suppose that we now express u^ν in terms of eigenvectors of $\frac{q}{m}F^\mu_{\ \nu}$, called U_α. Then

$$u^\nu = c^\alpha U_\alpha^{\ \nu}$$

where c^α is some vector of coefficients for eigenvectors. Therefore, the equation becomes

$$\frac{d}{d\tau}(c^\alpha U_\alpha^{\ \mu}) = \frac{q}{m}F^\mu_{\ \nu}(c^\alpha U_\alpha^{\ \nu}) = c^\alpha\frac{q}{m}F^\mu_{\ \nu}U_\alpha^{\ \nu}$$

Since U_α are eigenvectors of $\frac{q}{m}F^\mu_{\ \nu}$, we can say that

$$\frac{q}{m}F^\mu_{\ \nu}U_\alpha^{\ \nu} = \lambda_\alpha U_\alpha^{\ \mu}$$

where λ_α is the vector of eigenvalues corresponding to the eigenvectors (no summation here). Therefore, the equation becomes

$$\frac{d}{d\tau}(c^\alpha U_\alpha^{\ \mu}) = c^\alpha \lambda_\alpha U_\alpha^{\ \mu}$$

Since the fields are static, we can assume that the eigenvectors and eigenvalues are static as well, which leaves us with

$$U_\alpha^{\ \mu}\frac{dc^\alpha}{d\tau} = U_\alpha^{\ \mu}\lambda_\alpha c^\alpha$$

Since for most matrices, the eigenvectors are orthogonal, we can further say that

$$\frac{dc^\alpha}{d\tau} = \lambda_\alpha c^\alpha$$

with no summation. This is a simple differential equation, leading to

$$c^\alpha = A^\alpha e^{\lambda_\alpha \tau}$$

where A^α are integration constants. Therefore, we have

$$u^\mu = A^\alpha e^{\lambda_\alpha\tau}U_\alpha^{\ \mu} \tag{10}$$

where the sum runs over α.

2.1.1 Uniform field motion

Consider only electric field in only x direction, i.e. only $F^1_{\ 0} = F^0_{\ 1} = \frac{E_x}{c}$ components of Faraday tensor are non-zero. Therefore, the eigenvectors of $\frac{q}{m}F^\mu_{\ \nu}$ can be determined as eigenvectors of

$$\begin{pmatrix} 0 & \frac{qE_x}{mc} & 0 & 0 \\ \frac{qE_x}{mc} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These can be determined from the secular equation

$$\left(\frac{q}{m}F^\mu_{\ \nu} - \lambda_\alpha\delta^\mu_{\ \nu}\right)U_\alpha^\nu = 0$$

Starting with determination of the eigenvalues

$$0 = |\frac{q}{m}F^\mu_{\ \nu} - \lambda_\alpha\delta^\mu_{\ \nu}| = \begin{vmatrix} -\lambda_\alpha & \frac{qE_x}{mc} & 0 & 0 \\ \frac{qE_x}{mc} & -\lambda_\alpha & 0 & 0 \\ 0 & 0 & -\lambda_\alpha & 0 \\ 0 & 0 & 0 & -\lambda_\alpha \end{vmatrix} = (\lambda_\alpha)^2\begin{vmatrix} -\lambda_\alpha & \frac{qE_x}{mc} \\ \frac{qE_x}{mc} & -\lambda_\alpha \end{vmatrix} = (\lambda_\alpha)^2\left((\lambda_\alpha)^2 - \frac{q^2E_x^2}{m^2c^2}\right)$$

Hence we have two degenerate solutions for $\lambda_\alpha = \pm 0 = 0$. These correspond to eigenvectors

$$U_2^{\ \mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$U_3^{\ \mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The other two non-degenerate solutions have eigenvalues $\lambda_\alpha = \pm \frac{qE_x}{mc}$. The eigenvectors can be found from

$$\begin{pmatrix} \mp \frac{qE_x}{mc} & \frac{qE_x}{mc} & 0 & 0 \\ \frac{qE_x}{mc} & \mp \frac{qE_x}{mc} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_\alpha^0 \\ U_\alpha^1 \\ U_\alpha^2 \\ U_\alpha^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can see that these equations are satisfied for

$$U_\alpha^0 = \pm U_\alpha^1$$

Hence we have other two orthogonal eigenvectors

$$U_0^{\ \mu} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$U_1^{\ \mu} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

with eigenvalues $\lambda_0 = \frac{qE_x}{mc}$ and $\lambda_1 = -\frac{qE_x}{mc}$. Therefore, the solution for the fourvelocity is

$$u^\mu = \begin{pmatrix} A^0 e^{qE_x\tau/(mc)} + A^1 e^{-qE_x\tau/(mc)} \\ A^0 e^{qE_x\tau/(mc)} - A^1 e^{-qE_x\tau/(mc)} \\ A^2 \\ A^3 \end{pmatrix}$$

Suppose that our initial conditions are $u^\mu = (\gamma(0)c, 0, \gamma(0)v_y, \gamma(0)v_z)$. These can be satisfied by setting $A^0 = A^1 = \frac{\gamma(0)c}{2}$, $A^2 = \gamma(0)v_y$, $A^3 = \gamma(0)v_z$. Then

$$u^\mu = \begin{pmatrix} \gamma(0)c\cosh\left(\frac{qE_x\tau}{mc}\right) \\ \gamma(0)c\sinh\left(\frac{qE_x\tau}{mc}\right) \\ \gamma(0)v_y \\ \gamma(0)v_z \end{pmatrix}$$

Now, we have a velocity described in terms of proper time, but not in terms of time passed in the original reference frame. Therefore, we need an expression for the proper time τ in terms of the time t. Easiest way to obtain this is by integrating $u^0 = \frac{d(ct)}{d\tau}$, which leads to (assuming that $t = 0$ is the starting point)

$$ct = \int d\tau\gamma(0)c\cosh\left(\frac{qE_x\tau}{mc}\right) = \frac{\gamma(0)mc^2}{qE_x}\sinh\left(\frac{qE_x\tau}{mc}\right)$$

Therefore

$$\tau = \frac{mc}{qE_x}\sinh^{-1}\left(\frac{qE_x t}{\gamma(0)mc}\right)$$

As

$$\cosh\left(\frac{qE_x\tau}{mc}\right) = \cosh\left(\sinh^{-1}\left(\frac{qE_xt}{\gamma(0)mc}\right)\right) = \sqrt{1+\sinh^2\left(\sinh^{-1}\left(\frac{qE_xt}{\gamma(0)mc}\right)\right)} = \sqrt{1+\frac{q^2E_x^2t^2}{\gamma(0)^2m^2c^2}}$$

fourvelocity is

$$u^\mu = \begin{pmatrix} \gamma(0)c\sqrt{1+\frac{q^2E_x^2t^2}{\gamma(0)^2m^2c^2}} \\ \gamma(0)c\frac{qE_xt}{\gamma(0)mc} \\ \gamma(0)v_y \\ \gamma(0)v_x \end{pmatrix}$$

From definition of fourvelocity, we have

$$\gamma = \frac{u^0}{c} = \gamma(0)\sqrt{1+\frac{q^2E_x^2t^2}{\gamma(0)^2m^2c^2}}$$

and therefore the real space velocity is

$$\vec{v} = \frac{1}{\gamma}\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} c\frac{qE_xt}{\sqrt{\gamma(0)^2m^2c^2+q^2E_x^2t^2}} \\ v_y\frac{\gamma(0)mc}{\sqrt{\gamma(0)^2m^2c^2+q^2E_x^2t^2}} \\ v_z\frac{\gamma(0)mc}{\sqrt{\gamma(0)^2m^2c^2+q^2E_x^2t^2}} \end{pmatrix}$$

We should notice that as $t \to \infty$, $v_x \to c$ and $v_y, v_z \to 0$ - the transverse velocities are gradually reduced.

