PX263 – Physics of Fluids

fluid definition: any substance which cannot withstand a shear force without motion
e.g. liquids & gases.

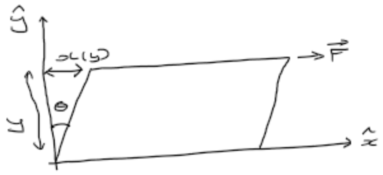
take a solid:

held fixed in place at bottom

tangent to area face

stress $\tau = \frac{F}{A}$ units of pressure (Pascals, Pa)

what solid now looks like in the x–y plane due to force $\vec{F} = F\hat{x}$



strain, $e = \frac{\partial x}{\partial y}$

$= \tan\theta \approx \theta$ for small displacements $x(y)$

& stress $\propto$ strain

$\tau = G e$

G ↳ shear modulus

remove stress & shape back to how it was originally

- if $\frac{\partial x}{\partial y}$ (strain) is constant for a fluid then no relative motion of layers of fluid

  $\Rightarrow$ no friction between layers

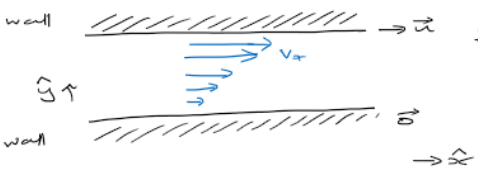
  $\Rightarrow \tau = 0$ or $\tau \neq Ge$

for a fluid to have stress, $e = e(t)$ & $\tau = \mu \frac{de}{dt}$ where $e = \frac{\partial x}{\partial y}$ ∴

μ ↓ viscosity

$$\frac{de}{dt} = \frac{d}{dt}\frac{\partial x}{\partial y} = \frac{\partial v_x}{\partial y}$$

so $\tau = \mu \frac{\partial v_x}{\partial y}$



(external stress)

if we then made $\vec{u} = \vec{0}$ (static wall) then fluid will continue same as before

newtonian fluid $\mu \neq \mu(v)$ & vice versa

v - velocity of the fluid

☆ 2 fluid descriptions:

- Lagrangian fluid element: section of the fluid which moves with the fluid; may deform
  - → Lagrangian fluid element moves with the fluid so no flux of fluid enters or leaves. mass constant

full solution, must solve for all elements
→ analytically complicated.

- Eulerian fluid element: fixed at some initial time ($t=0$) & do not move. fluid now enters & leaves the element
  mass not conserved in the element (but obviously total mass across all elements is conserved)

Lagrangian

not fixed in space
same fluid velocity field lines.

DEFORMED!!!

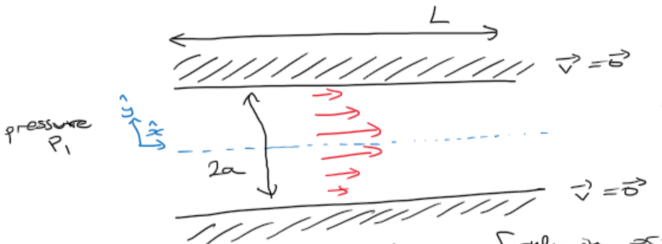
Eulerian

NOT DEFORMED

fixed in space
different fluid vector field

Example: Laminar, viscous, steady flow

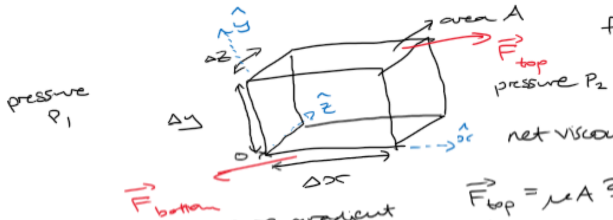
- Laminar ↓ smooth flow surface
- viscous ↓ friction between fluid layers $\mu \neq 0$
- steady ↘ $\vec{v}(\vec{r})$ fluid velocity vector flow field independent of time



the difference in pressure causes such a velocity field as shown in red. such that $P_1 > P_2$

assume $\vec{v} = v_x(y)\,\hat{x}$ [only in x-direction, & magnitude varies with y-component]

lets look at a fluid element



pressure gradient along x-axis due to differences in P_1 & P_2

force due to viscous shear

$$F = A\mu \frac{\partial v_x}{\partial y}$$

since $\tau = \frac{F}{A}$

$\tau = \mu \frac{\partial v_x}{\partial y}$

net viscous force: $\vec{F} = \vec{F}_{top} + \vec{F}_{bottom}$

$$\vec{F}_{top} = \mu A \frac{\partial v_x}{\partial y}$$

$$\vec{F}_{bottom} = -\mu A \frac{\partial v_x}{\partial y}$$

where $\frac{\partial v_x}{\partial y}$ not equal at top & bottom

$A = \Delta x \Delta z$

assume μ constant & used definition of the derivative

$$\therefore \vec{F} = \left[\mu \frac{\partial v_x}{\partial y}\Big|_{y+\Delta y} - \mu \frac{\partial v_x}{\partial y}\Big|_{y}\right] \Delta x \Delta z\, \hat{x}$$

