1 Plasma Characteristics

A plasma is a state of matter in which the atoms of the matter are ionized and relatively free to move - the dominant interaction of the particles, ions and electrons, is by the macroscopic electrodynamics forces created by differences in concentrations of the ions and electrons. Usually, however, this is not satisfied perfectly - the particles still have some contact interactions when they collide with each other, not all the atoms may be ionized and the concentration of the charged particles might be too low.
We will now discover a few basic properties of plasma, and use these to figure out reasonable criteria for systems to be in a state of plasma.

1.1 Definitions

A few metrics should be defined for description of plasma - the number concentration of particles of certain species x is $n_x = \frac{dN_x}{dV}$, where dN_x is the number of particles x in an infinitesimal volume dV. So, for example, the concentration of electrons is n_e. This can be related to mass density as

$$\rho = m_x n_x$$

and to charge density

$$\rho_q = q_x n_x$$

Consider now that particle x is the ionized state of particle y (for example $y = H$, $x = H^+$). The degree of ionisation of y is then defined as

$$\alpha_y = \frac{n_x}{n_x + n_y}$$

Importantly, the degree of ionisation alone does not determine the quality of the plasma - $\alpha = 1$, typical for the Sun, and $\alpha = 0.01$, typical for Earth's ionosphere, can still represent a good plasma.
Usually, ions and electrons move at very different velocities in the plasma, and therefore we usually assign different temperature to each species, and denote it as T_x, T_y etc.
The usual property of plasma that it tries to restore the equilibrium state, in which all electrons and ions are equally spread so that there is no overall motion without external force.

1.2 Electron-Plasma Oscillations

This is the classic phenomena in plasma physics. Consider that for some reason, the equilibrium of plasma is perturbed by a slab of electrons moving away from the ions, as in Fig. 1

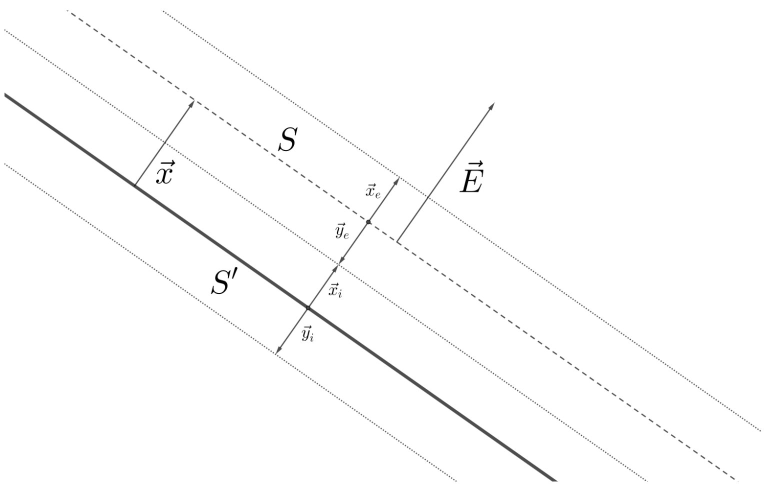

Figure 1: The full line represents a midplane of ions, the dashed line represents the midplane of electrons. The displacement vector $\vec{x}$ points towards the midplane of electrons and $\vec{E}$ is perpendicular to this plane. The dotted lines represent three different integration planes.

The slab has thickness $x = |\vec{x}_e| + |\vec{y}_e| = |\vec{x}_i| + |\vec{y}_i|$, with $|\vec{x}_e| = |\vec{x}_i| = |\vec{y}_i| = |\vec{y}_e| = \frac{x}{2}$. The total charge inside the electron slab due to the displaced electrons is

$$Q_e = -en_e S(|\vec{x}_e| + |\vec{y}_e|) = -en_e Sx$$

where S is the surface of the slab boundary. This means that the charge in the slab of the now stranded ions is

$$Q_i = en_e Sx$$

as overall, the plasma is assumed to be electrically neutral. Consider now the electron slab alone. Using Gauss's law, if we integrate over the boundary of the electron slab, we expect $\vec{E}_-$ to be symmetrical around the midplane of the slab (plane in the middle of the slab) and perpendicular to the midplane and boundaries of the slab. Therefore, the integral of the electric field over the boundary of the electron slab is

$$\iint_{S_e} \vec{E}_- \cdot d\vec{S} = 2E_- S = \frac{Q_e}{\epsilon_0}$$

Hence, we expect

$$E_- = \frac{Q_e}{2S\epsilon_0} = -\frac{en_e x}{2\epsilon_0}$$

and in between the slabs, we expect

$$\vec{E}_- = -\frac{en_e x}{2\epsilon_0}\hat{y}_e = \frac{en_e x}{2\epsilon_0}\hat{x}_i$$

where $\hat{y}_e$ is the unit vector in direction of $\vec{y}_e$. Outside the slabs, we expect

$$\vec{E}_- = -\frac{en_e x}{2\epsilon_0}\hat{x}_e = -\frac{en_e x}{2\epsilon_0}\hat{x}_i$$

Similarly, but with exchanged sign, for the ions we expect

$$\vec{E}_+ = \frac{en_e x}{2\epsilon_0}\hat{x}_i$$

in between the slabs and

$$\vec{E}_+ = \frac{en_e x}{2\epsilon_0}\hat{y}_i = -\frac{en_e x}{2\epsilon_0}\hat{x}_i$$

Therefore, the total field in between the slabs is

$$\vec{E} = \frac{en_e x}{2\epsilon_0}(\hat{x}_i + \hat{x}_i) = \frac{en_e x}{\epsilon_0}\hat{x}_i$$

The field outside the slabs in the direction of $\hat{y}_i$ is

$$\vec{E} = \frac{en_e x}{2\epsilon_0}\hat{y}_i - \frac{en_e x}{2\epsilon_0}\hat{y}_e = 0$$

and similarly in the direction of $\hat{x}_e$ outside the slabs, field also goes to zero. The force on an electron in between the slabs is then

$$\vec{F} = m\vec{a} = -e\vec{E}$$

where m is the mass of the electron and $\vec{a}$ is the acceleration of the electron. Usually, electrons are much lighter than the ions, and therefore we can neglect the motion of the ions with respect to electrons, i.e.

$$\vec{a} = \frac{d^2\vec{x}}{dt^2}$$

as the separation is only due to the motion of the electrons. Hence

$$\frac{d^2\vec{x}}{dt^2} = -e\frac{en_e}{m\epsilon_0}x\hat{x}_i = -\frac{e^2 n_e}{m\epsilon_0}\vec{x}$$

which is an equation of simple harmonic motion with frequency

$$\omega_{ep} = \sqrt{\frac{e^2 n_e}{m\epsilon_0}} \tag{1}$$

This is called the electron-plasma frequency, and these oscillations are called the electron plasma oscillations. They can be viewed as effort of plasma to balance equilibrium but overshooting the equilibrium position due to the electrons' non-zero inertia.

1.3 Debye Screening

While electron-plasma oscillations correspond to dynamic perturbations to local charge density of the plasma, Debye screening is an electrostatic phenomenon. Consider that we introduce some extra static charge q into the plasma. This creates electrostatic potential in the plasma ϕ. The electrons and ions than move in order to minimize their potential energy. The situation is described by the Poisson equation

$$\nabla^2 \phi = -\frac{\rho_q}{\epsilon_0}$$

We will describe a simple 1D model, in which we assume that $\rho_q = e(n_i - n_e)$ and try to describe the potential far away from the inserted charge q. At non-zero temperature, the excess concentration of electrons and ions around the inserted charge will correspond to Boltzmann distribution as

$$n_e = n_0 e^{-\frac{(-e\phi)}{k_B T_e}} = n_0 e^{\frac{e\phi}{k_B T_e}}$$

where n_0 is the equilibrium concentration of electrons. Similarly,

$$n_i = n_0 e^{-\frac{e\phi}{k_B T_i}}$$

Therefore

$$\rho_q = e n_0 \left(e^{\frac{-e\phi}{k_B T_i}} - e^{\frac{e\phi}{k_B T_e}} \right)$$

Phenomenologically, we expect that if the charge is presented into the plasma, other charges of opposite signs will gather around it, decreasing the overall charge enclosed in the volume around the initial charge. Therefore, we expect the excess concentration of electrons and ions to decrease as we move further away from the charge, until it reaches the equilibrium value. Therefore, we expect $\phi \to 0$ in great distances from the inserted charge. In these distances, we can approximate

$$\rho_q = e n_0 \left(1 - \frac{e\phi}{k_B T_i} - \left(1 + \frac{e\phi}{k_B T_e} \right) \right) = -e n_0 \frac{e\phi}{k_B} \left(\frac{1}{T_i} + \frac{1}{T_e} \right)$$

which is linear in ϕ. Therefore, Laplace's equation linearizes to (in 1D)

$$\frac{d^2}{dx^2}\phi = \frac{e^2 n_0}{k_B \epsilon_0} \left(\frac{1}{T_i} + \frac{1}{T_e} \right) \phi = k^2 \phi$$

Which can be solved by

$$\phi = A e^{kx} + B e^{-kx}$$

Given our boundary conditions that $\phi(x \to \infty) \to 0$, we have to set $A = 0$, and thus we have

$$\phi \propto e^{-kx} = e^{-\frac{x}{\lambda_D}}$$

where the decay length λ_D is the Debye decay length.

$$\lambda_D = \sqrt{\frac{k_B \epsilon_0}{e^2 n_0} \left(\frac{T_e T_i}{T_e + T_i} \right)} \tag{2}$$

For the case we discussed before, when the motion of the ions is negligable, we would have concluded that the ions stay fixed and do not change number density, i.e. $n_i \to n_0$, which happens for $T_i \to \infty$. And we can see that for this case $\lambda_D(T_i \to \infty)$ becomes

$$\lambda_D = \sqrt{\frac{k_B \epsilon_0 T_e}{e^2 n_0}}$$

Usually, we would write $n_e = n_0$, as we usually mean the equilibrium concentration of electrons when talking about plasma properties.

1.4 Plasma Criteria

In order for plasma to behave like plasma, the electron-plasma oscillations and Debye screening must look like material properties, not like dynamic processes of the entire system. This means that we require the system to be much larger than the Debye screening length λ_D.

$$L \gg \lambda_D$$

Furthermore, we require that the concentration of electrons and ions is sufficient to behave like plasma. This usually means that we need Debye screening to effectively take place, which can only happen if the continuum approximation for the potential ϕ is valid, which occurs when the number of charges around the inserted charge is big. The number of charges around the inserted charge is called the Debye number and can be found as number of charges in sphere of radius equal to Debye length

$$N_D = \frac{4}{3}\pi\lambda_D^3 n_e = \frac{4}{3}\pi\frac{(\epsilon_0 k_B T_e)^{3/2}}{e^3\sqrt{n_e}}$$

This is also called the plasma parameter, and we require $N_D \gg 1$.
Lastly, we require that the electrons are pretty much free to move. This corresponds to case when the mean time in bewteen collisions τ is much bigger than a time of one cycle of electron-plasma oscillations, i.e.

$$\tau \gg \frac{2\pi}{\omega_{ep}} = 2\pi\sqrt{\frac{\epsilon_0 m}{e^2 n_e}}$$

2 Plasma Dynamics

This section explores further how plasma behaves when different forces act on it.