2.2 Field Transformations

Since Faraday tensor is a proper fourtensor, we can use Lorentz transformation to obtain the expressions for the fields $\vec{E}$ and $\vec{B}$ in a moving frame of reference. We use

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}F^{\mu\nu} = \Lambda^{\mu'}_{\mu}F^{\mu\nu}\Lambda^{\nu'}_{\nu} =$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma\beta\frac{E_x}{c} & -\gamma\frac{E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \gamma\frac{E_x}{c} & -\gamma\beta\frac{E_x}{c} & -B_z & B_y \\ \gamma\frac{E_y}{c} - \gamma\beta B_z & \gamma B_z - \gamma\beta\frac{E_y}{c} & 0 & -B_x \\ \gamma\frac{E_z}{c} + \gamma\beta B_y & -\gamma B_y - \gamma\beta\frac{E_z}{c} & B_x & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \gamma^2(\beta^2-1)\frac{E_x}{c} & \gamma\left(\beta B_z - \frac{E_y}{c}\right) & -\gamma\left(\beta B_y + \frac{E_z}{c}\right) \\ \gamma^2(1-\beta^2)\frac{E_x}{c} & 0 & \gamma(-B_z+\beta\frac{E_y}{c}) & \gamma(B_y+\beta\frac{E_z}{c}) \\ \gamma\left(-\beta B_z + \frac{E_y}{c}\right) & \gamma\left(B_z - \beta\frac{E_y}{c}\right) & 0 & -B_x \\ \gamma\left(\beta B_y + \frac{E_z}{c}\right) & \gamma\left(-B_y - \beta\frac{E_z}{c}\right) & B_x & 0 \end{pmatrix}$$

Using $\gamma^2(1-\beta^2) = 1$ and $\beta = \frac{v_x}{c}$, we can write

$$F^{\mu'\nu'} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{\gamma}{c}\left(E_y - v_xB_z\right) & -\frac{\gamma}{c}\left(E_z + v_xB_y\right) \\ \frac{E_x}{c} & 0 & -\gamma(B_z - \frac{v_xE_y}{c^2}) & \gamma(B_y + \frac{v_xE_z}{c^2}) \\ \frac{\gamma}{c}\left(E_y - v_xB_z\right) & \gamma\left(B_z - \frac{v_xE_y}{c^2}\right) & 0 & -B_x \\ \frac{\gamma}{c}\left(E_z + v_xB_y\right) & \gamma\left(-B_y - \frac{v_xE_z}{c^2}\right) & B_x & 0 \end{pmatrix}$$

Hence we have

$$E'_x = cF^{1'0'} = E_x$$

the field parallel to the velocity of the new frame remains unchanged. For the components perpendicular to x, following vector identity follows

$$\vec{E}'_\perp = \gamma\vec{E}_\perp + \gamma\vec{v}\times\vec{B}$$

Similarly, the component of $\vec{B}$ parallel to the velocity $\vec{v}$ remains unchanged

$$B'_x = B_x$$

and the components perpendicular transform as

$$\vec{B}'_\perp = \gamma \vec{B}_\perp - \gamma \frac{\vec{v} \times \vec{E}}{c^2}$$

2.2.1 Spin-Orbit Coupling

Consider a particle moving in a frame of reference S at velocity $\vec{v}$. In this frame, only $\vec{E}$ field exists, i.e. $\vec{B} = 0$. However, in the rest frame of the particle, additional $\vec{B}'$ field occurs

$$\vec{B}' = -\gamma \frac{\vec{v} \times \vec{E}}{c^2} \approx -\frac{\vec{v} \times \vec{E}}{c^2}$$

for non-relativistic approximation. This is the formula used for derivation in quantum physics of atoms.

2.2.2 Ampere's Law

Consider a particle moving in positive z direction along a cylindrical wire which carries current I in the positive z direction. This creates magnetic field $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$, where $\hat{\phi}$ is the unit vector in the azimuthal direction and r is the distance of the particle from the wire. Therefore, the force on the particle is (as electric field $\vec{E} = 0$)

$$F = q\vec{v} \times \vec{B} = qv\frac{\mu_0 I}{2\pi r}\hat{z} \times \hat{\phi} = -\frac{\mu_0 qvI}{2\pi r}\hat{r}$$

Lets say that the current I is created by charges drifting at the same speed as the particle - by speed v in positive z direction. Then, in the rest frame of the particle, the charges are stationary and no $\vec{B}$ field exists. However, there is extra electric field created in perpendicular direction

$$\vec{E}'_{\perp,extra} = \gamma \vec{v} \times \vec{B}$$

Therefore, there is an extra force on the particle due to electrostatic interaction

$$\vec{F}' = \gamma q \vec{v} \times \vec{B} = -\gamma \frac{\mu_0 qvI}{2\pi r}\hat{r}$$

Therefore, in non-relativistic limit, the behaviour of the particle will be the same in both frames of reference, as $\vec{F}' \approx \vec{F}$.

3 Relativistic Description of Electromagnetic Fields

Now that we have described the motion of a particle in an electromagnetic field, we need to describe how these fields are created in the first place. To make a useful description, we need to rewrite Maxwell equations in the relativistic formalism. To be able to do this, we need to develop a few tools for handling the fields. Lets start by defining a fourgradient - a differential operator defined as

$$\partial_\mu = \left(\frac{\partial}{\partial(ct)}, \nabla\right)$$

where ∇ is the 3D gradient. To see whether this is a true fourvector, consider taking a scalar product (effective fourdivergence) with a fourposition

$$\partial_\mu x^\mu = \frac{\partial(ct)}{\partial(ct)} + \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 4$$

This is definitely a Lorentz invariant, and therefore fourgradient is a valid fourvector. We should note the contravariant-covariant switch implied - $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and inversly

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial(ct)}, -\nabla\right)$$

Importantly, we can now define the d'Alembertian (wave) operator as

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial (ct)^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

Since the wave operator is created by scalar product of two fourvectors, it is effectively the magnitude of the fourgradient, and we can therefore see what we proved in the very beginning, that is that the d'Alembertian is invariant under Lorentz transformation.
Last thing we need to mention is the Levi-Civita symbol, which will help us describe 3D vector products in index notation. Levi-civita symbol ϵ_{ijk} for indices $i, j, k \in \{0, 1, 2, 3\}$ is equal to 1 for $(i, j, k) = (1, 2, 3)$ and for any cyclic permutation of this order. Furthermore, it is equal to -1 for $(i, j, k) = (2, 1, 3)$ and any cyclic permutation of this order. For any other combination, mainly if either $i = j$ or $j = k$, the symbol goes to 0. As a consequence, exchanging two indices leads to an exchange of the sign of Levi-Civita symbol (hence, exchanging the order twice leads to no change of the sign). With these tools prepared, we can start to translate the Maxwell equations and the constituent relations into relativistic language.

3.1 Continuity Equation

The continuity equation is particularly easy to spot as a starting point. It can be rewritten as

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = \frac{\partial (c\rho)}{\partial (ct)} + \nabla \cdot \vec{j} = \partial_\mu j^\mu$$

where ρ is the charge density, $\vec{j}$ is the current density and we newly defined the fourcurrent $j^\mu = (\rho c, \vec{j})$. Since the scalar product of the fourcurrent with fourgradient goes to zero, we can see that fourcurrent is a valid fourvector. Therefore, we have a relativistic description of the sources of electromagnetic fields. Now, we must relate this fourvector to the Faraday tensor via Maxwell equations.

