PX145: Physics Foundations

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Disclaimer

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1 Introduction

PX145 is the first physics module taught at Warwick University and is meant to gently bridge the gap between A level difficulty maths and University level. Along the way ideas such as complex number notation of wave functions, and the differential form of simple harmonic motion are discussed, but for now lets start at the very best place to start... the beginning.

1.1 Module Topics

This module is split into three main sections, the first being Dimensional Analysis. This simple tool can be used to postulate a relationship between variables and arrive at a formula which can be tested and verifed experimentally. Secondly, this module covers the basic principles of Simple Harmonic Motion; starting from the very basics and ending with various types of damping. Finally, the module ends with an introduction to waves and outlines the basic ways of describing various types of wave function in both trigonometric and imaginary maths notation. Imaginary numbers are introduced (though rigorous coverage of these is included in MA108 Mathematics for Scientists) and beat patterns are discussed.

1.2 Why these topics for foundations?

This may seem like a quite random assortment of topics to include in a foundations module, however, these topics form a basis for most of physics. First year physics can be divided into two main sections; that concerned with point particles and that concerned with fields.
The difference between the mathematics included in the two lies in the number of variables required for the functions. Physics of particles generally requires functions of only a single variable, such as velocity being position as a function of time. Field theories require more than one variable input per function, often even with functions of entire vectors of variables.
Examples of 1^{st} year modules that are based primarily with dealing with point like particles are :

- Mechanics - Particles with velocities $\ll$ speed of light
- Relativity - Particles with velocities $\simeq$ speed of light
- Particle Physics

Examples of 1^{st} year modules that are based primarily with dealing with fields are:

- Electricity and Magnetism
- Waves

As you notice some 1^{st}year modules aren't explicitly characterised by either such as Astrophysics or Thermal Physics; and some begin to find links between the two areas of physics, such as Quantum Phenomena. On the whole, neither is more or less difficult or important than the other, and inevitibly both are required to fully understand the physics description of the world we live in.

1.3 Dimensions and Units

Firstly, a distinct clarification between two terms often confused at A level is required. Dimensional analysis is the first topic in this module and a definition must be laid down of the difference between Dimensions and Units. The dimension of a quantity is formed by the structure of physics and is the same regardless of which system of measurement we use. Units, on the other hand, are chosen by the measuring equipment used, and reflect the quantitive answer, the actual numerical value. This is easier shown with a demonstration.

To help illustrate this point, consider for a moment the quantity known as Length. The dimensions of this unit would be exactly that, Length, however the units for this quantity depend on who you are talking to! The standard measurement for length in the UK is metres, but imperial measures such as miles, furlongs, inches, and feet are still all regularly used. Units are chosen by the measurer whereas the dimensions are the same regardless.
All of the quantities measured in physics can be divided into a list of a few fundamental ones, and a set of derived quantities. It is important to realise that this division is not unique and is purely chosen by physicists, whilst there is a common list (that shown below) some physicists prefer to use a different set. The most common examples of this is that whilst in the list below Current is chosen as the fundamental quantity, and Charge is shown as $Current \times Time$, often this is the other way around (Charge as the fundamental quantity and Current as $Charge \div Time$.

Fundamental Quantity	**Symbol for Dimension**	**Standard Units (S.I.)**
Length	L	m (metres)
Mass	M	kg (kilograms)
Time	T	s (seconds)
Current	I	A (Amps)
Temperature	θ	K (Kelvin)
Amount of Substance	mol	mol(Mole)

Derived Quantity	**Symbol for Dimension**	**Standard Units**
Area	L^2	m^2
Volume	L^3	m^3
Velocity	LT^{-1}	ms^{-1}
Acceleration	LT^{-2}	ms^{-2}
Density	ML^{-3}	kgm^{-3}
Force	MLT^{-2}	$kgms^{-2}$ or N (Newtons)
Work / Energy	ML^2T^{-2}	kgm^2s^{-2} or J (Joules) or Nm
Charge	IT	As or C (Columbs)

You may ask the question as to why these are the units which are usually chosen. Well the answer is simple; physicists are lazy. The choice of units for quantities is generally that in which the fundamental laws of physics are the easiest to write down in. Units such as the furlong and a temperature scale in Farenheit, lead to unsightly constants of proportionality, whereas written in SI units, these all turn out to be 1.

2 Dimensional Analysis

2.1 Notation

Throughout this revision guide I will use the standard notation used during the lectures to denote the dimensions of a derived quantity.

$$[\alpha] = \text{dimensions of the derived quantity } \alpha$$

2.2 Calculating Derived Dimensions

To calculate the dimensions for a derived quantity the following recipe can be used:

1. Consider a formula linking the quantity in question to other fundamental quantities. Sometimes this is not immediately apparent and an intermediate forumla linking it to other derived quantities which you can then work out secondary forumlae for is required. (If this sounds confusing, see the example below and it will become more clear!)
2. Write each of the fundamental quantities as dimensions
3. Ensure all powers are collected and fractions are cancelled to give the neatest possible solution
4. Thats it! Your result is the dimensions of your derived quantity.