2.1 External Magnetic Field

When an external static magnetic field $\vec{B}_0$ is present, the equation of motion of the electrons in plasma becomes

$$m\frac{d\vec{v}}{dt} = -e\vec{v} \times \vec{B}_0$$

where $\vec{v}$ is the velocity of the electron.
By taking a dot product with $\vec{v}$, we can show that

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{m}{2}\frac{d(\vec{v}\cdot\vec{v})}{dt} = \frac{m}{2}\frac{d(|\vec{v}|^2)}{dt} = -e\vec{v}\cdot(\vec{v}\times\vec{B}_0) = 0$$

Therefore, the magnitude of $\vec{v}$ does not change, only its direction. By separating $\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$ to components parallel and perpendicular to $\vec{B}_0$, we have (as these two vectors are perpendicular to each other)

$$m\frac{d\vec{v}_{||}}{dt} = -e(\vec{v}_{||} \times \vec{B}_0) = 0$$

$$m\frac{d\vec{v}_{\perp}}{dt} = -e(\vec{v}_{\perp} \times \vec{B}_0) = -evB_0\hat{r}$$

where $\hat{r}$ is the unit vector in direction of $\vec{v}_{\perp} \times \vec{B}_0$.
Consider now a case when $B_0||\hat{k}$. Then, $\vec{v}_{\perp} = (v_x, v_y, 0)$ and we have

$$m\frac{dv_x}{dt} = -ev_yB_0$$

$$m\frac{dv_y}{dt} = ev_xB_0$$

By taking the time derivative of the second equation, we have

$$m\frac{d^2v_y}{dt^2} = eB_0\frac{dv_x}{dt}$$

Substituting in from the first equation

$$m\frac{d^2v_y}{dt^2} = -e^2B_0^2\frac{1}{m}v_y$$
$$\frac{d^2v_y}{dt^2} = -\frac{e^2B_0^2}{m^2}v_y$$

Therefore, we can write that

$$v_y = A\sin(\omega_{ec}t + \phi_0)$$

where ϕ_0 is the phase and

$$\omega_{ec} = \frac{eB_0}{m} \tag{3}$$

is the electron cyclotron frequency, or the gyral frequency. By substituting back to find v_x

$$v_x = \frac{m}{eB_0}\frac{dv_y}{dt} = \frac{1}{\omega_{ec}}A\omega_{ec}\cos(\omega_{ec}t + \phi_0) = A\cos(\omega_{ec}t + \phi_0)$$

Therefore, the electron travels with $\vec{v}_\parallel$ and rotates around the direction of $\vec{B}_0$ with frequency ω_{ec}. This motion is called the gyration. The radius of the gyration can be estimated as follows. Since the motion in x and y is oscillatory, $v_x^2 = \omega_{ec}^2x^2$ and $v_y^2 = \omega_{ec}^2y^2$ where $x^2 + y^2 = R^2$ where R is the radius of gyration. Therefore

$$v_\perp^2 = v_x^2 + v_y^2 = \omega_{ec}^2R^2$$

In a thermal plasma, the gyration takes two degrees of freedom, i.e. the expected energy is $\frac{1}{2}mv_\perp^2 \approx k_BT_e$. Therefore

$$R = \frac{v_\perp}{\omega_{ec}} \approx \sqrt{\frac{2k_BT_e}{m}}\frac{m}{eB_0} = \sqrt{\frac{2k_Bm}{e^2}}\frac{\sqrt{T}_e}{B_0}$$

In the lecture notes, we are provided with numerical result that differs from the one provided here - we are given

$$R = 2.4\times10^{-6}\sqrt{\frac{T[\mathrm{K}]}{11605}}\frac{1}{B_0[T]}\mathrm{m}$$

However, this result is relatively close to the result obtained before, so I thought I will still include it.

2.2 Drift Motion

Consider now that another force $\vec{F}$ acts on the electrons (with charge q) in the plasma, perpendicularly to direction of $\vec{B}_0$. For the sake of brevity, lets assume that $\vec{B}_0 = B_0\hat{k}$ and $\vec{F} = F\hat{\jmath}$. Then, the equation of motion becomes

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v}\times\hat{k}B_0 + \frac{\vec{F}}{m}$$

Lets now assume that this force changes the speed only very slowly. We can then express $\vec{v} = \vec{u} + \vec{v}_d$ where $\frac{d\vec{v}_d}{dt} = 0$. This leads to

$$\frac{d\vec{u}}{dt} = \frac{q}{m}\vec{u}\times\vec{B}_0 + \frac{q}{m}\vec{v}_d\times\hat{k}B_0 + \frac{\vec{F}}{m}$$

We can see that this becomes a steady equation of gyration if we have

$$\frac{qB_0}{m}\vec{v}_d\times\hat{k} = \frac{-F\hat{\jmath}}{m}$$
$$\vec{v}_d\times\hat{k} = \frac{-F}{qB_0}\hat{\jmath}$$

By taking a dot product with $\hat{\jmath}$

$$\hat{\jmath}\cdot(\vec{v}_d\times\hat{k}) = \vec{v}_d\cdot(\hat{k}\times\hat{\jmath}) = -\vec{v}_d\cdot\hat{\imath} = -\frac{F}{qB_0}$$

By taking a dot product with $\hat{\imath}$

$$\hat{\imath}\cdot(\vec{v}_d\times\hat{k}) = \vec{v}_d\cdot(\hat{k}\times\hat{\imath}) = \vec{v}_d\cdot\hat{\jmath} = 0$$

and we know that the motion along the direction of the field is unimpeded, as $\vec{F}$ is perpendicular to the field. Therefore $\vec{v}_d \cdot \hat{k} = 0$ and

$$\vec{v}_d = (\frac{F}{qB_0}, 0, 0) = \frac{F}{qB_0}\hat{\imath}$$

In order to represent this in basis-free notation, we recognize that $\vec{F} \times \vec{B}_0 = FB_0\hat{\jmath} \times \hat{k} = FB_0\hat{\imath}$, and therefore

$$\vec{v}_d = \frac{\vec{F} \times \vec{B}_0}{q|\vec{B}_0|^2} \tag{4}$$

Therefore, the particles in plasma gyrate as usual with $\vec{u}$ and also undergo drift motion described by $\vec{v}_d$.

2.2.1 Gradient Drift

Consider now a case when the static $\vec{B}$ field slowly changes, so that around any particular point $\vec{r}_0$ we can Taylor expand

$$\vec{B}(\vec{r}) \approx \vec{B}_0(\vec{r}_0) + \Big((\vec{r} \cdot \nabla)\vec{B}(\vec{r})\Big)\Big|_{\vec{r}=\vec{r}_0}$$

The equation of motion then becomes (in frame where $\vec{r}_0 = \vec{0}$)

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v} \times \vec{B}_0 + \frac{q}{m}\vec{v} \times ((\vec{r} \cdot \nabla)\vec{B})$$

In a case when the direction of $\vec{B}$ is always the same and $\vec{B}$ simply changes magnitude in some direction, we have $(\vec{r} \cdot \nabla)\vec{B} \parallel \vec{B}$, and therefore $\vec{v} \times ((\vec{r} \cdot \nabla)\vec{B}) \perp \vec{B}$, and we can therefore predict that the behaviour produced will be a resultant drift velocity of the particles (here, $\vec{F} = q(\vec{v} \times ((\vec{r} \cdot \nabla)\vec{B}))$. However, solving for the exact velocity is somewhat more involved, as the force is not constant (both $\vec{v}$ and $\vec{r}$ change). However, the resultant drift force is

$$\vec{v}_d \propto \frac{1}{q}\vec{B} \times (\nabla B)$$

where $B = |\vec{B}|$. Therefore, for positive particle in $\vec{B} = B(x)\hat{k}$ and $\nabla B(x) = B'(x)\hat{\imath}$, $B'(x) > 0$, the particle will drift in positive $\hat{\jmath}$ direction.

2.2.2 Curvature Drift

Consider now $\vec{B}_0$ field which does not change magnitude, but changes slowly direction with local radius of curvature R. Let $\hat{n}$ be the vector normal to field lines of $\vec{B}_0$ in the direction into the centre of curvature. The particle tries to keep its centre of gyration on the field line of $\vec{B}_0$ - this creates an effective centrifugal force on the particle

$$\vec{F}_c = -\frac{mv_\parallel^2}{R}\hat{n}$$

where $v_\parallel$ is the speed parallel to $\vec{B}_0$, which is constant. Therefore, we can directly say

$$\vec{v}_d = \frac{\vec{F} \times \vec{B}_0}{q|\vec{B}_0|^2} = -\frac{mv_\parallel^2}{qR|\vec{B}_0|^2}\hat{n} \times \vec{B}_0 = \frac{mv_\parallel^2}{qR|\vec{B}_0|^2}\vec{B}_0 \times \hat{n}$$

Therefore, particle drifts in direction perpendicular to $\vec{B}_0$ and perpendicular to the plane of the circle of curvature. Therefore, if $\vec{B}_0 = B_0\hat{\jmath}$ and $\hat{n} = -\hat{\imath}$, the particle would drift in positive $\hat{k}$ direction, given that q is positive.

We should note that both curvature drift and gradient drift are sign dependent and will lead to creation of currents in the plasma.

2.2.3 Ring Currents

Ring currents occur in Earth's Magnetosphere. The magnetic field lines go from pole to pole and the field gets weaker as we move further away from Earth's surface. Therefore, the gradient of the magnetic field points towards Earth, $\vec{B}$ approximately parallel to earths surface and radius of curvature vector $\hat{n}$ points towards the surface. Therefore, both drift velocities due to gradient and curvature currents are in the same direction, and therefore currents are induced in this part of magnetosphere. The nature of the current is such that it induces new magnetic field that opposes the original field $\vec{B}$, and therefore strong ring currents, as they are called, can reduce the magnetic field of Earth.

3 Optical Waves in Plasma

Now, lets consider electromagnetic waves incident on the plasma, with frequency high enough that the motion of the ions can be neglected compared to the motion of electrons, i.e. the frequency of the waves ω

$$\omega \gg \omega_{ic}, \omega \gg \omega_{ip}$$

where $\omega_{ic/ip}$ are ion cyclotron and ion-plasma frequencies.
The electromagnetic waves can be expressed as

$$\vec{E} = \text{Re}\left(\vec{E}_0 e^{i(\vec{k}\cdot vecr - \omega t)}\right)$$

where Re(z) denotes the real part of z and $\vec{E}_0$ is a constant, generally complex. Usually, I will drop real part and just assume that it is implied. Similarly, for $\vec{B}$

$$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Since plasma is a conductor, we can assume that we will have some form of Ohm's law

$$\vec{j} = \boldsymbol{\sigma}\vec{E}$$

where $\boldsymbol{\sigma}$ is generally a tensor. We now need to make this consistent with Maxwell equations. The fourth Maxwell equation is

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \boldsymbol{\sigma}\vec{E} + \frac{1}{c^2}(-i\omega)\vec{E} = \frac{1}{c^2}\left(\frac{1}{\epsilon_0}\boldsymbol{\sigma} - i\omega\mathbf{I}\right)\vec{E} = \frac{-i\omega}{c^2}\left(\mathbf{I} + \frac{i}{\omega\epsilon_0}\boldsymbol{\sigma}\right)\vec{E}$$

We usually write $\boldsymbol{\epsilon} = \mathbf{I} + \frac{i}{\omega\epsilon_0}\boldsymbol{\sigma}$, where $\mathbf{I}$ is the identity tensor.
Now, consider the effect of $\vec{E}$ and $\vec{B}$ on the change of position of the particle. From the 3rd Maxwell equation

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega\vec{B}$$

Hence

$$|\vec{E}| \approx \frac{\omega}{|\vec{k}|}|\vec{B}| = c|\vec{B}|$$

The Lorentz force on the electrons is then

$$\frac{d\vec{v}}{dt} = \frac{-e}{m}\vec{E} - \frac{e}{m}\vec{v} \times \vec{B}$$

$$\left|\frac{d\vec{v}}{dt}\right| \approx \frac{e}{m}|\vec{E}| + \frac{e}{m}\frac{|\vec{v}|}{c}|\vec{E}|$$

We can see that in a non-relativistic approach, the effect from the wave of the $\vec{B}$ field on the motion of the electrons is very small compared to the effect from the electric field $\vec{E}$. Therefore, we will usually neglect this effect.
In order to generalize, we could however impose some background field $\vec{B}_0$ which is not wavelike and therefore has an effect on the electron motion. We are therefore setting $\vec{B} \to \vec{B} + \vec{B}_0$. Then, the Lorentz force becomes

$$\frac{d\vec{v}}{dt} = -\frac{e}{m}\vec{E} - \frac{e}{m}\vec{v} \times \vec{B}_0$$

Lets now assume that $\vec{B}_0 = B_0\hat{z}$ and that the velocity of the electrons is also wavelike, i.e. $\vec{v} = \vec{v}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.
Then

$$-i\omega\vec{v} = -\frac{e}{m}\vec{E} - \frac{e}{m}\vec{v} \times \vec{B}_0$$

$$\vec{v} = -\frac{ie}{m\omega}\vec{E} - \frac{ie}{m\omega}\vec{v} \times \vec{B}_0$$