3.2 Maxwell Equations

First Maxwell equation is

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

where ϵ_0 is the permittivity of vacuum. Since $F^{00} = 0$, we can write

$$\nabla \cdot \vec{E} = \partial_0 F^{00} + c\partial_1 F^{10} + c\partial_2 F^{20} + c\partial_3 F^{30} = \frac{\rho}{\epsilon_0} = \mu_0 c^2 \rho = \mu_0 c j^0$$

Hence, since $\partial_0 F^{00} = 0 = c\partial_0 F^{00}$

$$c\partial_\mu F^{\mu 0} = \mu_0 c j^0$$
$$\partial_\mu F^{\mu 0} = \mu_0 j^0$$

So, we assigned the time-part of the fourcurrent, now we need to assign the space part. This will be due to some vector equation, specifically due to fourth Maxwell equation. It states that

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

In the x component (writing $(\vec{B})_z = B_z$ and similarly for other components and vectors)

$$\partial_2 B_z - \partial_3 B_y = \mu_0 j_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial (ct)}\left(-\frac{E_x}{c}\right) + \partial_2 B_z - \partial_3 B_y = \mu_0 j_x$$

Since $F^{11} = 0$, $\partial_1 F^{11} = 0$ and

$$\mu_0 j_x = \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$\partial_\mu F^{\mu 1} = \mu_0 j^1$$

we could continue on to show that same identity applies in other components of fourth Maxwell equation, leading to a result

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{11}$$

This is completed translation of the source equations, however, there are other two Maxwell equations that put restraints on $\vec{E}$ and $\vec{B}$. The second Maxwell equation is

$$\nabla \cdot B = 0$$

$$\partial_1 B_x + \partial_2 B_y + \partial_3 B_z = 0$$
$$-\partial_1 F^{23} - \partial_2 F^{31} - \partial_3 F^{12} = 0$$

Using components of $F_{\mu\nu}$ rather than components of $F^{\mu\nu}$ and multiplying the equation by -1, we obtain

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

Here, I used that

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} = \eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{-E_x}{c} & 0 & -B_z & B_y \\ \frac{-E_y}{c} & B_z & 0 & -B_x \\ \frac{-E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

Multiplying the equation by 2, we have

$$\partial_1(F_{23} + F_{23}) + \partial_2(F_{31} + F_{31}) + \partial_3(F_{12} + F_{12}) = 0$$

Finally, using the anti-symmetry of the Faraday tensor

$$\partial_1(F_{23} - F_{32}) + \partial_2(F_{31} - F_{13}) + \partial_3(F_{12} - F_{21}) = 0 \tag{12}$$

We can definitely recognize the structure of the vector product in this sum, but in order to be able to describe it with Levi-Civita symbol, we need to further explore third Maxwell equation in all its components. The third Maxwell equation is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

In x component

$$\partial_2 E_z - \partial_3 E_y + c\partial_0 B_x = 0$$
$$\partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$
$$-\partial_2 F_{30} + \partial_3 F_{20} + \partial_0 F_{32} = 0$$
$$-\partial_2 F_{30} - \partial_3 F_{02} - \partial_0 F_{23} = 0$$

Multiplying by -2 and using anti-symmetry of $F^{\mu\nu}$

$$\partial_0(F_{23} - F_{32}) + \partial_2(F_{30} - F_{03}) + \partial_3(F_{02} - F_{20}) = 0 \tag{13}$$

In y component

$$\partial_3 E_x - \partial_1 E_z + c\partial_0 B_y = 0$$
$$\partial_3 F^{10} - \partial_1 F^{30} + \partial_0 F^{13} = 0$$
$$-\partial_3 F_{10} + \partial_1 F_{30} + \partial_0 F_{13} = 0$$
$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0$$

$$\partial_0(F_{13} - F_{31}) + \partial_1(F_{30} - F_{03}) + \partial_3(F_{01} - F_{10}) = 0 \tag{14}$$

In the z component

$$\partial_1 E_y - \partial_2 E_x + c\partial_0 B_z = 0$$
$$\partial_1 F^{20} - \partial_2 F^{10} + \partial_0 F^{21} = 0$$
$$-\partial_1 F_{20} + \partial_2 F_{10} + \partial_0 F_{21} = 0$$
$$-\partial_0 F_{12} - \partial_1 F_{20} - \partial_2 F_{01} = 0$$

$$\partial_0(F_{12} - F_{21}) + \partial_1(F_{20} - F_{02}) + \partial_2(F_{01} - F_{10}) = 0 \tag{15}$$

By adding equations (12)-(15) together, we obtain

$$\partial_0(F_{12} - F_{21} + F_{13} - F_{31} + F_{23} - F_{32}) + \partial_1(F_{20} - F_{02} + F_{30} - F_{03} + F_{23} - F_{32}) +$$
$$+\partial_2(F_{01} - F_{10} + F_{30} - F_{03} + F_{31} - F_{13}) + \partial_3(F_{01} - F_{10} + F_{02} - F_{20} + F_{12} - F_{21}) = 0$$

This can be summarized by the Bianchi identity

$$\partial_\alpha F_{\beta\gamma}\epsilon_{\alpha\beta\gamma} = 0 \tag{16}$$

3.3 Potentials

Classically, when we solve Maxwell equations, we do so with the use of electromagnetic potentials - scalar potential ϕ and vector potential $\vec{A}$. Lets refresh our memory quickly - $\vec{A}$ is defined based on the second Maxwell equation, as the fact that $\nabla \cdot \vec{B} = 0$ dictates that there exists $\vec{A}$ such that

$$\vec{B} = \nabla \times \vec{A}$$

Substituting this into the third Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A}$$

$$\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

Since $\vec{E}+\frac{\partial \vec{A}}{\partial t}$ is curl free field, it implies that it is equal to gradient of some scalar function. In correspondence with electrostatics, this is defined to be $\nabla(-\phi)$, where ϕ is the scalar potential. Therefore

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

Substituting this result into the first Maxwell equations yields

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \vec{A} = \frac{\rho}{\epsilon_0}$$

And from the fourth Maxwell equation

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{j} + \frac{1}{c^2}\left(-\frac{\partial^2 \vec{A}}{\partial t^2} - \frac{\partial \nabla\phi}{\partial t}\right)$$

This can be either reorganized as

$$\nabla(\nabla \cdot \vec{A}) = \nabla^2 \vec{A} + \mu_0 \vec{j} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}$$

or as

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} - \nabla\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla \cdot \vec{A}\right) \tag{17}$$

We can substitute the first form into the first Maxwell equation if we take a gradient of the first Maxwell equation

$$-\nabla(\nabla^2\phi) - \frac{\partial}{\partial t}\nabla(\nabla\cdot\vec{A}) = \nabla\left(\frac{\rho}{\epsilon_0}\right)$$

$$-\nabla\left(\nabla^2\phi + \frac{\rho}{\epsilon_0}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} - \mu_0\vec{j}\right) - \nabla\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right)$$

$$\nabla\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi - \frac{\rho}{\epsilon_0}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} - \mu_0\vec{j}\right) \tag{18}$$

We can notice that both of these equations can be significantly simplified, if we fix the gauge invariance by requiring the so called Lorenz gauge condition

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{A} = 0 \tag{19}$$

This causes equation (17) to become simply

$$\Box\vec{A} = \mu_0\vec{j} \tag{20}$$

which in turn modifies (18) to

$$\Box\phi = \frac{\rho}{\epsilon_0} \tag{21}$$

We therefore have two decoupled symmetrical wave equations.