Worked Example 2.2.1

Here we take the derived quantity Pressure and calculate its units.

1. Pressure can be written as
$$Pressure = \frac{Force}{Area}$$
and since
$$Force = Mass \times Acceleration$$
$$Acceleration = \frac{Speed}{Time}$$
$$Speed = \frac{Distance}{Time}$$
$$\Rightarrow Pressure = \frac{Mass \times Distance}{Time^2 \times Area}$$
2. These fundamental quantities can be re-written as their dimensions:
$$Pressure = \frac{Mass \times Distance}{Time^2 \times Area} = \frac{ML}{T^2L^2}$$
3. Tidying this up we end up with
$$\frac{M}{T^2L}$$
4. Therefore the solution is $[Pressure] = MT^{-2}L^{-1}$

2.3 Buckingham Π Theorem

Being able to calculate the dimensions of a derived quantity is all well and good, but the real dimensional analysis lies in the Buckingham PI Theorem.

The theorem states that any physical law can be written in the form $q_1 = g(q_2, q_3, ..., q_N)$ or$f(q_1, q_2, ..., q_N) = 0$ where f and g are some given functions. This can be rearranged and expressed in terms of **dimensionless combinations of the original variables**, Π_i. This would give rise to the following:

$$\Pi_1 = \tilde{g}(\Pi_2, \Pi_3, ..., \Pi_{\tilde{N}}) \; or \; \tilde{f}(\Pi_1, \Pi_2, ..., \Pi_{\tilde{N}}) = 0^1$$

The second of these two forms turns out to be the most useful; especially when $\tilde{N}$ happens to equal 1. This implies that $\Pi_1 = a\, constant$ and this is the case we shall be looking at for most of the rest of this section.

2.4 The Recipe

Long and short, dimensional analysis is a tool. It can be written down as a recipe to be followed, and once you know what you are doing, can lead to some nice easy marks. Below is an outline of the recipe and then a brief example showing how it can be put into practice.

1. Postulate some law or relation and figure out the different variables likely to be involved.
2. Form dimensionless combinations of these variables (ie, the Π_1 to $\Pi_{\tilde{N}}$ mentioned above).
3. Solve the resulting simulatenous equations to find out the value of the powers.
4. If only 1 dimensionless variable exsists it must be equal to a constant.

This is the basis of dimensional analysis and becomes much clearer after a few questions have been attempted. These should be straightforward easy marks in the exam, just memorise the above steps.

Worked Example 2.4.1 Suppose we want to find the distance waves travel along a tight string. The variables likely to influence this property are the wave velocity, v, the tension of the string, F, and the mass per unit length of the string, ρ. We require

$$f(v, F, \rho) = 0$$

We want a dimensionless quantity Π with $\Pi = v^\alpha F^\beta \rho^\gamma$. The dimensions of each of the variables are as follows:

$$[v] = LT^{-1} \qquad [F] = MLT^{-2} \qquad [\rho] = ML^-1$$

If Π is to be dimensionless then we require that $L^\alpha T^{-\alpha} M^\beta L^\beta T^{-2\beta} M^\gamma T^{-\gamma}$ is also dimensionless. This gives us the following 3 simultaneous equations:

$$\beta + \gamma = 0$$

$$-\alpha - 2\beta = 0$$

$$\alpha + \beta - \gamma = 0$$

We can choose $\alpha = 1$ giving the solution $\beta = -\frac{1}{2}$ and $\gamma = \frac{1}{2}$. Therefore our dimensionless variable $\Pi = vF^{-\frac{1}{2}}\rho^{\frac{1}{2}}$.

[1] It should be noted that $\tilde{N}$ must be less than or equal to N as you cannot have more dimensionless combinations than variables that you have started with.

3 Simple Harmonic Motion

3.1 Introduction

Most A level courses (if not all) cover simple harmonic motion. This module is mostly revision of these basic A level concepts with a small amount of new material towards the end. Simple harmonic motion crops up in all the most unlikely places in physics and the equations and concepts around it are vital to understand some complex physical systems. The end of this chapter starts on the complex representation of the simple harmonic motion equations, and I would advise spending some time ensuring that you are confident with this material, as it can often simplify problems (ironically!) and save laborious trigonometric identity work.