Since $\vec{v} \times \vec{B}_0 = (v_y, -v_x, 0)B_0$, we have

$$v_z = -\frac{ie}{m\omega}E_z$$

$$v_y = -\frac{ie}{m\omega}E_y + \frac{ie}{m\omega}v_x B_0$$
$$v_x = -\frac{ie}{m\omega}E_x - \frac{ie}{m\omega}v_y B_0$$

Substituting from the last equation into the middle one

$$v_y = -\frac{ie}{m\omega}E_y + \frac{ieB_0}{m\omega}\left(-\frac{ie}{m\omega}E_x - \frac{ieB_0}{m\omega}v_y\right)$$

$$v_y = -\frac{ie}{m\omega}E_y + \frac{e^2B_0}{m^2\omega^2}E_x + \frac{e^2B_0^2}{m^2\omega^2}v_y$$

Remembering that $\frac{eB_0}{m} = \omega_{ec}$

$$v_y = -\frac{ie}{m\omega}E_y + \frac{\omega_{ec}^2}{\omega^2 B_0}E_x + \frac{\omega_{ec}^2}{\omega^2}v_y$$

$$v_y\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right) = \frac{\omega_{ec}^2}{\omega^2 B_0}E_x - \frac{ie}{m\omega}E_y = \frac{e}{m}\left(\frac{\omega_{ec}^2}{\omega^2}\frac{m}{eB_0}E_x - \frac{i}{\omega}E_y\right) = \frac{e}{m}\left(\frac{\omega_{ec}}{\omega^2}E_x - \frac{i}{\omega}E_y\right)$$

Hence

$$v_y = \frac{e}{m}\left(\frac{\omega_{ec}}{\omega^2\left(1 - \frac{\omega_{ec}^2}{\omega^2}\right)}E_x - \frac{i}{\omega\left(1 - \frac{\omega_{ec}^2}{\omega^2}\right)}E_y\right) = \frac{e}{m}\left(\frac{\omega_{ec}}{\omega^2 - \omega_{ec}^2}E_x - \frac{i\omega}{\omega^2 - \omega_{ec}^2}E_y\right)$$

Hence

$$v_x = \frac{-ie}{m\omega}E_x - \frac{i\omega_{ec}}{\omega} \times \frac{e}{m}\left(\frac{\omega_{ec}}{\omega^2 - \omega_{ec}^2}E_x - \frac{i\omega}{\omega^2 - \omega_{ec}^2}E_y\right) = \frac{e}{m}\left(\frac{-i}{\omega}\left(1 + \frac{\omega_{ec}^2}{\omega^2 - \omega_{ec}^2}\right)E_x - \frac{\omega_{ec}}{\omega^2 - \omega_{ec}^2}E_y\right)$$

Therefore, we have

$$v_x = \frac{e}{m}\left(\frac{i\omega}{\omega_{ec}^2 - \omega^2}E_x + \frac{\omega_{ec}}{\omega_{ec}^2 - \omega^2}E_y\right)$$
$$v_y = \frac{e}{m}\left(\frac{-\omega_{ec}}{\omega_{ec}^2 - \omega^2}E_x + \frac{i\omega}{\omega_{ec}^2 - \omega^2}E_y\right)$$
$$v_z = \frac{e}{m}\left(\frac{-i}{\omega}E_z\right)$$

This can be summarized as

$$\vec{v} = \frac{e}{m}\boldsymbol{\alpha}\vec{E} = \frac{e}{m}\begin{pmatrix} \frac{i\omega}{\omega_{ec}^2-\omega^2} & \frac{\omega_{ec}}{\omega_{ec}^2-\omega^2} & 0 \\ \frac{-\omega_{ec}}{\omega_{ec}^2-\omega^2} & \frac{i\omega}{\omega_{ec}^2-\omega^2} & 0 \\ 0 & 0 & \frac{-i}{\omega} \end{pmatrix}\vec{E}$$

The current induced by the wave can then be expressed as

$$\vec{j} = \rho_q\vec{v} = -en_e\frac{e}{m}\boldsymbol{\alpha}\vec{E} = -\frac{e^2n_e}{m}\boldsymbol{\alpha}\vec{E} = -\epsilon_0\omega_{ep}^2\boldsymbol{\alpha}\vec{E}$$

where ω_{ep} is the electron-plasma frequency $\omega_{ep} = \sqrt{\frac{e^2n_e}{m\epsilon_0}}$ With reference to the Ohm's law, $\vec{j} = \boldsymbol{\sigma}\vec{E}$

$$\boldsymbol{\sigma} = -\epsilon_0\omega_{ep}^2\boldsymbol{\alpha}$$

Therefore

$$\boldsymbol{\epsilon} = \mathbf{I} + \frac{i}{\omega\epsilon_0}\sigma = \mathbf{I} - \frac{i\omega_{ep}^2}{\omega}\boldsymbol{\alpha} = \begin{pmatrix} 1 + \frac{\omega_{ep}^2}{\omega_{ec}^2-\omega^2} & \frac{-i\omega_{ep}^2\frac{\omega_{ec}}{\omega}}{\omega_{ec}^2-\omega^2} & 0 \\ \frac{i\omega_{ep}^2\frac{\omega_{ec}}{\omega}}{\omega_{ec}^2-\omega^2} & 1 + \frac{\omega_{ep}^2}{\omega_{ec}^2-\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{ep}^2}{\omega^2} \end{pmatrix} = \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

And we now know that $\nabla \times \vec{B} = \frac{-i\omega}{c^2}\boldsymbol{\epsilon}\vec{E}$ in terms of $\vec{E}$.
If we take curl of the third Maxwell equation, we have

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right)$$

Using the fact that both $\vec{E}$ and $\vec{E}$ are wavelike

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = i\vec{k}(i\vec{k} \cdot \vec{E}) - (i|\vec{k}|)^2 \vec{E} = -\vec{k}(\vec{k} \cdot \vec{E}) + |\vec{k}|^2 \vec{E}$$

$$-\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) = i\omega \nabla \times \vec{B} = i\omega \left(\frac{-i\omega}{c^2} \boldsymbol{\epsilon} \vec{E}\right) = \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \vec{E}$$

Therefore, we have

$$-\vec{k}(\vec{k} \cdot \vec{E}) + |\vec{k}|^2 \vec{E} = \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \vec{E}$$

$$\vec{k}(\vec{k} \cdot \vec{E}) - |\vec{k}|^2 \vec{E} + \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \vec{E} = 0 \tag{5}$$

The electric field of an optical EM wave in plasma must obey this equation. If we are searching for linear waves, we need the equation to be obeyed for any vector $\vec{E}$ that we use as representation for the linear waves. This will clearly put some restrictions on the values of $\vec{k}$ and ω the wave can have - this equation will provide us with the dispersion relation. In order to find the dispersion relation, we need to rewrite the above equation in a tensor form. The first term in the equation becomes

$$\vec{k}(\vec{k} \cdot \vec{E}) = \begin{pmatrix} k_x(k_x E_x + k_y E_y + k_z E_z) \\ k_y(k_x E_x + k_y E_y + k_z E_z) \\ k_z(k_x E_x + k_y E_y + k_z E_z) \end{pmatrix} = \begin{pmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_x k_y & k_y^2 & k_y k_z \\ k_x k_z & k_y k_z & k_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

The second term is

$$|\vec{k}|^2 \vec{E} = (k_x^2 + k_y^2 + k_z^2)\vec{E} = \begin{pmatrix} k_x^2 + k_y^2 + k_z^2 & 0 & 0 \\ 0 & k_x^2 + k_y^2 + k_z^2 & 0 \\ 0 & 0 & k_x^2 + k_y^2 + k_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

We already know the tensor form for the third term, so we can write that

$$\vec{k}(\vec{k} \cdot \vec{E}) - |\vec{k}|^2 \vec{E} + \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \vec{E} = \begin{pmatrix} \frac{\omega^2}{c^2}\epsilon_1 - k_y^2 - k_z^2 & k_x k_y - i\frac{\omega^2}{c^2}\epsilon_2 & k_x k_z \\ k_x k_y + i\frac{\omega^2}{c^2}\epsilon_2 & \frac{\omega^2}{c^2}\epsilon_1 - k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & \frac{\omega^2}{c^2}\epsilon_3 - k_x^2 - k_y^2 \end{pmatrix} \vec{E} = 0$$

We can also rewrite this in terms of refractive index vector

$$\vec{N} = \frac{c}{\omega}\vec{k}$$

as

$$\frac{\omega^2}{c^2} \begin{pmatrix} \epsilon_1 - N_y^2 - N_z^2 & N_x N_y - i\epsilon_2 & N_x N_z \\ N_x N_y + i\epsilon_2 & \epsilon_1 - N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & \epsilon_3 - N_x^2 - N_y^2 \end{pmatrix} \vec{E} = \frac{\omega^2}{c^2} \mathbf{M} \vec{E} = 0$$

This equation applies for any $\vec{E}$ if $\det \mathbf{M} = 0$, i.e.

$$\begin{vmatrix} \epsilon_1 - N_y^2 - N_z^2 & N_x N_y - i\epsilon_2 & N_x N_z \\ N_x N_y + i\epsilon_2 & \epsilon_1 - N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & \epsilon_3 - N_x^2 - N_y^2 \end{vmatrix} = 0 \tag{6}$$

This is the required dispersion relation, which will help us predict the types and basic behaviour of optical waves that propagate through plasma.

3.1 Optical Eigenmodes

More precisely, we might be interested in the polarization of the waves given by the dispersion relation. To solve this, return to the original equation $\mathbf{M}\vec{E} = 0$. If $\vec{E}$ is the eigenvector of $\mathbf{M}$, we have a vector that can potentially always satisfy the dispersion relation, if its eigenvalue λ satisfies

$$\mathbf{M}\vec{E} = \lambda \vec{E} = 0$$

$$\lambda = 0$$

Therefore, we can solve $|\mathbf{M} - \lambda \mathbf{I}| = 0$ to find eigenvalues of $\mathbf{M}$, use these eigenvalues to find eigenmodes $\vec{E}$ which can propagate through the system, and solve $\lambda = 0$ to obtain dispersion relations of the eigenmodes.

3.2 No External Field

In the case when $\vec{B}_0 = 0$, $\omega_{ec} = 0$, and therefore

$$\epsilon_2 = \frac{\omega_{ep}^2 \frac{\omega_{ec}}{\omega}}{\omega_{ec}^2 - \omega^2} = 0$$

$$\epsilon_1 = 1 + \frac{\omega_{ep}^2}{\omega_{ec}^2 - \omega^2} = 1 - \frac{\omega_{ep}^2}{\omega^2} = \epsilon_3$$

Also, since $\vec{B}_0 = 0$, there is no preffered direction in the plasma, and therefore we should expect the same result from the dispersion relation. We can therefore choose that the wave propagates, for example, in the x direction, so that $N_x = \frac{ck}{\omega}$, $N_y = N_z = 0$. The secular equation becomes

$$0 = |\mathbf{M} - \lambda \mathbf{I}| = \begin{vmatrix} \epsilon_3 - \lambda & 0 & 0 \\ 0 & \epsilon_3 - N_x^2 - \lambda & 0 \\ 0 & 0 & \epsilon_3 - N_x^2 - \lambda \end{vmatrix} = (\epsilon_3 - \lambda)(\epsilon_3 - N_x^2 - \lambda)^2$$

