PX267 Hamiltonian Mechanics

- Action: $$A = \int_{path} \mathcal{L}\, dt$$ (functional)

↳ Hamilton's principle of least action:
Of all possible paths a dynamical system may take, the actual path is that which minimizes the action.

- Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$$

$\mathcal{L}(q, \dot{q}, t)$, variation $\eta(t)$ such that $\eta(t_0) = \eta(t_1) = 0 \Rightarrow \mathcal{L}(q+\alpha\eta, \dot{q}+\alpha\dot{\eta}, t)$. To minimize A, we require

$$0 = \left.\frac{dA}{d\alpha}\right|_{\alpha=0} = \int_{t_0}^{t_1} \frac{d}{d\alpha}\mathcal{L}(q+\alpha\eta, \dot{q}+\alpha\dot{\eta}, t)\Big|_{\alpha=0} dt = \int_{t_0}^{t_1}\left(\frac{\partial \mathcal{L}}{\partial q}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}(t)\right) dt$$

$$= \cancel{\eta(t)\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big|_{t_0}^{t_1}} + \int_{t_0}^{t_1} \eta(t) \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right)}_{=0} dt \qquad \forall \eta(t) \in C_0^2[t_0, t_1]$$

↳ canonical momentum: $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ ↳ canonical force: $F_i = \frac{\partial \mathcal{L}}{\partial q_i}$ $\boxed{\text{E-L} \Rightarrow \text{N2}}$

- Hamitonian: $$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \qquad \left(H = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}\right)$$

↳ Hamilton's equations: $$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

↳ If H does not depend explicitly on time, $\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i\left(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i\right) = 0 + \sum_i(\dot{q}_i\dot{p}_i - \dot{p}_i\dot{q}_i) = 0$

- Noether's theorem:
If a symmetry in the Lagrangian exists, there is a corresponding constant of the motion. (conserved quantity)

↳ Defn: Conserved quantities (integrals of motion) are functions of q_i and $\dot{q}_i$ that are constant. (for a Lagrangian $\mathcal{L}(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t)$)

↳ With n general coordinates, we can have (up to) 2n+1 conserved quantities:
- n linear momenta (homogeneous space)
- n angular momenta (isotropic space)
- energy (homogeneous time)

↳ The canonical momenta p_i that don't have their conjugate coordinates q_i appearing explicitly in the Lagrangian $\mathcal{L}$ are conserved.
(i.e. $\mathcal{L} \neq \mathcal{L}(q_i) \Rightarrow \frac{dp_i}{dt} = 0$)

No.

Date

Normal mode theory

- Inertia matrix: (mass) $M_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}$ $\left(T = \frac{1}{2} \sum_{i,j} \dot{q}_i M_{ij} \dot{q}_j = \frac{1}{2} \dot{\underline{q}} \cdot \underline{\underline{M}} \, \dot{\underline{q}} \right)$

  ↳ $\underline{\underline{M}}$ is symmetric: $M_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i} = M_{ji}$

  ↳ If V (in $\mathcal{L}$) is independent of velocities, $p_i = \sum_j M_{ij} \dot{q}_j$

  $$p_i = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left(\frac{1}{2} \sum_{j,k} \dot{q}_j M_{jk} \dot{q}_k \right) = \frac{1}{2} \sum_{j,k} M_{jk} \left(\frac{\partial \dot{q}_j}{\partial \dot{q}_i} \dot{q}_k + \frac{\partial \dot{q}_k}{\partial \dot{q}_i} \dot{q}_j \right) = \frac{1}{2} \sum_{j,k} M_{jk} (\delta_{ij} \dot{q}_k + \delta_{ik} \dot{q}_j)$$

  $$= \frac{1}{2} \sum_{j,k} (M_{ik} \dot{q}_k + M_{ji} \dot{q}_j) = \sum_j M_{ij} \dot{q}_j \qquad (M_{ij} = M_{ji}) \qquad \square$$

- Stiffness matrix: (force) $K_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{0 \,\leftarrow\, \text{equilibrium.}}$ $\left(F_i = - \sum_j K_{ij} q_j \right)$

  ↳ $\underline{\underline{K}}$ is also symmetric.

- Equations of motion for small oscillations: $\underline{\underline{M}} \, \ddot{\underline{x}} = - \underline{\underline{K}} \, \underline{x}$ ($\underline{\underline{M}}$ constant, ~ Hooke's)

  ↳ General solution: $\underline{x}(t) = \sum_\lambda A^{(\lambda)} \underline{X}^{(\lambda)} e^{i\omega^{(\lambda)} t}$

  - λ = index for different normal modes ← patterns of motion w/ same frequency.
  - $A^{(\lambda)}$ = amplitude
  - $\omega^{(\lambda)}$ = frequency (ω^2 = eigenvalue)
  - $\underline{X}^{(\lambda)}$ = "direction of polarisation" ($\underline{X}$ = eigenvector)

  $\omega^2 \underline{\underline{M}} \, \underline{X} = \underline{\underline{K}} \, \underline{X} \leftarrow (|\underline{\underline{K}} - \omega^2 \underline{\underline{M}}| = 0)$

  ↳ General motion of a system is a superposition of (at most n) normal modes, each with a corresponding amplitude $A^{(\lambda)}$.