3.2 The Problem

The simple harmonic oscillator equation can be physically thought of as finding the solution to the system of equations governing the motion of a mass attached to a string or oscillating spring (as shown below). As you will remember from A Level, the equation relating the force applied from the spring to the object is given by Hooke's Law:

$$F = -kx$$

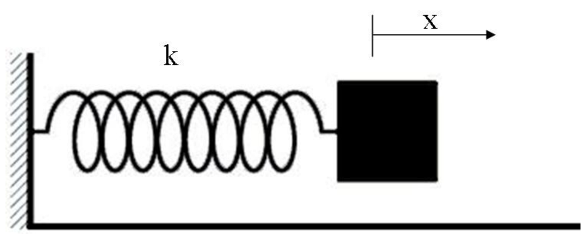

Figure 1: An example of a simple harmonic oscillator

Where F = force applied, k = spring constant and x = displacement from central equilibrium and m = the mass of the particle. Also we know from Newton's second law that:

$$F = ma = m\frac{d^2x}{dt^2}$$

and by equating these two we can arrive at the formula for undamped simple harmonic motion:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

We can make the following definition: $\frac{m}{k} = \omega^2$ giving the final equation of

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

To find a solution we can start by trying a sinusoidal function in the following format:

$$x(t) = A\cos(\alpha t + \phi)$$

$$\frac{dx}{dt} = -A\alpha sin(\alpha t + \phi)$$

$$\frac{d^2x}{dt^2} = A\alpha^2 \cos{(\alpha t + \phi)}$$

It turns out that for $\alpha = \omega$ we find a solution to the SHM equation. This leaves us with two constants: A (known as the Amplitude of the oscillations) and ϕ (known as the phase angle) which are determined by the initial conditions (usually an initial position and initial velocity) of the mass. ω is called the angular frequency of the system and is linked to the time period of each oscillation by the following formula, $T = \frac{2\pi}{\omega}$.
This is not the only version of the simple harmonic oscillator problem, with another common representation being a pendulum swinging freely under gravity. To solve this problem, and convert the equations into simple harmonic form, the small angle approximation for the sine function must be applied ($\sin{(\theta)} \approx \theta$).

3.3 Energy in a Simple Harmonic Oscillator

Kinetic Energy

To calculate the kinetic energy of the moving mass at a given time the following formula can be used:

$$KE = \frac{1}{2}m(\frac{dx}{dt})^2 = \frac{1}{2}m(-A\omega\sin{(\omega t + \phi)})^2$$

Since $\omega = \sqrt{\frac{k}{m}}$, this can be rewritten as:

$$KE = \frac{1}{2}kA^2\sin^2{(\omega t + \phi)}$$

Potential Energy

The potential engery stored in the spring must be equal to the work done on the spring against the restoring force. Since work done is the integral of force $\times$ (distance from point 1 to point 2), we can reach the following result:

$$PE = \int_0^x dx'F(x') = \int_0^x dx'(^2kx') = \frac{1}{2}kx^2$$

Since we have our solution as $x = A\cos{(\omega t + \phi)}$ we have the following defintion of PE:

$$PE = \frac{1}{2}kA^2\cos^2{(\omega t + \phi)}$$

Total Energy

The total energy in the system is a combination of the kinetic energy of the mass, and the potential energy stored in the spring. A point to note is that this energy should be independent of time, as there are no energy gains or losses from this idealised system.

$$KE + PE = \frac{1}{2}kA^2(\sin^2{(\omega t + \phi)} + \cos^2{(\omega t + \phi)})$$

$$KE + PE = \frac{1}{2}kA^2$$

3.4 Damped Harmonic Oscillation

This is all lovely in an ideal world, but in most systems you will come across there is a resistance (frictional) force involved. This is quite easy to account for by the addition of a first derivative term; the figure below shows an example of such a system.

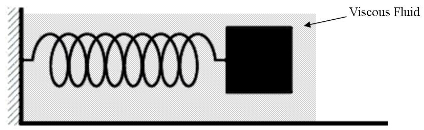

Figure 2: An example of a damped simple harmonic oscillator

This gives the new equation of motion as follows

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{where} \qquad \omega_0^2 = \frac{k}{m}$$

By going through some quite tricky algebra (which I'm going to skip out here, if you're interested try the solution $x(t) = Ae^{i\alpha t}$ and see what happens...), you reach the conclusion that the following is a solution to this equation:

$$x(t) = Ae^{i\alpha t} \qquad \text{where} \qquad \alpha = i\frac{b}{2m} \pm \omega' \qquad \text{with} \qquad \omega' = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

This quantity ω' defines what type of dampening occurs in the system. The type is determined by if the square root has 2 real, a repeated, or two imaginary roots. These cases correspond to underdamping, critical damping and overdamping respectively. Each of these cases is explained below and has a graph showing the system. Sketching one of these graphs is a favourite exam question so be sure to learn them well.