Therefore, we have one eigenvalue $\lambda_1 = \epsilon_3$ and two degenerate eigenvalues $\lambda_2 = \epsilon_3 - N_x^2$. The corresponding eigenmodes are given by $(\mathbf{M} - \lambda \mathbf{I})\vec{a} = 0$. For λ_1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -N_x^2 & 0 \\ 0 & 0 & -N_x^2 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = 0$$

which implies that the normalised eigenvector $\vec{a}_1$ will be

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

As the wave propagates in the x direction, this wave represents longtitudal waves - oscillation is in the direction of propagation.
To find the dispersion relation, we solve $\lambda_1 = 0$

$$\epsilon_3 = 1 - \frac{\omega_{ep}^2}{\omega^2} = 0$$

$$\omega = \omega_{ep} \tag{7}$$

So, this mode corresponds to oscillations at a single allowed frequency, ω_{ep}, therefore this behaviour corresponds to the electron-plasma oscillation. To further illustrate this, we can find that while the phase speed is

$$v_p = \frac{\omega}{k} = \frac{\omega_{ep}}{k}$$

the group speed is

$$v_g = \frac{d\omega}{dk} = 0$$

so at this wave, there is no real transport of matter.
The other two eigenvectors are given by

$$0 = (\mathbf{M} - \lambda_2 \mathbf{I})\vec{a}_2 = \begin{pmatrix} N_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

which is satisfied when $a_1 = 0$. Therefore, we can find two orthonormal eigenvectors

$$\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The wave propagates in the x direction, and the eigenmodes are in y and z directions - this corresponds to any transverse, linearly polarized wave. The dispersion relation is

$$\epsilon_3 - N_x^2 = 0$$

$$1 = \frac{\omega_{ep}}{\omega^2} + \frac{c^2k^2}{\omega^2}$$

$$\omega^2 = c^2k^2 + \omega_{ep}^2 \tag{8}$$

These waves are therefore dispersive, although in the limit of large k (small wavelength), $\omega^2 \to c^2k^2$, which is dispersion relation for elastic, i.e. non-dispersive waves.
Generally, the phase speed of these waves is given by

$$v_p = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_{ep}^2}{k^2}}$$

and the group speed

$$v_g = \frac{d\omega}{dk} = \frac{c^2k}{\sqrt{c^2k^2 + \omega_{ep}^2}} = \frac{c}{\sqrt{1 + \frac{\omega_{ep}^2}{c^2k^2}}}$$

Notice that the phase speed is bigger than c. This, however, does not contradict the special relativity, as any real transport of information, or matter, can happen only at the group speed, which is smaller than c.
The dispersion relations for the waves in zero external field are summarized in Fig. 2

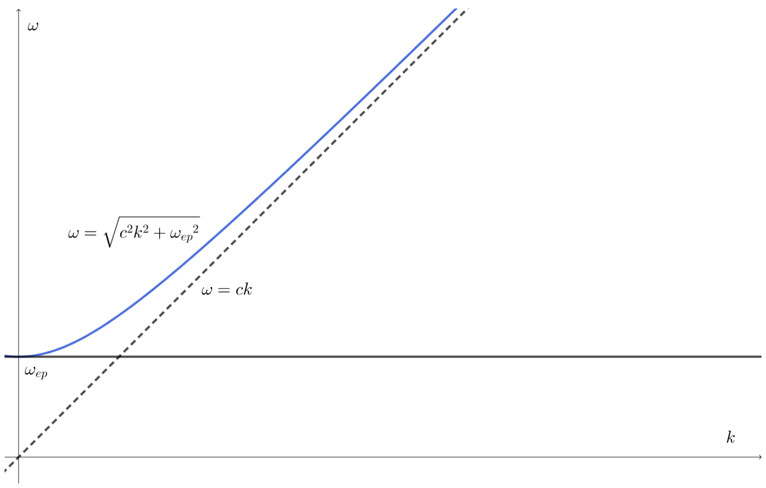

Figure 2: The blue line represents the dispersion curve of transverse waves in zero external field. As k increases for these waves, they start to behave like elastic waves and approach dispersion curve $\omega = ck$. The black line represents the electron-plasma oscillations.

One peculiar property of these waves is the fact that the product of phase and group speed goes to c^2

$$v_pv_g = c\sqrt{1 + \frac{\omega_{ep}^2}{c^2k^2}} \frac{c}{\sqrt{1 + \frac{\omega_{ep}}{c^2k^2}}} = c^2$$

If we have a wave incident on the plasma that does not satisfy the dispersion relation, we can have for example, for low frequency waves

$$c^2k^2 = \omega^2 - \omega_{ep}^2 < 0$$

Then, we have $k = i\frac{\sqrt{\omega_{ep}^2 - \omega^2}}{c}$

$$\vec{E} = \vec{E}_0 e^{ikx} e^{-i\omega t} = \vec{E}_0 e^{-i\omega t} e^{-x\sqrt{\omega_{ep}^2 - \omega^2}/c}$$

Thus we have an evanescent wave that exponentially decreases as it enters the plasma, with constant of decay k, which means that the wave penetrates the medium up to so called skin depth z_D

$$z_D = \frac{c}{\sqrt{\omega_{ep}^2 - \omega^2}}$$

The wave that is prevented from entering the medium is then usually in part absorbed in the evanescent wave, but also reflected.

3.2.1 Radio Waves in Earth's Atmosphere

In the Earth's atmosphere, the number of ionized particles rapidly increases in the ionosphere. This increase means that the concentration of electrons n_e here increases, and therefore ω_{ep} increases here as well. This means that wave that had real dispersion relation below ionosphere with $\omega^2 - (\omega'_{ep})^2 > 0$ (where ω'_{ep} is the electron-plasma frequency below ionosphere) can now have $\omega^2 - \omega_{ep}^2 < 0$ and therefore gets reflected by the ionosphere. This enables transmittion of radio waves by bouncing of ionosphere and surface of the Earth, repeatadly.

3.3 Waves Propagating Parallel to External Field

Consider now a wave that propagates in the same direction as a non-zero field $\vec{B}_0 = B_0\hat{z}$. This means that $N_z = \frac{ck}{\omega}$ and $N_x = N_y = 0$. The secular equation we have to solve is

$$\begin{vmatrix} \epsilon_1 - N_z^2 - \lambda & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 - N_z^2 - \lambda & 0 \\ 0 & 0 & \epsilon_3 - \lambda \end{vmatrix} = 0$$

$$(\epsilon_3 - \lambda)[(\epsilon_1 - N_z^2 - \lambda)^2 - \epsilon_2^2] = 0$$

We therefore again have a mode at $\lambda_1 = \epsilon_3$ and two other modes at $\lambda_2 = \epsilon_1 - N_z^2 - \epsilon_2$ and $\lambda_3 = \epsilon_1 - N_z^2 + \epsilon_2$. The eigenvector for λ_1 is

$$\begin{pmatrix} \epsilon_1 - N_z^2 - \epsilon_3 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 - N_z^2 - \epsilon_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

The equations for the first two components a_1 and a_2 can be rewritten as

$$Pa_1 - iQa_2 = 0$$

$$iQa_1 + Pa_2 = 0$$

where $P = \epsilon_1 - N_z^2 - \epsilon_3$ and $Q = \epsilon_2$ and generally, $P \neq Q$. Multiplying first equation by Q leads to

$$PQa_1 = iQ^2 a_2$$

Multiplying second equation by $-iP$ leads to

$$PQa_1 = iP^2 a_2$$

Equating these two, we recover $P^2 a_2 = Q^2 a_2$. For this to generally apply, we need $a_2 = 0$. This directly leads to $a_1 = 0$ and therefore, we have again

$$\vec{a}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with dispersion relation $\epsilon_3 = \lambda = 0$, i.e. $\omega = \omega_{ep}$. This is exactly the same electron-plasma oscillation mode we observed when $\vec{B}_0 = 0$. Perhaps this should not be too surprising, as this is a longtitudal mode with electrons moving parallel to the field. The force of this field on the electrons is then proportional to $\vec{v} \times \vec{B}_0$, which for $\vec{v} \parallel \vec{B}_0$ goes to zero - the electron plasma oscillation mode is undisturbed by the additional field. For $\lambda_2 = \epsilon_1 - N_z^2 - \epsilon_2$

$$\begin{pmatrix} \epsilon_2 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 - \epsilon_1 + N_z^2 + \epsilon_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

This is satisfied when $a_3 = 0$ and $a_1 = ia_2$, i.e. for case

$$\vec{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

This is somewhat non-standard, as the eigenvector is complex. To give interpretation to this, consider the form of the wave field $\vec{E}$

$$\vec{E} = E\vec{a}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{E}{\sqrt{2}} \begin{pmatrix} \exp\left(i(\vec{k}\cdot\vec{r}-\omega t)\right) \\ \exp\left(-i\frac{\pi}{2}\right)\exp\left(i(\vec{k}\cdot\vec{r}-\omega t)\right) \\ 0 \end{pmatrix} = \frac{E}{\sqrt{2}} \begin{pmatrix} \exp\left(i(\vec{k}\cdot\vec{r}-\omega t)\right) \\ \exp\left(i(\vec{k}\cdot\vec{r}-\omega t-\frac{\pi}{2})\right) \\ 0 \end{pmatrix}$$

where I used $-i = e^{-i\frac{\pi}{2}}$. This eigenvector implies that the eigenmode has a phase shift between the x and y components of the oscillating field of $\frac{\pi}{2}$ and that the fields have the same magnitude in both directions - the eigenmode is a circularly polarized light. In this case, this is the left-handed circularly polarized light. The dispersion relation is obtained by $\lambda_2 = 0$

$$\epsilon_1 - N_z^2 - \epsilon_2 = 0$$

$$\frac{c^2k^2}{\omega^2} = \epsilon_1 - \epsilon_2 = 1 + \frac{\omega_{ep}^2}{\omega_{ec}^2-\omega^2} - \frac{\omega_{ep}^2\frac{\omega_{ec}}{\omega}}{\omega_{ec}^2-\omega^2} = 1 + \omega_{ep}^2\frac{\omega-\omega_{ec}}{\omega(\omega_{ec}-\omega)(\omega_{ec}+\omega)} = 1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}$$

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}} \tag{9}$$

Before we discuss the properties of this wave, lets determine the other eigenmode first, as we will discover that they are related. For $\lambda_3 = \epsilon_1 - N_z^2 + \epsilon_2$

$$\begin{pmatrix} -\epsilon_2 & -i\epsilon_2 & 0 \\ i\epsilon_2 & -\epsilon_2 & 0 \\ 0 & 0 & \epsilon_3-\epsilon_1+N_z^2-\epsilon_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Again, we have $a_3 = 0$, but this time $ia_1 = a_2$, and therefore

$$\vec{a}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

which corresponds to right-hand circularly polarized light. The dispersion relation is

$$\lambda_3 = \epsilon_1 - N_z^2 + \epsilon_2 = 0$$

$$\frac{c^2k^2}{\omega^2} = 1 + \frac{\omega_{ep}^2}{\omega_{ec}^2-\omega^2} + \frac{\omega_{ep}^2\frac{\omega_{ec}}{\omega}}{\omega_{ec}^2-\omega^2} = 1 + \omega_{ep}^2\frac{\omega_{ec}+\omega}{\omega(\omega_{ec}-\omega)(\omega_{ec}+\omega)} = 1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec}-\omega)}$$

$$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec}-\omega)}} \tag{10}$$

The dispersion relations for these modes are summarized in Fig. 3.
Several things can be determined about these two wave modes. We will start by cut-off frequencies - frequencies when the k starts to become complex rather than real. For left-handed waves

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}} = 0$$

$$1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)} = 0$$

$$\omega^2 + \omega_{ec}\omega - \omega_{ep}^2 = 0$$

$$\omega = \frac{-\omega_{ec}}{2} + \frac{\sqrt{\omega_{ec}^2+4\omega_{ep}^2}}{2} = \frac{1}{2}\left(\sqrt{\omega_{ec}^2+4\omega_{ep}^2} - \omega_{ec}\right)$$

Where I chose the positive root. We call this frequency ω_1.

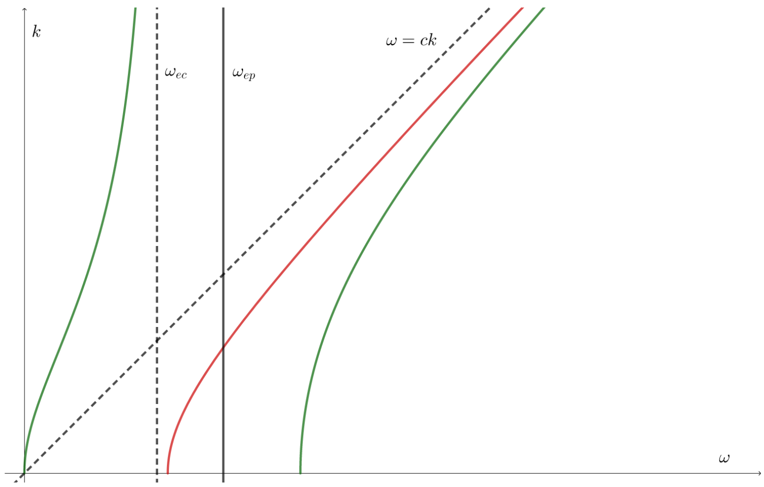

Figure 3: The red line represents the left-handed polarized waves, the green line represents the right-handed polarized waves. We can see that left-handed waves only have one branch, while right-handed waves have an extra, so called whistler branch. We also still observe electron-plasma oscillations (black line). Notice that the axes for ω and k are swapped.