$$\Rightarrow \quad \vec{F} = \mu \frac{\partial^2 v_x}{\partial y^2} \Delta x \Delta y \Delta z\, \hat{x} \quad ①$$

pressure force: $\vec{F}_p = \left[P(x) - P(x+\Delta x)\right] \Delta y \Delta z\, \hat{x}$

$$= -\frac{\partial P}{\partial x} \Delta x \Delta y \Delta z\, \hat{x} \quad ②$$

total force: $\vec{F}_{total} = \vec{F}_p + \vec{F}$ steady flow $\Rightarrow$ no acceleration

& use ①&② into

α

$$\Rightarrow \quad \frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

both sides functions of different variables
⇒ both equal to some constant

$$\mu \frac{d^2 v_x}{dy^2} = -Q$$

$$\Rightarrow v_x(y) = -\frac{Q}{2\mu} y^2 + Ay + B$$

use boundary conditions $v_x(a) = 0 = v_x(-a)$

$$\Rightarrow \quad v_x(y) = \frac{Q}{2\mu}(a^2 - y^2)$$

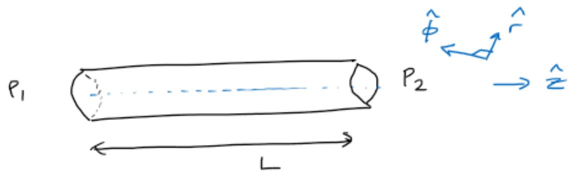
from pressure $Q = -\frac{dP}{dx} = \frac{P_1 - P_2}{L}$ — pressure difference over the length of the pipe.

example: Poiseuille flow
(cylindrically symmetric, steady, viscous, laminar flow)

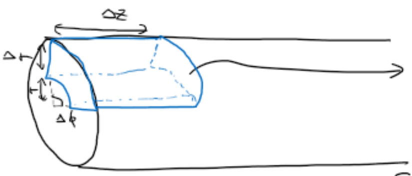
$$\vec{r} = (r\cos\phi, r\sin\phi, z)$$

velocity field $\vec{v} = v_z(r)\,\hat{z}$

assume fluid mass density ρ constant



similar analysis to first example.



area of element $= r\Delta\phi\Delta r$

viscous force $\vec{F}_v = \Delta z \Delta\phi\, \mu \left[(r+\Delta r)\frac{\partial v_z}{\partial r}\Big|_{r+\Delta r} - r\frac{\partial v_z}{\partial r}\Big|_{r} \right] \hat{z}$

Taylor expansion around r ... (ignore higher order terms)

$$\vec{F}_v = \mu \Delta z \Delta\phi \Delta r \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

Pressure force: $\vec{F}_p = r\Delta\phi\Delta r\left[P(z) - P(z+\Delta z)\right]\hat{z}$

$$\approx -r\Delta\phi\Delta z\Delta r \frac{\partial P}{\partial z}$$

force balance $\vec{F}_p + \vec{F}_v = \vec{0}$

$\Rightarrow \quad \frac{\partial P}{\partial z} = \mu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right)$

similar to planar flow example:

$\frac{\partial P}{\partial z} = -Q = \frac{P_2 - P_1}{L}$

$\frac{\mu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = -Q$

$\Rightarrow \quad \frac{\partial v_z}{\partial r} = -\frac{Qr}{2\mu} + \frac{A}{r}$ smooth solution at $r=0$ $\Rightarrow A = 0$

$$v_z(r) = -\frac{Qr^2}{4\mu} + B$$

$v_z(a) = 0$

$$\Rightarrow \quad v_z(r) = \frac{Q}{4\mu}(a^2 - r^2) = \frac{P_1 - P_2}{4\mu L}(a^2 - r^2)$$

flow rate: the mass of fluid flowing through area $r\Delta r\Delta\phi$ is

$\rho v_z r \Delta r \Delta\phi$ in unit time $(t=1)$

mass density

Total mass flow: $M = \int_0^{2\pi}\int_0^a \rho\, v_z(r)\, r\, dr\, d\phi$

$$= \frac{\pi\rho}{8\mu L}(P_1 - P_2)a^4$$

general equation for steady, viscous flow - only pressure & viscous forces:

$$\vec{\nabla} P = \mu \nabla^2 \vec{u}$$

i.e. planar flow between plates $\nabla^2 \vec{u} = \frac{\partial^2 v_x}{\partial y^2}\hat{x}$

cylindrical flow $\nabla^2 \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\hat{z}$

viscosity property of fluid,

gas $\mu \uparrow$ with T

liquid $\mu \downarrow$ with T

kinematic viscosity

$\nu = \frac{\mu}{\rho}$

Reynolds Number $Re = \frac{\rho_0 v_0 L_0}{\mu}$ velocity, length

subscript "0" denotes typical scale

viscous flow $\Rightarrow$ Re small

take Poiseuille flow: $\quad v_z = \dfrac{Q(a^2 - r^2)}{4\mu}$

average speed: $\quad v_0 = \dfrac{1}{\pi a^2} \int_0^{2\pi} \int_0^{a} u(r)\, r\, dr\, d\phi = \dfrac{Qa^3}{8\mu}$ $\qquad L_0 = 2a$ – diameter of the pipe

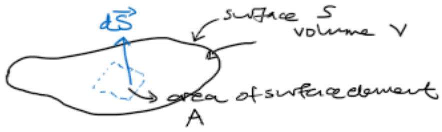
$$Re = \frac{Q d_0^3 \rho_0}{32\mu}$$ for water flow $\rightarrow$ laminar if $Re < 3200$

what about large Re?. Need equation set. –
for non-steady, non-laminar flows

NAVIER STOKES EQUATIONS.

we assume a fluid is a continuum $\rightarrow$ continuum hypothesis followed
$\rightarrow$ length scales of fluid $\quad L \gg \lambda_{mfp} \rightarrow$ mean free path
$\rightarrow$ average distance between collisions

Derivation of continuity equation:
$\rho(\vec{r}, t)$ – density



$d\vec{S} = A\hat{n}$ $\searrow$ outward normal unit vector

total outflow of mass = change of mass in volume V

$$\frac{\partial}{\partial t}\int \rho\, dV = -\oint_S \rho\, \vec{v} \cdot d\vec{S}$$ for a stationary surface & divergence theorem

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$ for any volume V
$\vec{v}$ = velocity field vector

$\Leftrightarrow$ conservation of mass

we want to apply $\vec{F} = m\vec{a}$ to a volume element
but $\vec{a} = \dfrac{D\vec{v}}{Dt}$ where $\dfrac{D}{Dt}$ is the total/advective derivative

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}$$

considering all forces:
1) Gravity: $\vec{F}_g = -mg\hat{z} = -\rho g \Delta x \Delta y \Delta z\ \hat{z}$
2) Pressure: $(\vec{F}_p)_x = -\dfrac{\partial p}{\partial x} \Delta x \Delta y \Delta z\ \hat{x}$
3) Viscous: arising from stress

$$\vec{F}_v = \left[\mu\nabla^2\vec{v} + \frac{\mu}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right]\Delta x\Delta y\Delta z$$

combine to give:

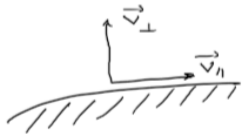
$$\rho\frac{\partial\vec{v}}{\partial t} + \underbrace{\rho(\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{inertial term}} = -\vec{\nabla}p + \mu\nabla^2\vec{v} + \frac{\mu}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \rho g\hat{z}$$

NAVIER STOKES EQUATIONS

assume element moving with fluid is adiabatic = heat input to system ignored

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$ where γ = ratio of specific heats.

boundary condition:



fluid in contact with a solid

$\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$

no flow through solid $\vec{v}_\perp = \vec{0}$

no slip condition: $\vec{v}_\parallel = \vec{0}$

if solid is moving $\vec{v}_b$

then $\vec{v} = \vec{v}_b$

incompressibility $\Rightarrow$ ρ constant so no need for continuity equation

& N.S. $\Rightarrow$ $\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}p + \mu\nabla^2\vec{v} - \rho g\hat{z}$

will only use this incompressible version of N.S. equation.

then Euler's equation:

$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}p - \rho g\hat{z}$$ where $\mu = 0$ $\Longleftrightarrow$ inviscid.

sometimes gravity is also ignored

i.e. $-\rho g\hat{z} = \vec{0}$

steady $\Rightarrow$ $\frac{\partial\vec{v}}{\partial t} = \vec{0}$ but $\frac{D\vec{v}}{Dt} \neq \vec{0}$ since advective term

note: $\frac{d\vec{v}}{dt} = \frac{D\vec{v}}{Dt}$ in notes.

incompressibility

$\Longleftrightarrow$ $\frac{\partial\rho}{\partial t}$ & from continuity equation

$\Rightarrow$ $\vec{\nabla}\cdot\vec{v} = 0$.