Underdamping

This occurs when $(\omega')^2$ has two real roots; ie: when $\omega_0^2 > \frac{b^2}{4m^2}$ The graph of this shows oscillations with decaying amplitude and increasingly longer periods.

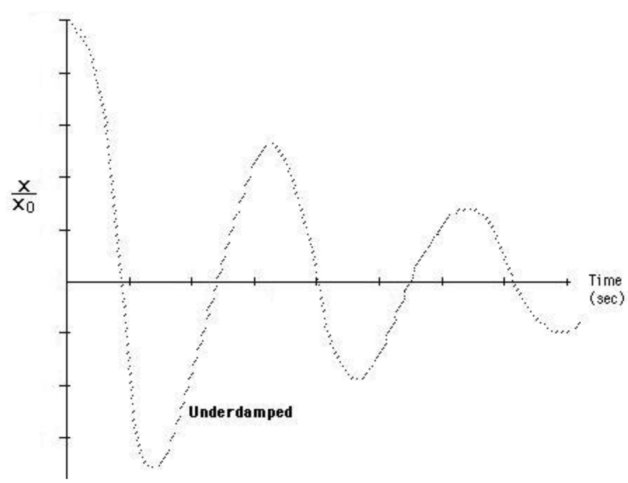

Figure 3: An example of a underdamped simple harmonic oscillator

Critical Damping

This occurs if $(\omega')^2$ has a repeated root, ie: when $\omega_0^2 = \frac{b^2}{4m^2}$. This is a special case and exactly one oscillation will occur before the system comes to a rest. The general solution to this is:

$$x(t) = Ae^{\frac{-bt}{2m}} + Bte^{\frac{-bt}{2m}} \qquad \text{where } A \text{ and } B \text{ are constants}$$

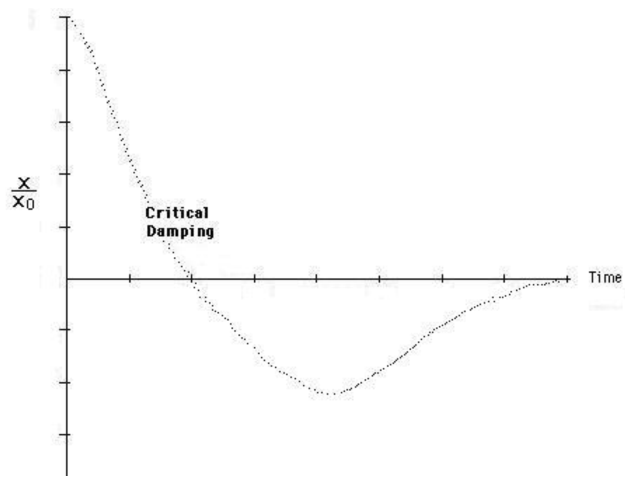

Figure 4: An example of a critically damped simple harmonic oscillator

Overdamping

This final case occurs when $(\omega')^2$ has imaginary roots. In this case the system has maximal damping and will not even complete one full oscillation. The graph shows the system slowly coming to a rest from its starting amplitude.

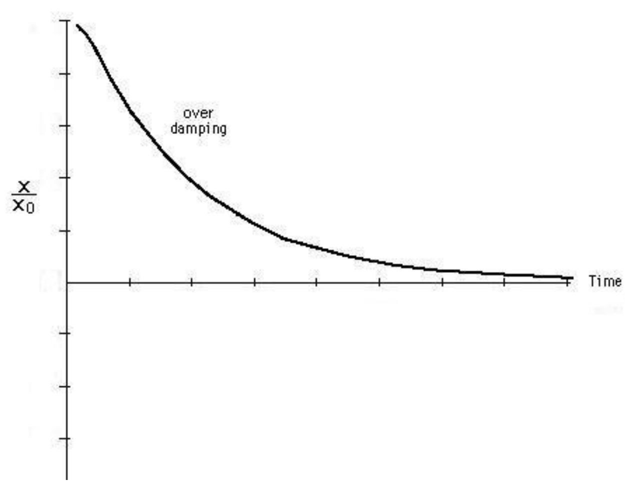

Figure 5: An example of an over damped simple harmonic oscillator

3.5 Forced Oscillations

We can force the harmonic system to oscillate by adding a forcing term on one side of the equation. The equation nows looks like:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = Fe^{i\omega t}$$ where ω is the frequency the system is being forced at.