3.3.1 Ensuring Gauge Condition

Consider now that we have some ϕ and $\vec{A}$ that do not satisfy the Lorenz gauge condition (19), and we have

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{A} = f$$

where f is some scalar field. Using the gauge transformations, we can transform to equivalent potentials ϕ' and $\vec{A}'$ which do satisfy the gauge condition. The gauge transformations are

$$\phi' = \phi - \frac{\partial\Lambda}{\partial t}$$

$$\vec{A}' = \vec{A} + \nabla\Lambda$$

where Λ is the scalar gauge. Substituting back to the definition of f.

$$f = \frac{1}{c^2}\frac{\partial\phi'}{\partial t} + \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} + \nabla\cdot\vec{A}' - \nabla^2\Lambda$$

Since in the new gauge, we assume that the Lorenz gauge condition is satisfied, we are left with

$$\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} - \nabla^2\Lambda = f$$

$$\Box\Lambda = f$$

Therefore, if the gauge condition is not satisfied, we can find a corresponding gauge Λ for gauge transformation by solving a wave equation, which is generally solvable. Therefore, we do not need to worry too much about the possibility of transforming into Lorenz gauge, knowing that we could always solve a wave equation to ensure that we are indeed working in the Lorenz gauge.

3.4 Fourpotential

We can notice that the Lorenz gauge can be rewritten as

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{A} = \frac{\partial}{\partial(ct)}\left(\frac{\phi}{c}\right) + \nabla\cdot\vec{A} = \partial_0\frac{\phi}{c} + \partial_1 A_x + \partial_2 A_y + \partial_3 A_z = \partial_\mu A^\mu = 0$$

Where $A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$ is a newly defined fourvector - so called fourpotential. Again, the scalar product with the fourgradient - which is a fourvector - results in 0 in Lorenz gauge, which is definitely a Lorentz invariant. Hence fourpotential is a valid fourvector. We now need to relate the fourpotential to the Faraday tensor. This can be done as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{22}$$

We clearly see that this is a anti-symmetrical combination, so basic requirements on form of F are satisfied. For the electric field components ($a \in \{1, 2, 3\}$)

$$E_a = cF_{0\alpha} = c\partial_0 A_a - c\partial_a A_0 = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

as expected. And, just to check, the magnetic field component

$$B_x = F_{32} = \partial_3 A_2 - \partial_2 A_3 = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y = (\nabla\times\vec{A})_x$$

as expected.
We can now substitute this into the equations (11) and (16) to find out the form of relativistic Maxwell equations in potential form. The first substitution leads to

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 j^\nu$$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = \mu_0 j^\nu$$

where I used $\partial_\mu\partial^\mu = \Box$. In the Lorenz gauge, $\partial_\mu A^\mu = 0$, hence

$$\Box A^\nu = \mu_0 j^\nu \tag{23}$$

summarizes both first and fourt Maxwell equations in terms of the fourpotential and fourcurrent. The Bianchi identity becomes

$$\partial_\alpha(\partial_\beta A_\gamma - \partial_\gamma A_\beta)\epsilon_{\alpha\beta\gamma} = 0$$

$$(\partial_\alpha\partial_\beta A_\gamma - \partial_\gamma\partial_\alpha A_\beta)\epsilon_{\alpha\beta\gamma} = 0$$

But, since the indices in $\partial_\alpha\partial_\beta A_\gamma$ and $\partial_\gamma\partial_\alpha A_\beta$ are just cyclic permutations of each other, we know that

$$\partial_\alpha\partial_\beta A_\gamma\epsilon_{\alpha\beta\gamma} = \partial_\gamma\partial_\alpha A_\beta\epsilon_{\alpha\beta\gamma}$$

And therefore we have that for any A_μ, the Bianci identity is satisfied. This means that we can summarize all of field equations in a single wave equation (23).
Therefore, all of electrodynamics can be summarized in terms of fourpotential A^μ as

$$\Box A^\mu = \mu_0 j^\mu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\frac{\partial p^\mu}{\partial\tau} = qF^{\mu\nu}u_\nu$$

These three equations form a complete relativistic theory of electrodynamics.

3.5 Solving the Fourpotential Wave Equation

The wave equation is solved by Green's function method. This involves Fourier transforming definition of the Green's function

$$\Box G(x^\mu, (x')^\mu) = \delta(x^\mu - (x')^\mu)$$

This leads to two possibilities for the Green's function

$$G(x^\mu, (x')^\mu) = \frac{1}{4\pi|\vec{r} - \vec{r}'|}\delta(c(t - t') \pm |\vec{r} - \vec{r}'|)$$

where $x^\mu = (ct, \vec{r})$ and $\vec{r}$ is the position vector. We usually choose the minus solution, as the plus solution would imply that events that happen in some time later than t can influence an event taking place at time t. We call this solution the causal solution. Note that this decision is purely based on our expectations, and in fact is not mathematically neccessary.
The fourpotential that solves the wave equation is then

$$A^\mu(x^\nu) = \int d^4(x')^\nu G(x^\nu, (x')^\nu)\mu_0 j^\mu((x')^\nu)$$

$$A^\mu(x^\nu) = \int d^4(x')^\nu \frac{\mu_0 j^\mu((x')^\nu)}{4\pi|\vec{r} - \vec{r}'|}\delta(c(t - t') - |\vec{r} - \vec{r}'|) = \int d^3r' \frac{\mu_0 j^\mu\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{4\pi|\vec{r} - \vec{r}'|}$$

Hence

$$A^\mu(t, \vec{r}) = \int d^3r' \frac{\mu_0 j^\mu\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{4\pi|\vec{r} - \vec{r}'|} \tag{24}$$

However, this is a somewhat hybrid description, as we describe a fourvector in terms of 3D vectors. We can factor this dependence out by writing the Green's function in relativistic terms as

$$G = \frac{1}{2\pi}\delta\left[(x^\nu - (x')^\nu)(x_\nu - (x')_\nu)\right]\Theta(c(t - t'))$$

where Θ is the Heapside step function (1 for non-negative argument, 0 everywhere else). Then

$$A^\mu(x^\nu) = \int d^4(x')^\nu \frac{\mu_0 j^\mu((x')^\nu)}{2\pi}\delta\left[(x^\nu - (x')^\nu)(x_\nu - (x')_\nu)\right]\Theta(c(t - t')) \tag{25}$$

This description is hybrid only due to our requirement of causality, encapsulated in the Heapside step function factor.