For right-handed waves, cut-off frequency is

$$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec} - \omega)}} = 0$$

$$1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec} - \omega)} = 0$$

$$\omega^2 - \omega_{ec}\omega - \omega_{ep}^2 = 0$$

$$\omega = \frac{\omega_{ec}}{2} + \frac{\sqrt{\omega_{ec}^2 + 4\omega_{ep}^2}}{2} = \frac{1}{2}\left(\sqrt{\omega_{ec}^2 + 4\omega_{ep}^2} + \omega_{ec}\right)$$

where I again chose the positive root. This frequency is called ω_2. We should note that $\omega_2 > \omega_1$. But, importantly, if we have $\omega < \omega_{ec}$ for right-handed waves, we have again a valid dispersion relation. This means that right-handed waves have a second branch of the dispersion relation from $\omega = 0$ to $\omega = \omega_{ec}$. These waves are called the whistler waves. We can see that at $\omega = 0$

$$\lim_{\omega\to 0} k = \lim_{\omega\to 0} \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec} - \omega)}} = \lim_{\omega\to 0} \frac{\omega}{c}\sqrt{\frac{\omega_{ep}^2}{\omega\omega_{ec}}} = 0$$

Other interesting property is that for right-handed waves, we can hit a resonance when $\omega = \omega_{ec}$, as then $k \to \infty$. This happens because the incident right-hand circularly polarized waves have the same frequency and orientation as gyration of the electrons in the plasma due to the external field. Therefore, the incident wave moves the electrons practically without any oposition - leads to resonance. This specific resonance is called electron cyclotron resonance. At this resonance, the incident wave tends to get absorbed by the plasma, rather then reflected.

We should also notice that for low frequencies, only the right-handed waves can propagate through the plasma. This is because the left-handed waves get diminished by the gyration of the electrons, which is in opposite direction. At very low frequencies, the whistler branch of the right-handed waves behaves as

$$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec} - \omega)}} \approx \frac{\omega}{c}\frac{\omega_{ep}}{\sqrt{\omega\omega_{ec}}} = \frac{1}{c}\sqrt{\omega}\frac{\omega_{ep}}{\sqrt{\omega}_{ec}}$$

$$\omega = \frac{\omega_{ec}}{\omega_{ep}^2}c^2k^2$$

Therefore, we have dispersive waves with $\omega \propto k^2$, and therefore the group speed is $v_g \propto k \propto \lambda^{-1}$. Therefore, longer wavelengths travel slower with these waves - this produces a characteristic whistle profile when a point signal propagates via this wave - hence the name whistler waves.

3.3.1 Faraday Rotation

Consider now a linearly polarized light propagating along the direction of the external magnetic field is incident on the plasma. Suppose that frequency $\omega > \omega_1$, hence the wave satisfies dispersion relation. How does this wave propagate through the plasma, since it is not the eigenmode of the dispersion equation?
We can express linearly polarized light as superposition of left-handed and right-handed circularly polarised light as

$$\vec{E}_{lin} = \vec{E}_{lh} + \vec{E}_{rh} = E_{lh}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{i(k_1 z - \omega t)} + E_{rh}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{i(k_2 z - \omega t)} = \begin{pmatrix} E_{rh}e^{ik_2 z} + E_{lh}e^{ik_1 z} \\ iE_{rh}e^{ik_2 z} - iE_{lh}e^{ik_1 z} \\ 0 \end{pmatrix} e^{-i\omega t}$$

where $k_1 = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}}$ and $k_2 = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec}-\omega)}}$. We can therefore see that as the linear light propagates throught the material, the angle of the polarisation changes. For the case when the light is initially perfectly linearly polarized, $E_{lh} = E_{rh}$. At $z = 0$, $\vec{E}_{lin}$ is

$$\vec{E}_{lin} = E_{rh}\begin{pmatrix} 1+1 \\ i-i \\ 0 \end{pmatrix} e^{-i\omega t} = 2E_{rh}e^{-i\omega t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, initially, the light is polarized along the x direction. The angle to the x axis after the light has travelled distance z in the medium is therefore given as $\Delta\phi = \tan^{-1}(E_y/E_x)$. Here

$$\frac{E_y}{E_x} = i\frac{e^{ik_2 z} - e^{ik_1 z}}{e^{ik_2 z} + e^{ik_1 z}} = i\frac{e^{ik_2 z} - e^{ik_1 z}}{e^{ik_2 z} + e^{ik_1 z}} \times \frac{e^{-i\frac{(k_1+k_2)}{2}z}}{e^{-i\frac{(k_1+k_2)}{2}z}} = i\frac{e^{i\frac{(k_2-k_1)}{2}z} - e^{-i\frac{(k_2-k_1)}{2}z}}{e^{i\frac{(k_2-k_1)}{2}z} + e^{-i\frac{(k_2-k_1)}{2}z}} = \tan\left(\frac{k_2-k_1}{2}z\right)$$

$$\Delta\phi = \frac{k_2 - k_1}{2}z \tag{11}$$

For high frequencies

$$k_2 - k_1 = \frac{\omega}{c}\left(\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec}-\omega)}} - \sqrt{1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}}\right) \approx \frac{\omega}{c}\left(1 + \frac{\omega_{ep}^2}{2\omega(\omega_{ec}-\omega)} - 1 + \frac{\omega_{ep}^2}{2\omega(\omega_{ec}+\omega)}\right) =$$

$$= \frac{\omega_{ep}^2}{2c}\left(\frac{1}{\omega_{ec}-\omega} + \frac{1}{\omega_{ec}+\omega}\right) = \frac{\omega_{ep}^2}{2c}\left(\frac{\omega_{ec}+\omega+\omega_{ec}-\omega}{\omega^2\left(\frac{\omega_{ec}^2}{\omega^2} - 1\right)}\right) \approx -\frac{\omega_{ep}^2\omega_{ec}}{c\omega^2}$$

Hence

$$\Delta\phi = -\frac{\omega_{ep}^2\omega_{ec}}{2c\omega^2}z$$

Using $\omega_{ep}^2 = \frac{e^2 n_e}{\epsilon_0 m}$ and $\omega_{ec} = \frac{eB_0}{m}$

$$\Delta\phi = -\frac{e^3 n_e B_0}{2c\epsilon_0 m^2\omega^2}z$$

Or, in infinitesimally

$$\frac{d\phi}{dz} = -\frac{e^3}{2c\epsilon_0 m^2\omega^2}n_e B_0 \tag{12}$$

Therefore, we can determine the concentration of plasma or external magnetic field by examining the polarization of light. If the light passes through part of plasma where n_e or B_0 is not uniform, we then have

$$\Delta\phi = \int_L \frac{d\phi}{dz}dz = \frac{-e^3}{2c\epsilon_0 m^2\omega^2}\int_L n_e B_0 dz$$

where L is the path the light travels.

3.4 Waves Propagating Perpendicular to External Field

In this case, assume $N_z = 0$ and choose for example $N_x = \frac{ck}{\omega}$ and $N_y = 0$ (the choice is arbitrary and does not influence the result). The secular equation becomes

$$\begin{vmatrix} \epsilon_1 - \lambda & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 - N_x^2 - \lambda & 0 \\ 0 & 0 & \epsilon_3 - N_x^2 - \lambda \end{vmatrix} = 0$$

$$(\epsilon_3 - N_x^2 - \lambda)\left[(\epsilon_1 - \lambda)(\epsilon_1 - N_x^2 - \lambda) - \epsilon_2^2\right] = 0$$

First solution is when $\lambda_1 = \epsilon_3 - N_x^2$. Then

$$\begin{pmatrix} \epsilon_1 + N_x^2 - \epsilon_3 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 - \epsilon_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

This can generally apply only when $a_1 = a_2 = 0$ and thus

$$\vec{a}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These waves are polarized in the same direction as external field B_0 and travel transversly to this polarization. The dispersion relation is

$$0 = \lambda_1 = \epsilon_3 - N_x^2 = 1 - \frac{\omega_{ep}^2}{\omega^2} - \frac{c^2k^2}{\omega^2}$$

$$\omega^2 = c^2k^2 + \omega_{ep}^2 \tag{13}$$

This is the same behaviour as in the zero field waves. These waves are therefore called the ordinary waves. Again, the oscillations of $\vec{E}$ are parallel to $\vec{B}_0$, therefore the motion of electrons $\vec{v} \parallel \vec{B}_0$, so it makes sense that the presence of the external field does not influence this motion.
For the other two eigenvalues, we have

$$(\epsilon_1 - \lambda)(\epsilon_1 - N_x^2 - \lambda) - \epsilon_2^2 = 0$$

$$\lambda^2 - \left(\epsilon_1 - N_x^2 + \epsilon_1\right)\lambda + \epsilon_1(\epsilon_1 - N_x^2) - \epsilon_2^2 = 0$$

$$\lambda = \frac{1}{2}\left(2\epsilon_1 - N_x^2 \pm \sqrt{(2\epsilon_1 - N_x^2)^2 + 4(\epsilon_2^2 - \epsilon_1^2 + \epsilon_1 N_x^2)}\right) = \frac{1}{2}\left(2\epsilon_1 - N_x^2 \pm \sqrt{N_x^4 + 4\epsilon_2^2}\right)$$

Therefore

$$\begin{pmatrix} \frac{N_x^2}{2} \mp \sqrt{\frac{N_x^4}{4} + \epsilon_2^2} & -i\epsilon_2 & 0 \\ i\epsilon_2 & -\frac{N_x^2}{2} \mp \sqrt{\frac{N_x^4}{4} + \epsilon_2^2} & 0 \\ 0 & 0 & \epsilon_3 - \epsilon_1 - \frac{N_x^2}{2} \mp \sqrt{\frac{N_x^4}{2} + \epsilon_2^2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Right away we see that $a_3 = 0$. However, requirements on a_1 and a_2 are not so obvious. The equations are

$$\left(\frac{N_x^2}{2} \mp \sqrt{\frac{N_x^4}{4} + \epsilon_2^2}\right) a_1 - i\epsilon_2 a_2 = 0$$

$$i\epsilon_2 a_1 - \left(\frac{N_x^2}{2} \pm \sqrt{\frac{N_x^4}{4} + \epsilon_2^2}\right) a_2 = 0$$

Multiplying second equation by $i\epsilon_2$

$$-\epsilon_2^2 a_1 - \left(\frac{N_x^2}{2} \pm \sqrt{\frac{N_x^4}{4} + \epsilon_2^2}\right)(i\epsilon_2 a_2) = 0$$

Substituting in for $(i\epsilon_2 a_2)$ from the first equation

$$-\epsilon_2^2 a_1 - \left(\frac{N_x^2}{2} \pm \sqrt{\frac{N_x^4}{4} + \epsilon_2^2}\right)\left(\frac{N_x^2}{2} \mp \sqrt{\frac{N_x^4}{4} + \epsilon_2^2}\right) a_1 = 0$$

$$\epsilon_2^2 a_1 + \left(\frac{N_x^4}{4} - \frac{N_x^4}{4} - \epsilon_2^2\right) a_1 = 0$$

But, we can see that this equation applies for any value of a_1. Therefore, there will be waves polarized somehow in the xy plane. But, before we explore these further, we will solve the dispersion relation, which will help us simplify the expression for the eigenmodes.
The dispersion relation is best obtained by setting $\lambda = 0$ in $(\epsilon_1 - \lambda)(\epsilon_1 - N_x^2 - \lambda) - \epsilon_2^2 = 0$, which leads to

$$\epsilon_1^2 - \epsilon_1 N_x^2 = \epsilon_2^2$$

$$N_x^2 = \frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_1} = \frac{1 + 2\frac{\omega_{ep}^2}{\omega_{ec}^2 - \omega^2} + \frac{\omega_{ep}^4}{(\omega_{ec}^2 - \omega^2)^2} - \frac{\omega_{ec}^2}{\omega^2} \times \frac{\omega_{ep}^4}{(\omega_{ec}^2 - \omega^2)^2}}{1 + \frac{\omega_{ep}^2}{\omega_{ec}^2 - \omega^2}}$$