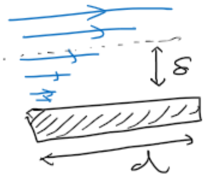
Reynolds number is the relative size of inertial term to viscous term in N.S.

1) $Re \gg 1$ assume $\mu = 0$

2) $Re \ll 1$ $\rho \frac{\partial \vec{v}}{\partial t} = -\nabla \vec{p} + \mu \nabla^2 \vec{v} - \rho g \hat{z}$

but must be careful near a solid boundary where $\vec{v} = \vec{0}$

$$\frac{\delta}{d} \sim \frac{1}{\sqrt{Re}}$$



but outside boundary layer $\mu = 0$ is a good model if $Re \gg 1$

streamline = a line which is everywhere tangent to the instantaneous velocity field $\vec{v}(\vec{r}, t)$

for incompressible, steady, non-viscous, inviscid flow, along a streamline

$$\rho \frac{v^2}{2} + P + \rho g z = \text{constant}$$

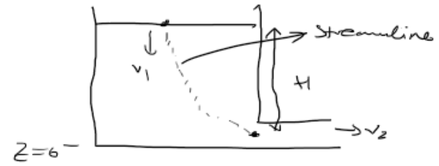
↳ some height above a reference height

from this we derive...

Archimedes Principle

net upward force = mass of water displaced $\times g$ — for a body submerged in a static fluid

water draining from a tank



assume $v_2 \gg v_1$

$P_1 = P_2 = P$ ↳ atmospheric pressure

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g H = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow v_2 = \sqrt{2gH}$$

- example of pitot tube in notes
- shallow water waves & aerofoil

} all Bernoulli equations

Inviscid N.S. equations (Euler equations)

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P \quad ; \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad ; \quad \frac{d}{dt}\left(\frac{P}{\rho^{\gamma}}\right) = 0$$

where assumed $\mu = 0$ (inviscid)

$\vec{g} = \vec{0}$ (ignored gravity)

adiabatic

$$\frac{d}{dt} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\nabla}\cdot\vec{v}$$

take a stationary, uniform background fluid

$$\rho = \rho_0, \ P = P_0, \ \vec{v} = \vec{0}$$

& apply a perturbation: $\rho = \rho_0 + \rho_1$; $P = P_0 + P_1$; $\vec{v} = \vec{v}_1$

with $\rho_1 \ll \rho_0$, $P_1 \ll P_0$, $|\vec{v}_1|$ small

put into continuity equation & linearise (ignore products of 'small' terms)
& some more substitutions we get

wave equation

$$\frac{\partial^2 v_{x}}{\partial t^2} - \frac{\gamma P_0}{\rho_0}\frac{\partial^2 v_x}{\partial x^2} = 0 \quad \text{where} \quad c_s = \sqrt{\frac{\gamma P_0}{\rho_0}} \ \text{soundspeed.}$$

i) where we assumed $\vec{\nabla}\cdot\vec{v} \neq 0$ (compressible)
ii) $|\vec{v}_1| \ll c_s$

irrotational $\Rightarrow \vec{\nabla}\times\vec{v} = \vec{\omega} = \vec{0}$

vorticity: $\vec{\omega} = \vec{\nabla}\times\vec{v}$ cross product

circulation around any closed loop Γ:

$$K = \oint_\Gamma \vec{v}\cdot\vec{dl} \quad \text{apply Stokes Theorem}$$

$$= \int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{S} \quad \text{if irrotational } K = 0.$$

Kelvin circulation theorem: circulation around any closed loop Γ moving with fluid is constant for inviscid flow

$$\frac{dK}{dt} = \frac{d}{dt}\left[\oint_\Gamma \vec{v}\cdot\vec{dl}\right] = 0$$

note: $\vec{\nabla}\cdot\vec{v}$ & $\vec{\nabla}\times\vec{v}$ different for cylindrical coordinates.