3.6 Fields of Moving Charged Particle

Consider a charge q moving at fourvelocity $(u_p)^\mu$ at fourposition $(x_p)^\nu$. The fourpotential due to the charge movement at position x^ν is

$$A^\mu(x^\nu) = \frac{\mu_0}{2\pi}\int d^4(x')^\nu j^\mu((x')^\nu)\delta\left[(x^\nu - (x')^\nu)(x_\nu - (x')_\nu)\right]\Theta(c(t - t'))$$

Relating to last years mathematical module, it can be shown that

$$\int g(x)\delta(f(x))dx = \sum_i \frac{g(x_i)}{\left|\frac{df}{dx}\right|_{x_i}}$$

where x_i are the positions of zeros of $f(x)$. Lets start with the integration with respect to time coordinate $(x')^0 = ct'$. The zero in the Dirac delta function occurs when $c^2(t - t')^2 = (x^a - (x')^a)^2$ where a runs over spatial indices. Due to Heapside step function, only the positive solution of $f = 0$ is taken into account, leaving us with zero at $ct' = ct - |\vec{r} - \vec{r}'|$, where $|\vec{r} - \vec{r}'| = \sqrt{(x^a - (x')^a)^2}$ (the sum is inside the square root and a runs only over the spatial indices). Therefore, the fourpotential is

$$A^\mu(x^\nu) = \frac{\mu_0}{2\pi}\int d^3(x')^\nu \left[\frac{j^\mu((x')^\nu)}{\left|\frac{d}{d(ct')}\left((x^\nu - (x')^\nu)(x_\nu - (x')_\nu)\right)\right|}\right]$$

where the square brackets indicate that the expression is taken at the retarded time $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$. Here

$$\frac{d}{d(ct')}(x^\nu - (x')^\nu)(x_\nu - (x')_\nu) = \frac{1}{c}\frac{d\tau'}{dt'}\frac{d}{d\tau'}(x^\nu - (x')^\nu)(x_\nu - (x')_\nu)$$

where τ' is the proper time of the charge moving. Therefore

$$\frac{dx^\nu}{d\tau'} = 0$$
$$\frac{d(x')^\nu}{d\tau'} = (u')^\nu$$
$$\frac{d\tau'}{dt'} = \frac{1}{\gamma}$$

And thus

$$\frac{d}{d(ct')}(x^\nu - (x')^\nu)(x_\nu - (x')_\nu) = \frac{2}{c\gamma}\left(\frac{dx^\nu}{d\tau'} - \frac{d(x')^\nu}{d\tau'}\right)(x_\nu - (x')_\nu) = -2\frac{1}{\gamma c}(u')^\nu(x_\nu - (x')_\nu)$$

And therefore

$$A^\mu(x^\nu) = \frac{\mu_0 c}{2\pi}\int d^3(x')^\nu\left[\frac{\gamma j^\mu((x')^\nu)}{2(u')^\nu(x_\nu - (x')_\nu)}\right]$$

Since we are interested in point charge, the current density has form

$$j^\mu((x')^\nu) = q(c,\vec{v})\delta^3((x')^\nu - (x_p)^\nu) = q\frac{1}{\gamma}(u_p)^\mu\delta^3((x')^\nu - (x_p)^\nu)$$

where $(x_p)^\nu$ is the actual position of the particle. Hence

$$A^\mu(x^\nu) = \frac{\mu_0 c}{2\pi}\left[\frac{\gamma q\frac{1}{\gamma}(u_p)^\mu}{2(u_p)^\nu(x_\nu - (x_p)_\nu)}\right] = \frac{\mu_0 qc}{4\pi}\left[\frac{(u_p)^\mu}{(u_p)^\nu(x_\nu - (x_p)_\nu)}\right]$$

Taking the required derivatives than leads to Faraday tensor

$$F^{\mu\nu} = \frac{\mu_0 qc}{4\pi}\left[\frac{((x^\mu - (x_p)^\mu)(u_p)^\nu - (x^\nu - (x_p)^\nu)u^\mu)\left(c^2 - (x^\alpha - (x_p)^\alpha)(a_p)_\alpha\right)}{((x^\alpha - (x_p)^\alpha)(u_p)_\alpha)^3} + \frac{(x^\mu - (x_p)^\mu)(a_p)^\nu - (x^\nu - (x_p)^\nu)(a_p)^\mu}{((x^\alpha - (x_p)^\alpha)u_\alpha)^2}\right]$$

Substituting $X^\mu = (x^\mu - (x_p)^\mu)$, $(x^\alpha - (x_p)^\alpha)(u_p)_\alpha = X \cdot u_p$ and $(x^\alpha - (x_p)^\alpha)(a_p)_\alpha = X \cdot a_p$, where $(a_p)^\mu$ is the fouracceleration of the particle. we can simplify the expression to

$$F^{\mu\nu} = \frac{\mu_0 qc}{4\pi}\left[\frac{(X^\mu(u_p)^\nu - X^\nu(u_p)^\mu)(c^2 - X\cdot a_p)}{(X\cdot u_p)^3} + \frac{X^\mu(a_p)^\nu - X^\nu(a_p)^\mu}{(X\cdot u_p)^2}\right] \tag{26}$$

3.6.1 Electric Field of Uniformly Moving Particle

Consider a particle moving uniformly - $a^\mu = 0$. Then, the electric field is given by

$$F^{a0} = \frac{\mu_0 qc}{4\pi}\left[\frac{(X^a u^0 - X^0 u^a)c^2}{(X\cdot u_p)^3}\right]$$

where a runs over the spatial indices. If we mark $\vec{r}$ and $\vec{r}_p$ as 3D positions of the field point and the particle (respectively) and $\vec{v}$ as the 3D velocity of the particle, we have

$$F^{a0} = \frac{\vec{E}}{c} = \frac{\mu_0 qc^3}{4\pi}\left[\frac{(\vec{r}-\vec{r}_p)\gamma c - c(t-t_p)\gamma\vec{v}}{(X\cdot u_p)^3}\right]$$

Taken at the retarded time requires $c(t-t_p) = |\vec{r}-\vec{r}_p|$. Writing a shorthand $\vec{R} = \vec{r}-\vec{r}_p$, we can also write

$$X\cdot u_p = \gamma c^2(t-t_p) - \gamma\vec{v}\cdot\vec{R}$$

Taken at the retarded time

$$X\cdot u_p = \gamma c|\vec{R}| - \gamma\vec{v}\cdot\vec{R} = \gamma c\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)$$

And thus

$$\frac{\vec{E}}{c} = \frac{\mu_0 q c^3}{4\pi} \frac{\gamma c \left(\vec{R} - \frac{R}{c}\vec{v}\right)}{\gamma^3 c^3 \left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^3}$$

$$\vec{E} = \frac{\mu_0 q c^2}{4\pi} \frac{\vec{R} - \frac{R}{c}\vec{v}}{\gamma^2 \left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^3} \tag{27}$$

Most interesting feature of this field is that it behaves as a field caused by a particle at position $\vec{R} - \frac{R}{c}\vec{v}$, which is the position extrapolated from the position of the particle at the retarded time and the instantaneus velocity of the particle at that time. This is called the pretorial understanding of the radiation field.

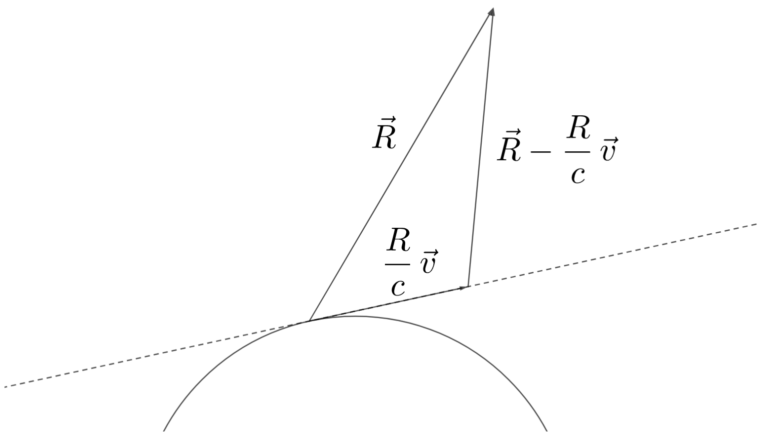

Figure 1: The curved line represents the trajectory of the particle, the dashed line is a tangent to the trajectory hence parallel with instantaneus velocity of the particle. The field at some point is given as by a particle in position predicted by extrapolating the instantaneus velocity.