This equation is rather complicated, so I present only the result. It follows that

$$k = \frac{1}{c}\sqrt{\frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}{\omega^2 - \omega_H^2}} \tag{14}$$

where ω_1 and ω_2 have the same meanings as before and $\omega_H = \sqrt{\omega_{ep}^2 + \omega_{ec}^2}$ is called the upper hybrid frequency. We can again see that there will be a hybrid resonance at $\omega = \omega_H$ and that ω_1 and ω_2 will play the roles of cut-off frequencies. At the resonance, the wave is again absorbed rather than reflected.
For the eigenmodes, it is sufficient to find out that

$$\frac{N_x^2}{2} = \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{2\epsilon_1}$$

$$\frac{N_x^4}{4} = \frac{\epsilon_1^2}{4} - \frac{\epsilon_2^2}{2} + \frac{\epsilon_2^4}{4\epsilon_1^2}$$

$$\frac{N_x^4}{4} + \epsilon_2^2 = \frac{\epsilon_1^2}{4} + \frac{\epsilon_2^2}{2} + \frac{\epsilon_2^4}{4\epsilon_1^2} = \left(\frac{\epsilon_1}{2} + \frac{\epsilon_2^2}{2\epsilon_1}\right)^2$$

Hence, we have two modes with the same dispersion relation. One mode is given by (using only part of the matrix that mixes components in xy plane, as $a_3 = 0$)

$$\begin{pmatrix} \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{2\epsilon_1} - \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{2\epsilon_1} & -i\epsilon_2 \\ i\epsilon_2 & -\frac{\epsilon_1}{2} + \frac{\epsilon_2^2}{2\epsilon_1} - \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{2\epsilon_1} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon_2^2}{\epsilon_1} & -i\epsilon_2 \\ i\epsilon_2 & -\epsilon_1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

which is satisfied when $a_2 = i\frac{\epsilon_2}{\epsilon_1}a_1$. So, we have

$$\vec{a}_1 = \begin{pmatrix} 1 \\ i\frac{\epsilon_2}{\epsilon_1} \\ 0 \end{pmatrix}$$

which corresponds to an elliptically polarized wave, with the exact ratio of E_x/E_y dependant on the frequency of the wave. This mode is right-hand elliptically polarized.
Similarly, for the other eigenmode

$$\begin{pmatrix} \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{2\epsilon_1} + \frac{\epsilon_1}{2} + \frac{\epsilon_2^2}{2\epsilon_1} & -i\epsilon_2 \\ i\epsilon_2 & -\frac{\epsilon_1}{2} + \frac{\epsilon_2^2}{2\epsilon_1} + \frac{\epsilon_1}{2} + \frac{\epsilon_2^2}{2\epsilon_1} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 & -i\epsilon_2 \\ i\epsilon_2 & \frac{\epsilon_2^2}{\epsilon_1} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

which is satisfied when $a_2 = -i\frac{\epsilon_1}{\epsilon_2}a_1$.

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -i\frac{\epsilon_1}{\epsilon_2} \\ 0 \end{pmatrix}$$

This mode is left-hand elliptically polarized.
We also see that both modes have oscillations in x direction and y direction - extraordinary waves are mix of transverse and longtitudal waves.
The overview of the dispersion relations is presented in Fig. 4

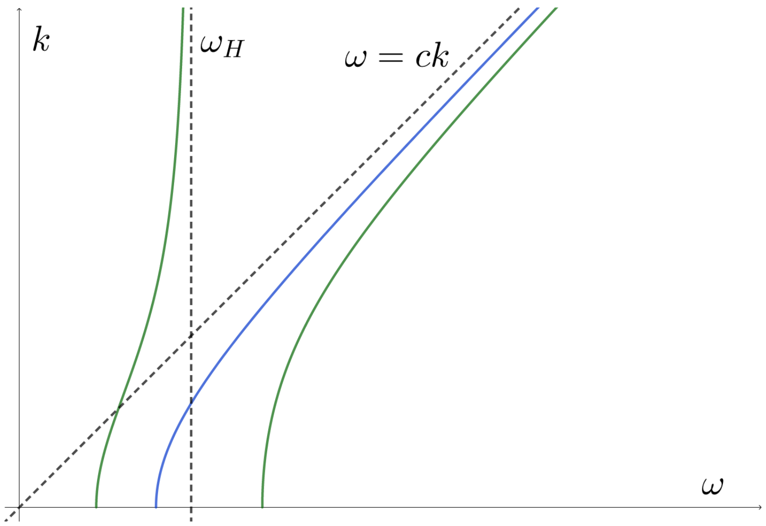

Figure 4: The dispersion relations in waves travelling perpendicularly to the external field. The blue line represents ordinary waves, the green line represents extraordinary waves.

3.4.1 Heating Plasma

We can remotly heat the plasma by radiating it at resonance frequencies. If we apply an EM wave parallel to the direction of the external $\vec{B}_0$ field, we can use the RH circularly polarized wave to hit the electron cyclotron resonance at $\omega = \omega_{ec}$. If we are providing radiation perpendicular to the external field $\vec{B}_0$, we can hit the upper hybrid resonance $\omega = \sqrt{\omega_{ep}^2 + \omega_{ec}^2}$

4 Plasma Kinetics

Here, we will briefly discuss some very general phenomena that apply to plasma that might not be ideal or not in perfect equilibrium.

4.1 Temperature Effects on Plasma

When there is a non-zero temperature present in the plasma, electrons tend to move around on their own. This creates an electronic pressure, which makes modifications to some of the observed waves in plasma. Generally, when there is a random fluctuation of electron movement, there is a gradient of the concentration of electrons established, and therefore a force $\vec{F} \propto -\nabla n_e$ is created. Modelling the electrons as an ideal gas, we can derive that

$$p = \frac{Nk_BT}{V} = n_e k_B T$$

where p is the pressure of the electrons and T their temperature. And thus

$$\vec{F} \propto -\nabla \frac{p}{k_BT}$$

Consider now that the pressure gradient exists on the scale of the waves $\lambda = \frac{2\pi}{k}$. Therefore

$$\vec{F} \approx \frac{1}{2\pi k_B T}\vec{k}p$$

Which leads to addition of effective sound waves into the system. These are longtitudal waves in the electron concentration, occuring without any external field. This means that they will modify the electron-plasma oscillation mode. This mode now becomes

$$\omega^2 = \omega^2 + c_s^2 k^2$$

where c_s is the speed of sound in the electron gas

$$c_s = \sqrt{\frac{\gamma_e k_B T}{m}}$$

where γ_e is the Poisson constant for the electron gas. This means that there is some dispersion even for the electron plasma oscillations, although for small temperatures, c_s is very small and therefore the effect is also very weak.

4.2 Vlasov Equation

If we want to describe certain system with most detail, we need a distribution function f associated with the system. This distribution function is usually the function of position $\vec{r}$, velocity $\vec{v}$ and time t and describes a probability that some part of the system is at position $\vec{r}$ with velocity $\vec{v}$ at time t. For a system of many particles, f can describe the number of particles that are at position $\vec{r}$ and have speed $\vec{v}$ at time t. We could then normalize f to correspond to probability that the particles are in such state by defining normalized distribution function

$$F(\vec{r}, \vec{v}, t) = \frac{1}{n_0} f(\vec{r}, \vec{v}, t)$$

where n_0 is the concentration of the particles at equilibrium.
We can retrieve information from the distribution function using different integral operations, such as

$$N(t) = \iiint_{space} \iiint_{velocity} f(\vec{r}, \vec{v}, t) d^3v d^3r$$

where N is the total number of particles in the system.
Very useful constraint on f can be made by requiring that the number of these particles is conserved, i.e.

$$\frac{dN}{dt} = 0$$

Since the integral in definition of N does not run over time, we can say that

$$0 = \frac{d}{dt} \iiint_{space} \iiint_{velocity} f(\vec{r}, \vec{v}, t) d^3v d^3r = \iiint_{space} \iiint_{velocity} \frac{df}{dt} d^3v d^3r$$

Therefore, we have the Boltzmann equation

$$\frac{df}{dt} = 0$$

Expanding the total differential, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dr_i}{dt} \frac{\partial f}{\partial r_i} + \frac{dv_i}{dt} \frac{\partial f}{\partial v_i} = 0$$

where Einstain summation convention is used and we sum over all spatial indices. We could also define that

$$\frac{\partial f}{\partial \vec{r}} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

and then rewrite the Boltzmann equation as

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{15}$$

where I used $\vec{v} = \frac{d\vec{r}}{dt}$ and Newton's second law $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$.
If we now assume that our system of particles is plasma which is collisionless, the only force acting on the particles is the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Then, for electrons with $q = -e$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} - \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{16}$$

This equation is called the Vlasov equation and is the basis of kinetic description of plasma. However, calculations with it are very complicated, and therefore we will discuss its use only very briefly.

4.3 Waves in Vlasov Equation

In a perfect equilibrium state with no external fields, we expect the system to be uniform and stationary, i.e.

$$f = f_0(\vec{v})$$

Consider now a small wave-like perturbation to the system, which adds a perturbation distribution function $f_1(\vec{r}, \vec{v}, t)$ to the total distribution function and similar perturbations to all other variables. The Vlasov equation then is

$$0 = \frac{\partial(f_0 + f_1)}{\partial t} + \vec{v} \cdot \frac{\partial(f_0 + f_1)}{\partial \vec{r}} - \frac{e}{m}\vec{E}_1 \cdot \frac{\partial(f_0 + f_1)}{\partial \vec{v}} = 0$$

where I used the fact that for wave-like perturbations, effect of $\vec{B}$ is much smaller than effect of $\vec{E}$. Using the definition of f_0 and retaining only the expressions up to the first order in perturbation, we have

$$0 = \frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{r}} - \frac{e}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}}$$

Using the wave-like nature of f_1, $\frac{\partial f_1}{\partial t} = -i\omega f_1$, $\frac{\partial f_1}{\partial \vec{r}} = i\vec{k} f_1$ and therefore, the Vlasov equation becomes

$$0 = -i\omega f_1 + \vec{v} \cdot i\vec{k} f_1 - \frac{e}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}}$$

We now however have an equation dependent on the field $\vec{E}_1$, which is created by the perturbation and we would like to determine from f_1. To get rid of this dependence, we will use the 1st Maxwell equation

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0} = -\frac{en_e}{\epsilon_0}$$

where n_e is the excess concentration of electrons. For our wavelike perturbation, this becomes

$$i\vec{k} \cdot \vec{E}_1 = -\frac{en_e}{\epsilon_0}$$

Here, however, we have only introduced another unknown - n_e, which is also perturbed. But, we can express the concentration of electrons n_e in integral form as

$$n_e = \iiint_{velocity} f_1 d^3v$$

To make the model simpler, consider now a 1D case. The Vlasov equation is

$$0 = -i\omega f_1 + iv_x k_x f_1 - \frac{e}{m}E_{1x}\frac{\partial f_0}{\partial v_x}$$

Hence

$$f_1 = \frac{e}{m}E_{1x}\frac{\partial f_0}{\partial v_x} \times \frac{1}{iv_x k_x - i\omega}$$

1st Maxwell equation is

$$ik_x E_{1x} = -\frac{e}{\epsilon_0}\iiint_{velocity} f_1 d^3v$$

Hence

$$E_{1x} = i\frac{e}{\epsilon_0 k_x}\iiint_{velocity} f_1 d^3v$$

Thus, by combining these two equations

$$E_{1x} = i\frac{e}{\epsilon_0 k_x}\iiint_{velocity} \frac{ie}{m(\omega - v_x k_x)}E_{1x}\frac{\partial f_0}{\partial v_x} d^3v$$

As only f_0 depends on the velocity v, we can factor out most of the terms inside the integral.

$$E_{1x} = -\frac{e^2}{\epsilon_0 k_x m}E_{1x}\iiint_{velocity} \frac{\frac{\partial f_0}{\partial v_x}}{\omega - v_x k_x} d^3v$$

Hence, we can divide by E_{1x}. Also, as we are in 1D model, we know that $f_0(\vec{v}) = \delta(v_y)\delta(v_z)f_{0x}(v_x)$, so we have

$$1 = -\frac{e^2}{\epsilon_0 k_x m}\int_{-\infty}^{\infty}\frac{\frac{\partial f_{0x}}{\partial v_x}}{\omega - v_x k_x}dv_x \tag{17}$$