A lot of messy algebra can be done to show that in this case the amplitude of the corresponding oscillations is directly linked to the driving frequency ω and the viscosity term b. The equation for the amplitude of the oscillator is :

$$|A| = \frac{F}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

As is clear from the graph below, there is a sharp peak in amplitude at the point where the forcing frequency ω is the same as the natural frequency ω_0 of the system, this phenomenon is called resonance. Many examples of this can be seen, such as rattles in a car at certain speeds, singers shattering glasses (in theory!) with their voice, etc... One place you will come across this later is in Electricity and Magnetism, where resonance is important in LCR circuits (more in Electricty and Magnetism).

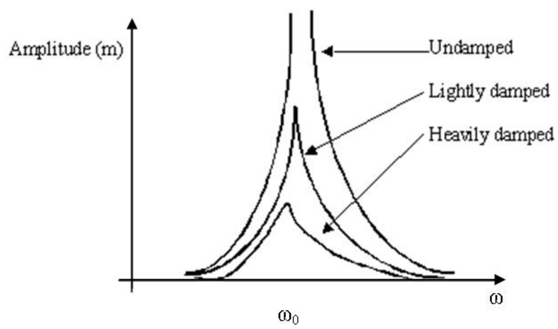

Figure 6: A graph demonstrating resonance peaks with varying amounts of dampening

4 Waves

4.1 Introduction

Waves play a major role in physics, and as you progress through your degree will become even more increasingly important. Waves come in many shapes and forms, ranging from ones you have heard of before (light, sound, water waves, waves on a tight string) to application of wave theory which you may not have (such as imaginary waves in quantum mechanics - more on this later!). In general, any wave travelling right can be described by the following function

$u(x,t) = f(x - vt) = \cos(kx - vt)$ Where x is the spatial co-ordinate and t is time.

Often we look at sinusoidal waves, as these are the easy to understand and very common.

A Few Definitions

A is the amplitude of the wave.
λ is the wavelength.
$k = \frac{2\pi}{\lambda}$ is known as the wave number. This is often more useful than the wavelength.
f is the frequency of the wave.
$T = \frac{1}{f}$ is the time period of the wave.
$\omega = f\,2\pi$ is the angular frequency of the wave.
$v = \frac{\omega}{k} = \frac{f}{\lambda}$ is the velocity of the wave.

Principle of Superposition

When two waves approach each other the displacement is the sum of their individual displacements. I.e.:

$$y_1(x,t) + y_2(x,t) = y(x,t)$$

4.2 Boundary Conditions

The easiest way to start to analysing the different boundary conditions is to use the example of waves on a tight string. There are two main types of boundary conditions possible for each end: fixed to a point (ie a wall) or free to move in space.

4.2.1 Fixed End

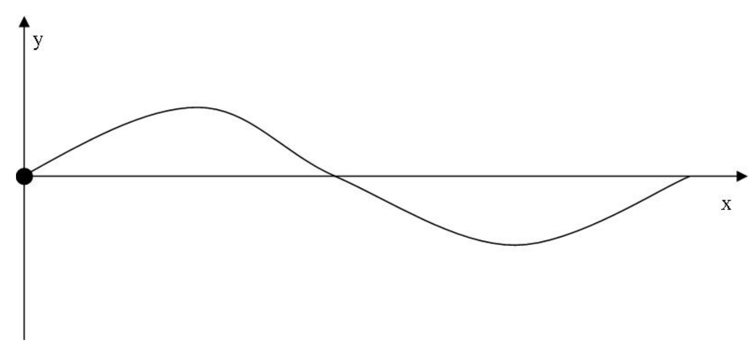

Figure 7: A graphical representation of the fixed end boundary condition

This boundary condition can be summed up by the above graph and one equation:

$$y(0,t) = 0 \qquad \forall t$$

When dealing with boundary conditions, the important thing to figure out is what happens to a wave travelling along the string when it hits the boundary. In this case since the end point is fixed, say to a wall, it cannot continue on through the wall and is reflected. By using the property of superposition and some algebraic manipulation the equation for the reflected wave form ends up as:

$$y(x,t) = -f(x+vt) = -r\cos(kx+vt) \qquad \text{Where } r \text{ is the reflection coefficient.}$$

$f(x+vt)$ is the left travelling incoming wave and $-f(-x+vt)$ is a right travelling reflection wave that is inverted. This describes a standing wave, as the pattern formed is a stationary pattern of waves. A second way of looking at this to to think of a standing wave not as a stationary wave, but as a travelling wave coming into the wall and being reflected back out.