Lets call the vector of extrapolated particle position $\vec{r} = \vec{R} - \frac{R}{c}\vec{v}$. We can write

$$\vec{R} = \vec{r} + \frac{R}{c}\vec{v}$$

$$R^2 = r^2 + 2\frac{R}{c}\vec{v}\cdot\vec{r} + R^2\frac{v^2}{c^2}$$

$$\frac{R^2}{\gamma^2} - 2\frac{R}{c}\vec{v}\cdot\vec{r} - r^2 = 0$$

$$R = \frac{\gamma^2}{2}\left(2\frac{\vec{v}\cdot\vec{r}}{c} \pm \sqrt{4\frac{(\vec{v}\cdot\vec{r})^2}{c^2} + 4\frac{r^2}{\gamma^2}}\right) = \gamma^2\left(\frac{\vec{v}\cdot\vec{r}}{c} \pm \sqrt{\frac{(\vec{v}\cdot\vec{r})^2}{c^2} + \frac{r^2}{\gamma^2}}\right)$$

Hence

$$R - \frac{\vec{v}\cdot\vec{R}}{c} = R - \frac{\vec{v}\cdot(\vec{r} + \frac{R}{c}\vec{v})}{c} = R\left(1 - \frac{v^2}{c^2}\right) - \frac{\vec{v}\cdot\vec{r}}{c} = \frac{R}{\gamma^2} - \frac{\vec{v}\cdot\vec{r}}{c}$$

Substituting for R

$$R - \frac{\vec{v}\cdot\vec{R}}{c} = \pm\sqrt{\frac{(\vec{v}\cdot\vec{r})^2}{c^2} + \frac{r^2}{\gamma^2}}$$

$$\gamma^2\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^2 = \frac{\gamma^2(\vec{v}\cdot\vec{r})^2}{c^2} + r^2$$

Decomposing $\vec{r}$ into orthogonal components - one parallel to $\vec{v}$, $r_\parallel$ and other perpendicular to $\vec{v}$, $r_\perp$, we have

$$\vec{v}\cdot\vec{r} = v r_\parallel$$

and

$$r^2 = r_\perp^2 + r_\parallel^2$$

Hence

$$\gamma^2\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^2 = \gamma^2\frac{v^2}{c^2}r_\parallel^2 + r_\parallel^2 + r_\perp^2 = r_\parallel^2(\gamma^2\left(1-\frac{1}{\gamma^2}\right)+1) + r_\perp^2 = \gamma^2 r_\parallel^2 + r_\perp^2$$

And so we can write

$$\vec{E} = \frac{\mu_0 q c^2}{4\pi}\frac{\gamma\vec{r}}{\left(\gamma^2 r_\parallel^2 + r_\perp^2\right)^{\frac{3}{2}}}$$

Or, in terms of permitivity $\epsilon_0 = \frac{1}{\mu_0 c^2}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{r}}{(\gamma^2 r_\parallel^2 + r_\perp^2)^{\frac{3}{2}}}$$

Therefore, field in direction parallel to velocity v gets reduced as $\vec{E} \propto \frac{1}{\gamma^2}$, while field perpendicular to the velocity v gets enhanced as $\vec{E} \propto \gamma$. This concentrates the radiation generated by the particle in a sort of disc perpendicular to the particle velocity, propagating in space - this is the generation of relativistic electromagnetic radiation.

4 Applications of Relativistic Electrodynamics

4.1 Undulator

Undulator is a high precision radiation generator, consisting of an electron accelerator and undulator chamber, where a magnetic field B is set in an stationary pattern where it oscillates as we move through the chamber with wavenumber k_0.
We should note that because the phase of a wave needs to be a Lorentz invariant, we have

$$\phi = \omega t - \vec{k}\cdot\vec{r} = \frac{\omega}{c}(ct) - \vec{k}\cdot\vec{r} = x^\mu k_\mu$$

where $k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$ is a fourwavevector (or wave fourvector). Again, the scalar product with fourposition is a phase of the wave - Lorentz invariant - hence fourwavevector is a proper fourvector.
Therefore, in a rest frame of the electron incident on the undulator chamber, the magnetic wave transforms as (for simplified 1D case)

$$\frac{\omega}{c} = \gamma\left(\frac{\omega_0}{c} - \beta k_0\right) = -\gamma\beta k_0$$

where $\omega_0 = 0$ is the frequency of the wave in the undulator chamber rest frame. Also

$$k = \gamma(k_0 - \beta\frac{\omega_0}{c}) = \gamma k_0$$

In this frame, the electron will then radiate light at frequency $\omega_L' = \omega$ and with wavenumber $k_L' = \frac{\omega_L'}{c} = \frac{\omega}{c}$. This light can be backwards Lorentz transformed as a wave to obtain the wavenumber and frequency of the light in the undulator rest frame

$$\frac{\omega_L}{c} = \gamma\left(\frac{\omega_L'}{c} + \beta k_L'\right) = \gamma(1+\beta)(-\gamma\beta k_0)$$

Then, as we can without any loss of generality say that $\omega_L = |\omega_L|$

$$\omega_L = \gamma^2(1+\beta)ck_0 = \sqrt{\frac{1+\beta}{1-\beta}}ck_0$$

And in high relativistic limit, $\beta \to 1$ and $\omega_L \to 2\gamma^2 ck_0$.

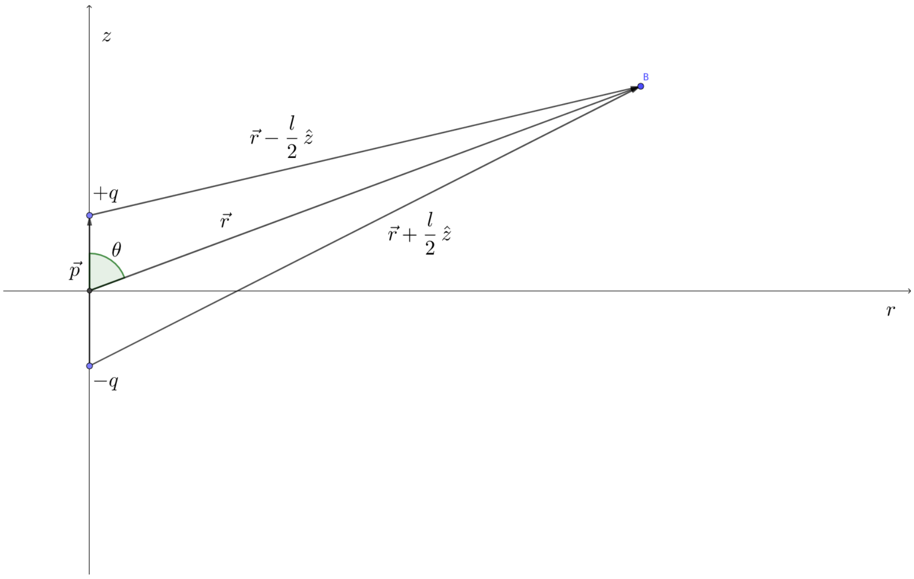

Figure 2: A oscillating dipole moment is created by current flowing along z axis from $z=-l/2$ to $z=l/2$ periodically, storing oscillating charge q and $-q$ on the opposite ends of the origins of the current.