This equation sets up a condition for the relation between ω and k_x - this is the dispersion relation in terms of distribution function f. If we change to normalized distribution function $F = \frac{f_{0x}}{n_0}$

$$1 = -\frac{e^2 n_0}{\epsilon_0 k_x m}\int_{-\infty}^{\infty}\frac{\frac{\partial f_{0x}/n_0}{\partial v_x}}{\omega - v_x k_x}d_x^v = -\frac{\omega_{ep}^2}{k_x}\int_{-\infty}^{\infty}\frac{\frac{\partial F}{\partial v_x}}{\omega - v_x k_x}$$

We can integrate by parts

$$1 = -\frac{\omega_{ep}^2}{k_x}\left(\left[\frac{F}{\omega - k_x v_x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{k_x F}{(\omega - k_x v_x)^2}dv_x\right)$$

For a bounded velocity profile, $f_{0x}(v_x = \pm\infty) = 0$, and therefore we have

$$1 = \omega_{ep}^2\int_{-\infty}^{\infty}\frac{F}{(k_x v_x - \omega)^2}dv_x \tag{18}$$

It can be shown that for a certain sensible choice of F, we can then obtain dispersion relation as

$$1 - \frac{\omega_{ep}^2}{\omega^2} - \frac{c_s^2 k_x^2}{\omega^2} - i\pi\frac{\omega_{ep}^2}{k_x^2}\frac{\partial F}{\partial v_x}\Big|_{v_x = \frac{\omega}{k_x}} = 0$$

We can see that there are contributions from the electron-plasma oscillations, sound waves and some additional contribution which makes ω generally complex. This gives rise to phenomena such as Landau damping and bump-on-tail instability.
We can see that the imaginary part of ω will be proportional to $\frac{\partial F}{\partial v_x}$. This means that any wave-like quantity (for example the electric field) will follow

$$E_{1x} \propto e^{-i(i\,\mathrm{Im}(\omega)t)} = e^{\mathrm{Im}(\omega)t} \propto e^{\frac{\partial F}{\partial v_x}t}$$

Therefore, if the distribution function is steadily decreasing with v_x, as is the case for example for Boltzmann distribution F, any wavelike perturbation will tend to decay away with characteristic time

$$t_D = \frac{1}{\mathrm{Im}(\omega)}$$

But, if at some point there is a bump in the distribution function, then for some v_x, the derivative is positive, and therefore we have exponential growth of the wave-like perturbation. This is called the bump-on-tail instability.

4.3.1 Origin of Imaginary Part

Consider integrating (18) as a contour integral, assuming that closing the contour by a half-circle through the upper half plane does not change the integral.
Then

$$\int_{-\infty}^{\infty}\frac{F}{(k_x v_x - \omega)}dv_x = \frac{1}{k_x}\int_C\frac{F}{(z-\omega)^2}dz$$

where $z = k_x v_x$. This contour goes through one singularity that lies at $z = \omega$. This is a pole of second order (assuming that F is bounded along real line), therefore the residue of the integrand at this point is

$$\mathrm{Res}\left[\frac{F}{(z-\omega)^2}, z = \omega\right] = \frac{\partial F}{\partial z}\Big|_{z=\omega}$$

Hence the value of the integral is

$$\int_C\frac{F}{(z-\omega)^2}dz = \frac{1}{2}\times 2\pi i\frac{\partial F}{\partial z}\Big|_{z=\omega} = i\pi\frac{\partial v_x}{\partial z}\frac{\partial F}{\partial v_x}\Big|_{v_x=\frac{\omega}{k}} = \frac{i\pi}{k_x}\frac{\partial F}{\partial v_x}\Big|_{v_x=\frac{\omega}{k}}$$

where the factor of one half is due to the fact that the function goes through the pole. Thus

$$\int_{-\infty}^{\infty} \frac{F}{(k_x v_x - \omega)} dv_x = i\pi \frac{1}{k_x^2} \frac{\partial F}{\partial v_x}\Big|_{v_x = \frac{\omega}{k}}$$

which is exactly the expression we see in the correct form, but with missing real parts. These are probably missing because F is not fully analytic function and the integral of the remainder of F that makes it analytic produces the real parts.

5 Magneto-Hydro Dynamics

Magneto-Hydro dynamics is a discipline studying behaviour of plasma on big scales and on timescales long enough that any motion of electrons could reach equilibrium. Generally, we could describe it in the Vlasov distribution function formalism, but this is very complicated, so we will rather use formalism of classical fluid dynamics.
In order for this to be appropriate, we assume mainly 3 things

1. Characteristic time of investigated processes is much larger than the time of electron motion processes (e.g. $\frac{2\pi}{\omega_{ep}}$, $\frac{2\pi}{\omega_{ec}}$)
2. Characteristic spatial scale of the problem is much larger than the scale of electron processes (e.g. radius of gyration)
3. Bulk velocities of the plasma are non-relativistic

Because we are at such long timescales, we can approximate plasma as a very good conductor in sense that any imbalance in charge density will tend to be negated by electron flow almost immediately. Therefore, the internal EM fields will be dominated by the magnetic $\vec{B}$ field.
As a quick illustration, consider a that a field $\vec{E}'$ is created in the rest frame of the plasma. By Ohm's law

$$\vec{j} = g\vec{E}'$$

where $\vec{j}$ is the current density and g is the conductivity - some very large number. Hence, any field in the rest frame of the plasma is

$$\vec{E}' = \frac{1}{f}\vec{j} \approx 0$$

This is especially important if we switch to some other frame of reference, where we observe fields $\vec{E}$ and $\vec{B}$. The non-relativistic Lorentz transformations lead to

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

which implies

$$\vec{E} \approx -\vec{v} \times \vec{B} \tag{19}$$

Therefore, we can describe the behaviour of plasma at these scales using only the magnetic field $\vec{B}$.

5.1 Equations of Magneto-Hydro Dynamics

Now, lets start building the differential equations needed to describe the plasma. Lets start with the classical fluid equations. There are two equations in play.First one is the continuity equation, which is straightforward

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{20}$$

where ρ is the mass density and $\vec{v}$ is the speed of the plasma flow. Second equation is the Navier-Stokes' equation. In our case, we will suppose that the plasma is collision less and hence inviscid, which will transform the equation into Euler's equation, which states

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = \vec{\Phi}$$

where $\vec{\Phi}$ is the force density in the fluid. There are two main forces acting on the plasma - the force due to pressure gradient and the Lorentz force due to field $\vec{B}$. Therefore, we can write

$$\vec{\Phi} = -\nabla p + \frac{q\vec{v} \times \vec{B}}{V} = -\nabla p + \vec{j} \times \vec{B}$$

where p is the pressure and V is some volume. Therefore

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = -\nabla p + \vec{j} \times \vec{B} \tag{21}$$

Finally, we need a way how to relate the pressure and the mass density of the processes. Usually, this is done via some equation of state. In our case, this can be an equation of ideal gas

$$p = \frac{2k_B}{m_i} \rho T$$

where m_i is the mass of the ion (which dominates the mass of the particles) and the factor of 2 is present because pressure is due to both ions and electrons.
But, more generally, for adiabatic processes in the plasma (flowing without external heating), we can write

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0 \tag{22}$$

Therefore, we have our fluid equations. Now, we need to explore Maxwell equations. Again, the first Maxwell equation is not of interest, as any excess charge density ρ_q will be quickly negated by electron flows. Second Maxwell equation is sometimes useful, but as we will see, not always neccessary. Third Maxwell equation is very interesting, as it describes how the magnetic field is induced in the plasma. Using $\vec{E} = -\vec{v} \times \vec{B}$, we have

$$\nabla \times \vec{E} = -\nabla \times (\vec{v} \times \vec{B}) = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{23}$$

Fourth Maxwell equation again has a term

$$\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = -\frac{1}{c^2} \frac{\partial (\vec{v} \times \vec{B})}{\partial t}$$

But, since time scales are relatively long and $|\vec{v}| \ll c$, we can neglect this term. The terms that are left are

$$\nabla \times \vec{B} \approx \mu_0 \vec{j}$$

Now, we have all the tools needed. If we substitute this last result into the Euler's equation (21), we have a set of two scalar and two vector equations, with two scalar and two vector unknowns, which is solvable. The set is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{24}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} \tag{25}$$

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0 \tag{26}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{27}$$

These equations form a closed set. Therefore, all the other equations, such as the state equation, fourth Maxwell equation or Ohm's law can be viewed as only constitutive relations - useful for calculating variables other than the ones used in equations above, which are p, ρ (scalars), $\vec{B}$ and $\vec{v}$ (vectors).
Clearly, solving these equations is very hard, as the equations are generally non-linear. We can however do several simplifications.

5.2 Static Equilibrium

Consider a motionless plasma in static equilibrium. This means that any time derivatives, both partial and total, equate to zero, and the velocity of the plasma is zero. This means that (24),(26) and (27) all become identically zero. The only equation left is therefore the Euler's equation (25), which now has form

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B}$$

Using identity

$$\nabla(\vec{B} \cdot \vec{B}) = 2(\vec{B} \cdot \nabla)\vec{B} + 2\vec{B} \times (\nabla \times \vec{B})$$

we have

$$(\nabla \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \nabla)\vec{B} - \frac{1}{2}\nabla(\vec{B} \cdot \vec{B})$$

and therefore

$$\nabla p = -\frac{1}{2\mu_0}\nabla(|\vec{B}|^2) + (\vec{B} \cdot \nabla)\vec{B}$$

$$\nabla \left(p + \frac{|\vec{B}|^2}{2\mu_0} \right) = (\vec{B} \cdot \nabla)\vec{B} \tag{28}$$

This enables us to define two specific variables - the magnetic pressure $p_m = \frac{|\vec{B}|^2}{2\mu_0}$ and magnetic tension $\vec{T}_B = (\vec{B} \cdot \nabla)\vec{B}$. These both act effectively as extra pressure or tension on the plasma, and their estimates are a good way how to predict qualitative behaviour of plasma.

5.2.1 Plasma Jet

Suppose we have a cylindrical stream of plasma which carries current $\vec{j}$. The magnetic field induced is cylindrical and decreases as we move away from the cylinder. This means that the gradient of the magnetic field is directed inwards, and so is the magnetic tension on the plasma. Therefore, the magnetic tension of the plasma can balance the thermal pressure of the plasma and prevent purely radial diffusion of the jet. The jet is however not stable to other deformations, as we will see later.

5.2.2 Plasma β

Plasma β is a dimensionless parameter which characterizes whether the thermal or magnetic terms dominate the behaviour of the plasma. We can estimate it by comparing terms in static Euler equation

$$\nabla p = \frac{1}{\mu_0}(\nabla \vec{B}) \times \vec{B}$$

For a spatial scale λ, this becomes

$$\frac{p}{\lambda} \approx \frac{B^2}{\mu_0 \lambda}$$

$$\frac{p\mu_0}{B^2} = 1$$

Therefore we define

$$\beta = \frac{p\mu_0}{B^2} \tag{29}$$

with $\beta \ll 1$ implying that magnetic effects dominate, while $\beta \gg 1$ implying that thermal effects dominate.

5.2.3 Sunspots

Sunspots are places on the Sun where the magnetic field emerges from the surface. The field emerges radially from the surface and does not change much radially. This means that

$$(\vec{B} \cdot \nabla)\vec{B} = (B_r \frac{\partial}{\partial r})\vec{B} \approx 0$$

so the magnetic tension is zero. The static equation for the plasma then becomes

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=const.$$

Labeling p_E the pressure outside the sunspot, p_0 the pressure inside the sunspot and B_0 the magnetic field inside the sunspot (assuming that outside the magnetic field is zero), we have

$$p_E=p_0+\frac{B_0^2}{2\mu_0}$$

This means that the pressure at the sunspot is lower than in its surroundings, and the sunspot is supported by the magnetic pressure. Alternatively, we can use the state equation to write (assuming that the density ρ is somewhat similar, which is a reasonable assumption)

$$2\frac{k_B}{m_i}\rho T_0=2\frac{k_B}{m_i}\rho T_E-\frac{B_0^2}{2\mu_0}$$

where T_0 is the temperature inside the sunspot and T_E temperature outside. So

$$T_0=T_E-\frac{m_iB_0^2}{4\mu_0k_B\rho}$$

This means that the temperature in the sunspot is lower than outside - hence why it appears black.