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & r v_\theta & v_z \end{vmatrix}$$

Proof of Kelvin's circulation theorem in notes.

for irrotational flow: $\vec{\nabla}\times\vec{v} = \vec{0}$

we can define $\underbrace{\vec{v} = \vec{\nabla}\phi}_{\text{potential flow.}}$ since $\vec{\nabla}\times(\vec{\nabla}\phi) = \vec{0} \quad \forall \phi$

& if incompressible we get $\nabla^2\phi = 0$ Laplace equation

examples: 1) Uniform flow 2) Point source in notes

for 2D incompressible flow $\vec{\nabla}\cdot\vec{v} = 0$ or. $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

define ψ s.t. $v_x = \frac{\partial \psi}{\partial y}$; $v_y = -\frac{\partial \psi}{\partial x}$

$\therefore \vec{\nabla}\cdot\vec{v} = \frac{\partial^2\psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

if also irrotational $\vec{\nabla}\times\vec{v} = \vec{0}$

then $\nabla^2\psi = 0$ Laplace equation $\psi :=$ streamfunction

where ψ is constant are streamlines

take Bernoulli equation:

$$P + \frac{1}{2}\rho v^2 + \rho g z = \text{constant}$$

for: · steady flow · incompressible · inviscid

& take N.S. without viscosity

$$\rho\frac{d\vec{v}}{dt} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P - \rho g\hat{z}$$

assuming incompressiblity, & irrotational $\Rightarrow \vec{v} = \vec{\nabla}\phi$

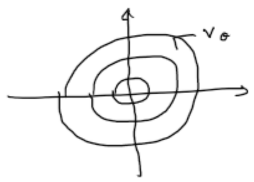
we get a generalised Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \frac{P}{\rho} + gz = \text{constant}$$

where we are not restricted to:

- steady flow
- flow along streamlines

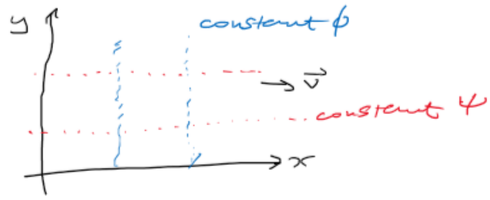
free potential vortex: flow with circular paths around an axis & $\vec{\nabla} \times \vec{v} = \vec{0}$ away from axis



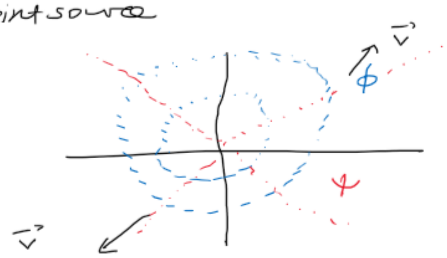
$\Rightarrow v_\theta = \frac{K}{2\pi r}$ K - constant

- can build complicated flows from a superposition of more simple flows

1) Uniform flow: $\vec{v} = v\hat{x}$, $\phi = vx$; $\psi = vy$



2) Point source

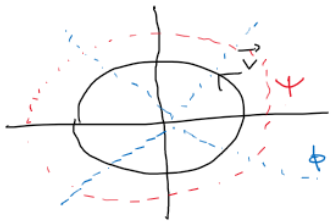


$\vec{v} = \frac{q}{2\pi r}\hat{r}$ $\phi = \frac{q}{2\pi}\ln(r)$

$\psi = \frac{q\theta}{2\pi}$

3) Potential vortex $\quad \vec{v} = \frac{K}{2\pi r}\hat{\theta} \quad\quad \phi = \frac{K\theta}{2\pi}$

$$\psi = -\frac{K}{2\pi}\ln(r)$$



4) Potential dipole flow (in notes) $\quad\quad \vec{\nabla}\cdot\vec{v} = 0$

$$\vec{\nabla}\times\vec{v} = \vec{0}$$

$$\phi = \frac{a\cos\theta}{r} \quad\quad \psi = -\frac{a\sin\theta}{r}$$

$$v_r = -\frac{a\cos\theta}{r^2} \quad\quad v_\theta = -\frac{a\sin\theta}{r^2}$$

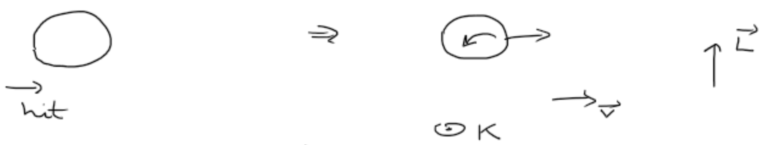
& satisfy $\nabla^2\phi = 0$ & $\nabla^2\psi = 0$

then in your notes is just application of the tools we have just stated.

Magnus effect: force (lift) per unit length on any 2D body in relative motion in a fluid with velocity $\vec{v}_0$ & circulation $\vec{K}$

$$\vec{L} = \rho\, \vec{v}_0 \times \vec{K}$$

↳ velocity of fluid in the object rest frame



- example of airflow around a rotating cylinder in notes → Using Kelvin's circulation theorem
- Similarly done for two interacting vortex lines → Magnus effect.
- potential flow over a wing