4.2 Hertzian Dipole Radiation

The fourpotential of the Hertzian dipole can be determined from the hybrid integral representation (24).

$$A^\mu = \frac{\mu_0}{4\pi}\int d^3r' \left[\frac{j^\mu}{|\vec{r}-\vec{r}'|}\right]$$

Hence, for the scalar potential

$$A^0 = \frac{\phi}{c} = \frac{\mu_0}{4\pi}\int d^3r' \left[\frac{c\rho}{|\vec{r}-\vec{r}'|}\right] = \frac{\mu_0 c}{4\pi}\left(\frac{q\left(t-\frac{\left|\vec{r}-\frac{l}{2}\hat{z}\right|}{c}\right)}{\left|\vec{r}-\frac{l}{2}\hat{z}\right|} - \frac{q\left(t-\frac{\left|\vec{r}+\frac{l}{2}\hat{z}\right|}{c}\right)}{\left|\vec{r}+\frac{l}{2}\hat{z}\right|}\right)$$

Now, we will assume that the size of the dipole l is relatively small compared to the distance at which we search for the potential, i.e. $\frac{l}{r} \ll 1$. Then

$$\left|\vec{r}-\frac{l}{2}\hat{z}\right| = \sqrt{\left(\vec{r}-\frac{l}{2}\hat{z}\right)^2} = \sqrt{r^2 - l\vec{r}\cdot\hat{z}+\frac{l^2}{4}} = r\sqrt{1-\frac{l\cos\theta}{r}+\frac{l^2}{4r^2}} \approx r\sqrt{1-\frac{l\cos\theta}{r}} \approx r - \frac{l\cos\theta}{2}$$

Hence

$$\phi \approx \frac{\mu_0 c^2}{4\pi}\left(\frac{q\left(t-\frac{r-\frac{l\cos\theta}{2}}{c}\right)}{r-\frac{l\cos\theta}{2}} - \frac{q\left(t-\frac{r+\frac{l\cos\theta}{2}}{c}\right)}{r+\frac{l\cos\theta}{2}}\right) \approx \frac{1}{4\pi\epsilon_0}\left(-\frac{\partial}{\partial r}\left(\frac{q\left(t-\frac{r}{c}\right)}{r}\right)\right) l\cos\theta$$

Here

$$-\frac{\partial}{\partial r}\frac{q\left(t-\frac{r}{c}\right)}{r} = -\frac{\frac{\partial q(t-r/c)}{\partial r}r - q(t-r/c)}{r^2}$$

Using

$$\frac{\partial q(t-r/c)}{\partial r} = \frac{\partial q(t-r/c)}{\partial(t-r/c)}\frac{\partial(t-r/c)}{\partial r} = -\frac{1}{c}\dot{q}\left(t-\frac{r}{c}\right)$$

where $\dot{q}$ is the time derivative of $q(t)$, we arrive at

$$\phi = \frac{1}{4\pi\epsilon_0 c} l\cos\theta\left(\frac{\dot{q}(t-r/c)r + cq(t-r/c)}{r^2}\right) = \frac{\cos\theta}{4\pi\epsilon_0 c}\left[\dot{q}\frac{l}{r} + q\frac{cl}{r^2}\right]$$

where square brackets indicate taken at retarded time $t-\frac{r}{c}$. Since we assumed that r is big, we can neglect the second part of the expression, so we get

$$\phi = \frac{\cos\theta}{4\pi\epsilon_0 cr} l[\dot{q}]$$

Using definition of dipole moment $\vec{p} = ql\hat{z}$ and its magnitude $p = ql$, we have

$$\phi = \frac{\cos\theta}{4\pi\epsilon_0 cr}[\dot{p}] \tag{28}$$

Similarly, we can obtain the vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{j}(t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} dz \frac{\dot{q}(t - \frac{|\vec{r}-\vec{r}'|}{c})\hat{z}}{|\vec{r}-\vec{r}'|} \approx \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} dz \frac{\dot{q}(t - \frac{r}{c})\hat{z}}{r} = \frac{\mu_0}{4\pi r} l[\dot{q}\hat{z}]$$

Therefore

$$\vec{A} = \frac{\mu_0}{4\pi r}[\dot{\vec{p}}] \tag{29}$$

Hence the magnetic field is (using curl in spherical coordinates, with $\vec{A} = A_z(r)\hat{z} = A_z(r)(\cos\theta\hat{r} - \sin\theta\hat{\theta})$

$$\vec{B} = \nabla\times\vec{A} = \hat{\phi}\frac{1}{r}\left(\frac{\partial(-rA_z(r)\sin\theta)}{\partial r} - \frac{\partial(A_z(r)\cos\theta)}{\partial\theta}\right) = \frac{\hat{\phi}}{r}\frac{\mu_0}{4\pi}\left(-\sin\theta\frac{\partial}{\partial r}[\dot{p}] + \sin\theta\frac{[\dot{p}]}{r}\right)$$

Again, we can use

$$-\frac{\partial}{\partial r}[\dot{p}] = -\frac{\partial(t-r/c)}{\partial r}\frac{\partial}{\partial(t-r/c)}[\dot{p}] = \frac{1}{c}[\ddot{p}]$$

And thus

$$\vec{B} = \frac{\mu_0\hat{\phi}}{4\pi}\left(\sin\theta\frac{[\ddot{p}]}{cr} + \sin\theta\frac{[\dot{p}]}{r^2}\right)$$

And since r is again big, we can approximate

$$\vec{B} \approx \frac{\mu_0\sin\theta}{4\pi rc}[\ddot{p}]\hat{\phi} \tag{30}$$

Similarly, we can find the electric field

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

Here

$$\frac{\partial\vec{A}}{\partial t} = \frac{\partial(t-r/c)}{\partial t}\frac{\partial\vec{A}}{\partial(t-r/c)} = \frac{\mu_0}{4\pi r}[\ddot{\vec{p}}] = \frac{\mu_0}{4\pi r}[\ddot{p}](\cos\theta\hat{r} - \sin\theta\hat{\phi})$$

and

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = \frac{\cos\theta}{4\pi\epsilon_0 c}\frac{\partial}{\partial r}\left(\frac{\dot{p}\left(t-\frac{r}{c}\right)}{r}\right)\hat{r} - \frac{[\dot{p}]}{4\pi\epsilon_0 cr^2}\sin\theta\hat{\theta}$$

Using the same differentiation and approximation as in the case for $\frac{\partial}{\partial r}\left(\frac{q(t-r/c)}{r}\right) \approx \frac{-1}{c}\frac{[\dot{q}]}{r}$, we have

$$\nabla\phi \approx -\frac{\cos\theta}{4\pi\epsilon_0 c^2 r}[\ddot{p}]\hat{r} - \frac{[\dot{p}]}{4\pi\epsilon_0 cr^2}\sin\theta\hat{\theta} \approx -\frac{\mu_0\cos\theta}{4\pi r}[\ddot{p}]\hat{r}$$

as r is big. Therefore

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = \frac{\mu_0\cos\theta}{4\pi r}[\ddot{p}]\hat{r} - \frac{\mu_0\cos\theta}{4\pi r}[\ddot{p}]\hat{r} + \frac{\mu_0\sin\theta}{4\pi r}[\ddot{p}]\hat{\theta}$$

Hence

$$\vec{E} = \frac{\mu_0\sin\theta}{4\pi r}[\ddot{p}]\hat{\theta} \tag{31}$$

The Poynting vector is

$$\vec{N} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0}{16\pi^2 r^2 c}\sin^2\theta[\ddot{p}]^2\hat{\theta}\times\hat{\phi} = \frac{\mu_0}{16\pi^2 r^2 c}\sin^2\theta[\ddot{p}]^2\hat{r}$$