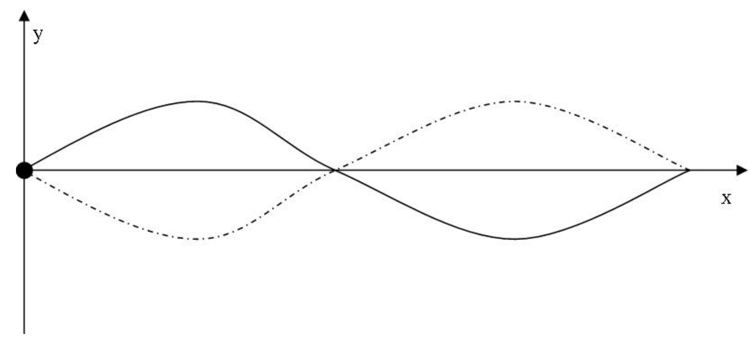

Figure 8: A standing wave from a fixed end, dotted line for reflected wave

Note:- remember when dealing with sinusodial waves (sin and cos functions), it can often be easier to rewrite them in terms of exponential functions. For the fixed end boundary condition

$$\text{Incoming Wave}^3\text{: } y_1(x,t) = e^{-i(kx+\omega t)}$$

$$\text{Reflected Wave: } y_2(x,t) = r\,e^{i(kx-\omega t)}$$

Where r is the reflection coefficient.

4.2.2 Free End

Modelling the free end condition is a little more tricky. The end of the string must be allowed to move freely in the y direction, yet have no movement in the x direction. The easiest way to think of this is as the end of the string being a ring, that is looped around a frictionless pole. This allows freedom of movement in the y direction as required, but since the pole is fixed, as is the x co-ordinate.

This condition required to fill this is similar to the one for a fixed end, except this time it's the first derivative:

$$\frac{\partial y}{\partial x} = 0 \qquad \text{at } x = 0$$

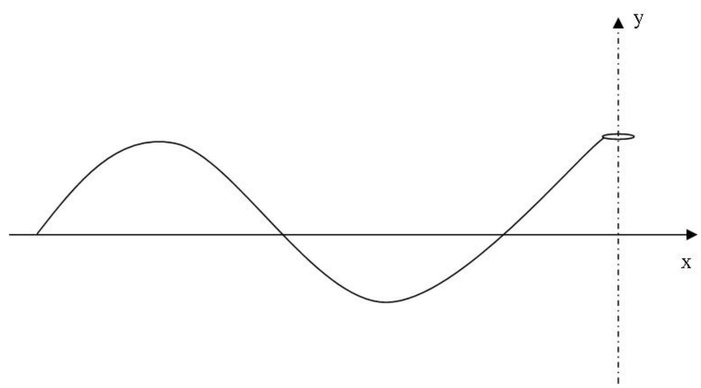

Figure 9: The free end boundary condition

Likewise with the fixed end, the wave front will get reflected when it hits the boundary (in this case the pole). Since the ring is free to move up and down the pole, as the wave front comes in, the ring gets moves up to the top (or bottom). This causes the reflected wavefront to be the same orientation as the original one, unlike in the case of the fixed end, where the reflected wave was inverted. This gives us the following equations (in exponential form) for the waves:

Incoming Wave: $y_1(x,t) = e^{i(kx-\omega t)}$

Reflected Wave: $y_2(x,t) = r\,e^{i(-kx-\omega t)}$

Where r is the reflection coefficient

Once again, this set of constraints produces a standing wave, remarkably similar to the fixed end condition.

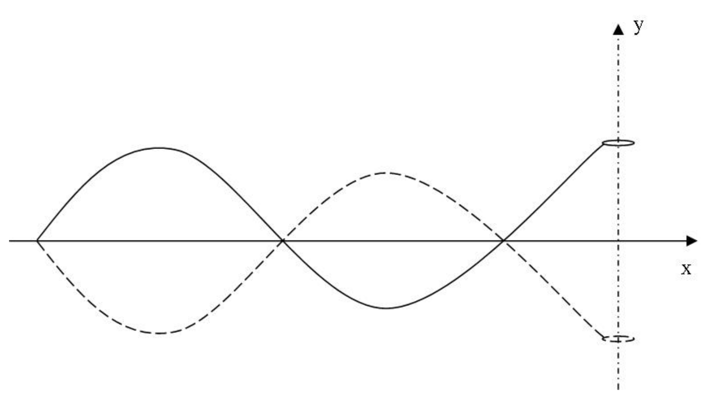

Figure 10: A standing wave from a free end, dotted line for reflected wave

4.3 Beats

The best example of what a beat is can be found from looking at sound waves produced by musical instruments. When you listen to two or more instruments playing the same note, often you can hear a slower beating sound in amongst the note. This is produced from the superposition of two waves which have very small variations in the wave number. The derivation of this is easiest in complex notation.
If we call the two different wave numbers:

$$k_1 = k + \Delta k \qquad\qquad k_2 = k - \Delta k$$

Where Δk is a small amount and k is the average wave number of the two waves we find the angular frequencies are as follows:

$$\omega_1 = \omega + \Delta\omega \qquad\qquad \omega_2 = \omega - \Delta\omega$$

The superposition of these two waves is then:

$$y = e^{i(k+\Delta k)x - i(\omega+\Delta\omega)t} + e^{i(k-\Delta k)x - i(\omega-\Delta\omega)t}$$

$$y = e^{i(kx-\omega t)}[e^{i(\Delta kx - \Delta\omega t)} + e^{-i(\Delta kx - \Delta\omega t)}]$$

By changing back into trigonometric functions[4]:

$$y = 2\cos(kx - \omega t)\cos(\Delta kx - \Delta\omega t)$$

This result shows us that the wave is comprised of two parts, called the beat wave and the carrier signal. Looking at the beat wave we can see that the velocity is as follows:

$$v_g = \frac{\Delta\omega}{\Delta k}$$

Since Δk and $\Delta\omega$ are very small (ie: $\Delta \to 0$) this is approximately equal to the following expression and this is the common definition of group velocity of a large number of waves.

$$v_g = \frac{d\omega}{dk}$$

An interesting point to note is that if the small difference in the wave numbers is very much smaller than the average wave number, then the wave length of the beat wave is very long.

$$\text{If } \Delta k << k \text{ then } \lambda_{beat} = \frac{2\pi}{\Delta k} >> \lambda$$

See figure 11 for a schematic diagram of what a wave with beats might look like.
Defintion: Waves are said to disperse if their angular frequency has a **non-linear** dependence on the wave number. If the dependance is linear (ie: $\omega = \alpha k$) then this is not classed as a dispersion relation. The relationship is as follows:

$$\omega(k) = \alpha\sqrt{k} + \beta \text{ is a dispersion relation for constants } \alpha \text{ and } \beta$$

4.4 Normal Modes

If we take the earlier example of a fixed end, and apply this idea to both ends of our string, we get a system of standing waves. By applying the conditions that :

$$x(0,t) = 0 \qquad \forall t$$

[4]Remember... Re[y] as always...

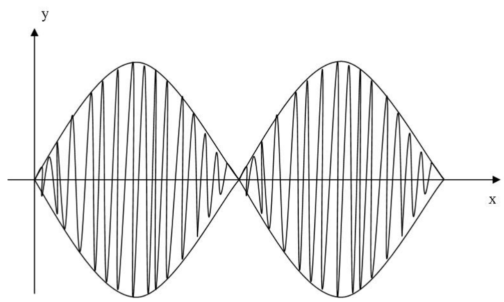

Figure 11: A schematic of a beat wave

$$x(L,t) = 0 \qquad \forall t \qquad \text{where } L = \text{ length of string}$$

A solution to these conditions is as follows:

$$y(x,t) = |A| \sin{(kx)} e^{i(\omega t + \phi)} \qquad \text{Where } \phi \text{ is a phase angle}$$

After going through the algebra it can be shown that a multiple of the wavenumber of the standing wave must fit exactly into the length of string. I.e.:

$$k_n = \frac{n\pi}{L} => \omega_n = v\frac{n\pi}{L} \qquad \text{Where } n \text{ is any integer}$$

This family of allowed solutions are called the normal modes, or in musical terms, the harmonics of the string. When $n = 1$ this is called the fundamental frequency or the first harmonic.When $n = 2$ this wave is called the second harmonic and so on...

Definition: Nodes are points which do not move at all.

Definition: Antinodes are points which have maximal amplitude.

The following diagram shows this information. Strings do not have to vibrate only in one mode.

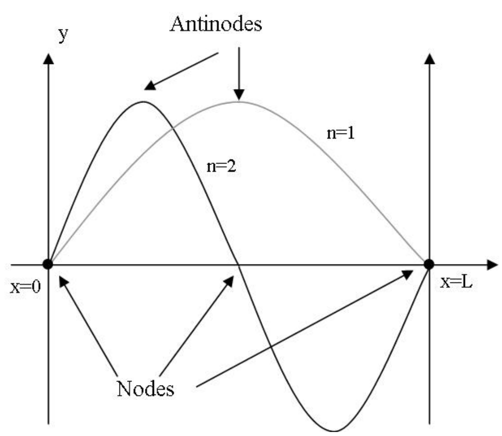

Figure 12: A diagram of normal modes and nodes

At any given time a string can be in a superposition of any number of normal modes. This can be written as:

$$y(x,t) = \sum_{n=1}^{\infty} |A_n| \sin \frac{nx}{L} e^{i(\omega_n t + \phi_n)}$$

4.5 Pipes

Okay, so if you were anything like me, you got to this point in the lectures, saw the horrendous algebra, and fell asleep! Fear not, waves in cylindrical pipes are no more complicated than on a string, in fact the concepts are exactly the same, just with the messy algebra around them.
The main concepts involve moving a section of air along the pipe. For simplicity's sake we'll call this volume element a small cylinder with a cross sectional area of A, a start point of x, an end point of Δx and a volume of V. The confusion comes when we try and work out how this has moved. The key comes in remembering this is air and can expand so we have to take into account that after a time t the air has not only moved forward but also expanded by an amount δV.
If we allow all this information about expansion and how it relates to time to be tied up nicely in a function we call $u(x,t)$ we can write the change in volume (after the algebra!) as follows:

$$\delta V = A\Delta x \frac{[u(x+\Delta x,t) - u(x,y)]}{\Delta x}$$

This information is summed up in the diagram below.