The energy therefore flows radially outwards. To calculate the energy transmitted into the solid angle $d\Omega$, we need to calculate corresponding dP

$$dP = \vec{N}\cdot d\vec{S} = \vec{N}\cdot r^2 d\Omega\hat{r} = \frac{\mu_0}{16\pi^2 c}\sin^2\theta[\ddot{p}]^2 d\Omega$$

Hence

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\sin^2\theta[\ddot{p}]^2$$

Or, taking the average value

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\sin^2\theta < [\ddot{p}]^2 >$$

Also, we can notice that for oscillatory p,

$$< [\ddot{p}]^2 >= \omega^2 < [\dot{p}]^2 >= \omega^2 < [l\dot{q}] >^2 = \omega^2 l^2 < I^2 >$$

where I is the current flowing through the dipole. As $\omega^2 = 4\pi^2\frac{c^2}{\lambda^2}$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{4}\frac{l^2}{\lambda^2}\sin^2\theta < I^2 >$$

We can recognize $\mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0 \approx 377\Omega$ is the impedance of vacuum. The common way how to adress differential radiation power in description of antennae is to use

$$\frac{dP}{d\Omega} = R_{rad} < I^2 > \frac{1}{4\pi}G(\theta,\phi) \tag{32}$$

where R_{rad} is the radiation resistance and G is a gain function, normalized as

$$\int\int_\Omega G(\theta,\phi)d\Omega = 4\pi$$

where the integration runs over the whole solid angle. In our case, we can see that $G(\theta,\phi) = C\sin^2\theta$, where C is some normalisation constant. Then

$$4\pi = \int_0^{2\pi}\int_0^{\pi} C\sin^3\theta d\theta d\phi = 2\pi C\int_0^{\pi}\sin^3\theta d\theta$$

Here

$$\int_0^{\pi}\sin^3\theta d\theta = \int_0^{\pi}\sin\theta\frac{1-\cos(2\theta)}{2}d\theta = \frac{1}{2}\left(\int_0^{\pi}\sin\theta d\theta - \int_0^{\pi}\sin\theta\cos(2\theta)d\theta\right) =$$
$$= \frac{1}{2}\left(2 - \int_0^{\pi}\frac{(\sin(3\theta)-\sin\theta)}{2}d\theta\right) = \frac{1}{2}\left(2 + \frac{1}{2}\int_0^{\pi}\sin\theta d\theta - \frac{1}{2}\int_0^{\pi}\sin(3\theta)d\theta\right) =$$
$$= \frac{1}{2}\left(3 - \frac{1}{2}\left[\frac{1}{3}\cos(3\pi)\right]_{\pi}^{0}\right) = \frac{1}{2}\left(3-\frac{1}{3}\right) = \frac{4}{3}$$

Therefore

$$4\pi = \frac{8\pi}{3}C$$
$$C = \frac{3}{2}$$

Therefore, we can rewrite

$$\frac{dP}{d\Omega} = R_{rad} < I^2 > \frac{3}{8\pi}\sin^2\theta$$

Comparing this with

$$\frac{dP}{d\Omega} = \frac{Z_0}{4}\frac{l^2}{\lambda^2}\sin^2\theta < I^2 >$$

Leads to

$$R_{rad} = \frac{2\pi}{3}Z_0\frac{l^2}{\lambda^2} \tag{33}$$

Careful analysis of the hertzian dipole problem shows that we infact also need requirement that $l \ll \lambda$, therefore we can see that the overall power transmitted by Hertzian dipole is relatively small.

We can also make the Hertzian dipole to absorb incident radiation. I will state the antenna reciprocal theorem without proof now. This combines the angular gain $G(\theta, \phi)$ of the given antena (not just Hertzian dipole) with the incident flux of energy N_{in} to determine the power absorbed by the antenna P_{in}

$$P_{in} = N_{in}\frac{\lambda^2}{4\pi}G(\theta, \phi) \tag{34}$$

where λ^2 is the wavelength of the radiation. However, this theorem only holds when the impidances (radiation resistances) are matched. Also, polarization of the incident radiation plays a role - to resolve it, simply find which part of the radiation electric field is along the direction of the dipole.
The overall power emmited by the Hertzian dipole is

$$P = \int\int_\Omega \frac{dP}{d\Omega}d\Omega = R_{rad} < I^2 > \frac{1}{4\pi}\int\int_\Omega G(\omega, \phi) = R_{rad}[I]^2 = \frac{2\pi}{3}Z_0\frac{l^2}{\lambda^2} < I^2 >$$

We can go back to expression using dipole moment rather than the current, as $< I^2 >=< \dot{q}^2 >= \omega^2 < q^2 >$
And

$$P = \frac{2\pi}{3}Z_0\frac{1}{\lambda^2}\omega^2 l^2 < q^2 >$$

and using $\omega = \frac{2\pi c}{\lambda}$ and $Z_0 = \frac{1}{\epsilon_0 c}$

$$P = \frac{1}{6\pi\epsilon_0 c^3}\omega^4 < p^2 > \tag{35}$$

4.2.1 Rayleigh scattering

We can approximate molecules in the atmosphere for certain range of wavelengths λ as Hertzian dipoles absorbing and reemitting radiation. The total scattering power is the same as for the Hertzian dipole, and goes as $P \propto \lambda^{-4}$ - scattering is much stronger for smaller wavelengths (blue light). As Hertzian dipoles, molecules can only scatter light into polarization parallel to the axis of the dipole (in electric field). Therefore, if we look at light which travelled horizontally from the Sun and then was scattered at 90 degrees vertically towards Earth's surface (and our eye), the light should be almost entirely polarized parallel to the surface and normal to the direction from us to the Sun. As a special case, if we look directly above us at light scattered in the morning, when the Sun is close to east, the light we observe should be polarized in the north-south direction.
The Rayleigh scattering is the reason why the sky is observed to be blue - the blue light is the one most scattered by atmospheric molecules. Also, it is responsible for red sunsets, as in that case, the light travels longer through the atmosphere and therefore the blue light is scattered out of the spectrum.

4.2.2 Thompson Scattering

When a free electron is displaced by vector $\vec{x}$ from some initial position in matter, it leaves an effective hole behind, creating an effective dipole moment $\vec{p} = q\vec{x}$. If the electron is then left to radiate, it radiates as dipole with power

$$P = \frac{1}{6\pi\epsilon_0 c^3} < \ddot{p}^2 >= \frac{1}{6\pi\epsilon_0 c^3}q^2 < a^2 >$$

where a is the acceleration of the electron. We can use this to calculate the power scattering cross-section of Thompson scattering σ_T. This is defined as

$$P = \sigma_T N_{in}$$

We can assume that the light incident on the molecule has $N_{in} = |\frac{1}{\mu_0}\vec{E} \times \vec{B}| = \frac{1}{\mu_0 c}E^2 = \epsilon_0 c E^2$ (using that for light, $E = cB$). Also, in a simple model, we can assume that the dipole moment is caused by free electrons accelerated under the electric field as

$$|m\vec{a}| = |q\vec{E}|$$

where m is the mass of the electron and $\vec{a}$ is the acceleration of the electron. Hence

$$< \ddot{p} >^2 = q^2 < a >^2 = \frac{q^4 E^2}{m^2}$$

And therefore, we have identity

$$P = \frac{1}{6\pi\epsilon_0 c^3}\frac{q^4}{m^2}E^2 = \sigma_T \epsilon_0 c E^2$$

$$\frac{e^4}{6\pi\epsilon_0^2 c^4 m^2} = \sigma_T$$

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 = \frac{8\pi}{3}r_0^2 \tag{36}$$

where $r_0 \approx 2.8fm$ is the classical radius of an electron.