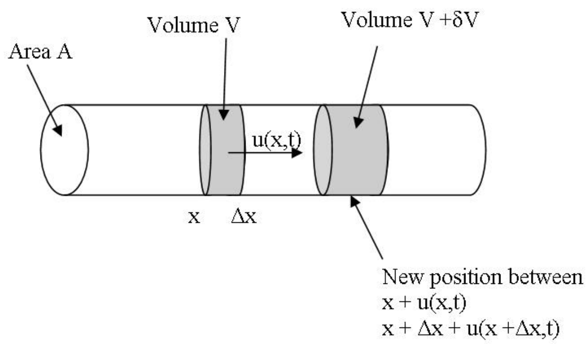

Figure 13: A basic pipe

The important results to remember from this are:

- For $x \to 0$: $\delta V = V \frac{\partial u(x,t)}{\partial x}$
- $\delta P = -B\frac{\partial u(x,t)}{\partial x}$ Where B = the Bulk Modulus of the material the wave is moving through u is the displacement from the equilibrium and P is the pressure. The velocity of the wave in the material, v, depends only on the Bulk Modulus, B, and the density in the material. Infact $v = \sqrt{\frac{B}{\rho}}$ (this can be shown by dimensional analysis, if you're bored try it, or else look it up from your lectures!).

4.5.1 Boundary Conditions in Pipes

Generally a pipe has two ends, and they can be one of two states, open or closed. These directly related to the fixed and free end of the string. A good example of both of these is a stopped pipe.

This pipe has a open end boundary condition at the top and a closed end boundary condition at the bottom. Looking at the change in pressure, we note that at the open end the pressure must be the same as the outside pressure. ie $\delta P(L,t) = 0$. By using $\delta P = -B\frac{\partial u(x,t)}{\partial x}$ we can calculate the boundary condition for this end:

$$\text{For an open end of a pipe we require that } \frac{\partial u(L,t)}{\partial x} = 0$$

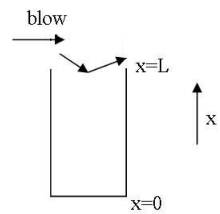

Figure 14: A stopped pipe

At the closed end there is no room for the particles to be displaced (due to the wall!) so we can write the closed end condition as:

$$u(0, t) = 0$$

These are very similar to the conditions for a tight string, and by running through the algebra it can be shown that once again standing waves are produced. We find that the displacement function looks like

$$u(x, t) = |A| \sin(kx) \cos(\omega t + \phi)$$

Where k, ω and ϕ are the wave number, angular frequency and phase angle respectively.

The diagram below shows the first and third normal modes of a closed ended pipe.

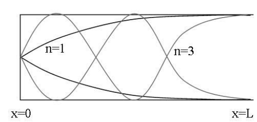

Figure 15: Normal modes of a stopped pipe

4.5.2 Organ Pipe

Organ pipes are slightly different from the stopped pipes mentioned earlier. The main difference is that they effectly have two open ends, as shown below.

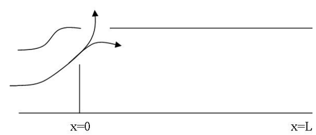

Figure 16: An organ pipe with two open ends

This change in boundary conditions changes the displacement function and we get a very similar function, but this time with a cos instead of a sin.

$$u(x,t) = |A| \cos(kx) \cos(\omega t + \phi)$$

This gives a significantly different shape to the normal modes of the waves. As both ends are now open, we have a wave that has maximum displacement (antinodes) at both ends. The first normal mode is shown below.

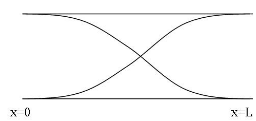

Figure 17: First normal mode of an organ pipe with two open ends

...And thats all folks!

Thats it! Everything there is to know about physics foundations. I would strongly reccommend going over the examples given in lectures, as this guide covers only the main material and misses out a lot of the algebraic derivations. This module sets the foundations for a lot of other work you will do and come across in your degree, so ensure you learn the equations and concepts thoroughly. All that leaves me to say is : Good luck for the exam!