5.3 Freezing-in of the Magnetic Field

The movement of plasma induces currents in the plasma, causing generation of magnetic field. A curious result of this is that the fieldlines of $\vec{B}$ appear to always follow the plasma flow - they are effectively frozen in the plasma. Microscopically, this can be explained by the fact that the particles always follow the field lines via gyration - they always flow along the fieldlines of $\vec{B}$. In fact, we cannot really distinguish what is the cause and effect - whether the particles are stuck following the fieldlines or the fieldlines are frozen in the plasma.
To illustrate this, consider a $\vec{B}=B_0\hat{j}$ field and velocity field $\vec{v}=\alpha y\hat{i}$ - velocity in x direction steadily increasing as y increases. The induction equation then states

$$\frac{\partial\vec{B}}{\partial t}=\nabla\times(\alpha yB_0\hat{k})=\alpha B_0\hat{i}$$

This means that $\vec{B}$ is increasing in the x direction - same direction as the speed $\vec{v}$ is increasing in. We should note that this is only very crude illustration.

5.4 MHD Waves

In order to discover which waves can exist in the plasma on these scales, we need to linearize the MHD equations. Lets then suppose we have some equilibrium density ρ_0, pressure p_0 and magnetic field $\vec{B}_0$, all of which are static and uniform. Furthermore, the equilibrium velocity $\vec{v}_0=0$. Then, we allow for some small wave-like perturbations to all these variables, i.e. $\vec{B}=\vec{B}_0+\vec{B}_1$, $\rho=\rho_0+\rho_1$, $p=p_0+p_1$ and $\vec{v}=\vec{v}_1$. Assume that the wave propagates in the z direction, and that the equilibrium $\vec{B}_0$ field is at an angle θ to this direction. Without any loss of generality, we can chose that $\vec{B}_0$ lies in the xz plane, so that $\vec{B}_0=(B_0\sin\theta,0,B_0\cos\theta)$. We substitute this into our MHD equations and retain only terms that are up to the first order in perturbative terms.
The continuity equation becomes

$$\frac{\partial(\rho_0+\rho_1)}{\partial t}+\nabla\cdot((\rho_0+\rho_1)(\vec{v}_1))=0$$

As ρ_0 is static and uniform, we have

$$\frac{\partial\rho_1}{\partial t}+\rho_0\nabla\cdot\vec{v}_1=0$$

where I neglected the second order term $\rho_1\vec{v}_1$. Applying the wave-like nature of ρ_1 and $\vec{v}_1$, the equation becomes

$$-\omega\rho_1+\rho_0kv_{1z}=0 \tag{30}$$

The adiabatic equation becomes

$$\frac{d}{dt}\left(\frac{p_0+p_1}{(\rho_0+\rho_1)^\gamma}\right)=0$$

$$\frac{\frac{dp_1}{dt}(\rho_0+\rho_1)^\gamma-(p_0+p_1)\frac{d(\rho_0+\rho_1)^\gamma}{dt}}{(\rho_0+\rho_1)^{2\gamma}}=0$$

This can only apply if the numerator is equal to zero. Using the wave nature of p_1

$$-i\omega p_1(\rho_0+\rho_1)^\gamma-(p_0+p_1)\gamma(\rho_0+\rho_1)^{\gamma-1}\frac{d\rho_1}{dt}=0$$

Binomially expanding the density terms up to first order in $\frac{\rho_1}{\rho_0}$ and using wave nature of ρ_1

$$-i\omega p_1\rho_0^\gamma\left(1+\gamma\frac{\rho_1}{\rho_0}\right)-(p_0+p_1)\gamma\rho_0^{\gamma-1}\left(1+(\gamma-1)\frac{\rho_1}{\rho_0}\right)(-i\omega\rho_1)=0$$

Retaining only the terms up to first perturbation order

$$-i\omega p_1\rho_0^\gamma-p_0\gamma\rho_0^{\gamma-1}(-i\omega\rho_1)=0$$

$$\omega p_1-\omega\gamma p_0\frac{\rho_1}{\rho_0}=0 \tag{31}$$

Similarly, we can modify the Euler's equation

$$(\rho_0+\rho_1)\left(\frac{\partial(\vec{v}_0+\vec{v}_1)}{\partial t}+((\vec{v}_0+\vec{v}_1)\cdot\nabla)(\vec{v}_0+\vec{v}_1)\right)=-\nabla(p_0+p_1)+\frac{1}{\mu_0}(\nabla\times(\vec{B}_0+\vec{B}_1))\times(\vec{B}_0+\vec{B}_1)$$

$$(\rho_0+\rho_1)\left(\frac{\partial\vec{v}_1}{\partial t}+(\vec{v}_0\cdot\nabla)\vec{v}_1\right)=-\nabla p_1+\frac{1}{\mu_0}(i\vec{k}\times\vec{B}_1)\times\vec{B}_0$$

$$\rho_0(-i\omega)\vec{v}_1+\rho_0(\vec{v}_0\cdot(i\vec{k}))\vec{v}_1=-i\vec{k}p_1+\frac{1}{\mu_0}(i(-kB_{1y})\hat{i}+i(kB_{1x})\hat{j})\times(B_0(\sin\theta\hat{i}+\cos\theta\hat{k}))$$

As $\vec{v}_0=0$

$$-\rho_0\omega\vec{v}_1=-\vec{k}p_1+\frac{1}{\mu_0}B_0\left(-kB_{1y}\cos\theta\hat{i}\times\hat{k}+kB_{1x}\sin\theta\hat{j}\times\hat{i}+kB_{1x}\cos\theta\hat{j}\times\hat{k}\right)$$

$$-\rho_0\omega\vec{v}_1+k\hat{k}p_1-\frac{1}{\mu_0}B_0\left(kB_{1x}\cos\theta\hat{i}+kB_{1y}\cos\theta\hat{j}-kB_{1x}\sin\theta\hat{k}\right)=0$$

Hence we have three component equations. In x

$$-\rho_0\omega v_{1x}-\frac{1}{\mu_0}kB_0B_{1x}\cos\theta=0 \tag{32}$$

In y

$$-\rho_0\omega v_{1y}-\frac{1}{\mu_0}kB_0B_{1y}\cos\theta=0 \tag{33}$$

In z

$$-\rho_0\omega v_{1z}+kp_1+\frac{1}{\mu_0}kB_0B_{1x}\sin\theta=0 \tag{34}$$

The last equation left to linearize is the induction equation. This is

$$\frac{\partial(\vec{B}_0+\vec{B}_1)}{\partial t}=\nabla\times((\vec{v}_0+\vec{v}_1)\times(\vec{B}_0+\vec{B}_1))$$

$$-i\omega\vec{B}_1=\nabla\times(\vec{v}_1\times\vec{B}_0)$$

Using a vector calculus identity

$$-i\omega\vec{B}_1=(\nabla\cdot\vec{B}_0)\vec{v}_1-(\nabla\cdot\vec{v}_1)\vec{B}_0+(\vec{B}_0\cdot\nabla)\vec{v}_1-(\vec{v}_1\cdot\nabla)\vec{B}_0$$

$$-i\omega\vec{B}_1 = i(\vec{B}_0 \cdot \vec{k})\vec{v}_1 - i(\vec{k} \cdot \vec{v}_1)\vec{B}_0$$
$$-\omega\vec{B}_1 - B_0\cos\theta k\vec{v}_1 + kv_{1z}\vec{B}_0 = 0$$

We therefore have again three component equations. In x

$$-\omega B_{1x} - B_0\cos\theta k v_{1x} + kv_{1z}B_0\sin\theta = 0 \tag{35}$$

In y

$$-\omega B_{1y} - B_0\cos\theta k v_{1y} = 0 \tag{36}$$

In z

$$-\omega B_{1z} = 0 \tag{37}$$

Therefore, we have 8 linearized equations (30)-(37). These together form a dispersion relation relating to a 8-component vector of perturbations - $(p_1, \rho_1, \vec{B}_1, \vec{v}_1)$. Generally, we would have to solve for an 8x8 matrix eigenvectors, but we can simplify the problem by noticing that equations (33) and (36) form a closed set for B_{1y} and v_{1y}. Furthermore, we can also notice that (37) only applies for arbitrary B_{1z} when $\omega = 0$ - this means that only a translational mode (no oscillations) is possible for the waves in B_{1z} - we can disregard it as well.
Therefore, we are left with 2 separate dispersion relations, one with 2x2 matrix and other with 5x5 matrix.

5.5 Alfvén Waves

The 2x2 matrix dispersion represents the Alfvén waves in plasmas. The dispersion relation is

$$\begin{pmatrix} \rho_0\omega & \frac{1}{\mu_0}kB_0\cos\theta \\ kB_0\cos\theta & \omega \end{pmatrix}\begin{pmatrix} v_{1y} \\ B_{1y} \end{pmatrix} = 0$$

This is a bit problematic equation as the different components of the vector have different dimensions. We can correct this by setting the vector to $(\sqrt{\rho_0}v_{1y}, \frac{B_{1y}}{\sqrt{\mu_0}})$, which leads to

$$\begin{pmatrix} \sqrt{\rho_0}\omega & \frac{1}{\sqrt{\mu_0}}kB_0\cos\theta \\ \sqrt{\frac{1}{\rho_0}}kB_0\cos\theta & \sqrt{\mu_0}\omega \end{pmatrix}\begin{pmatrix} \sqrt{\rho_0}v_{1y} \\ \frac{B_{1y}}{\sqrt{\mu_0}} \end{pmatrix} = 0$$

Now, we have the vector component dimensions the same, but the dimensionality of different matrix rows is again different. This can be corrected by multiplying the second row by factor $\frac{1}{\sqrt{\mu_0}}$ and first row by factor $\frac{1}{\sqrt{\rho_0}}$. Then, we have

$$\begin{pmatrix} \omega & \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta \\ \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & \omega \end{pmatrix}\begin{pmatrix} \sqrt{\rho_0}v_{1y} \\ \frac{B_{1y}}{\sqrt{\mu_0}} \end{pmatrix} = 0$$

Solving the secular equation

$$\begin{vmatrix} \omega - \lambda & \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta \\ \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & \omega - \lambda \end{vmatrix} = 0$$
$$(\omega - \lambda)^2 = \frac{k^2B_0^2\cos^2\theta}{\mu_0\rho_0}$$
$$\lambda = \omega \pm \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta$$

Therefore the eigenmodes are given by

$$\begin{pmatrix} \mp\frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta \\ \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & \mp\frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta \end{pmatrix}\begin{pmatrix} \sqrt{\rho_0}v_{1y} \\ \frac{1}{\sqrt{\mu_0}}B_{1y} \end{pmatrix} = 0$$

which is satsified by

$$\frac{1}{\sqrt{\mu_0}}B_{1y} = \pm\sqrt{\rho_0 v_{1y}}$$

which means that the eigenvectors are

$$\vec{a}_{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

which means that the velocity v_{1y} either oscillates together with the oscillations in B_{1y} or in exactly opposite direction. Since the wave propagates in the z direction, Alfvén waves are transverse waves. Since there are no oscillations in ρ or p, the waves are also incompressive. The dispersion relation is set by $\lambda = 0$ and is

$$\omega^2 = \frac{k^2 B_0^2 \cos^2\theta}{\mu_0 \rho_0} \tag{38}$$

Again, as B is frozen in plasma, the oscillations in v and B in the same direction make sense. The oscillations in opposite directions than probably represent the solution for opposite charge of the ions.
The physical reason behind formation of the Alfvén waves is again the magnetic tension force, which tries to restore any gradients in $\vec{B}$ back towards $\vec{B}_0$.
The phase speed of Alfvén waves is

$$v_p = \frac{\omega}{k} = \frac{B_0 \cos\theta}{\sqrt{\mu_0 \rho_0}}$$

Usually, we define $c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$ and thus $v_p = c_A \cos\theta$.
Alfvén waves are ellastic and non-dispersive for a fixed direction of the propagation/field. However, there is one particularity of the Alfvén waves - although the phase velocity is in the direction of z, the group velocity for Alfvén waves is always in the direction of $\vec{B}_0$ - hence these two velocities do not need to coincide. To illustrate this, imagine several strings in a row. Lets say we have wave packets travelling along these strings, which are not running in parallel, but along a oblique line with respect to the strings. The phase speed of each packet is clearly the speed at which it travels down the string. The group velocity of all packets is the velocity at which the line connecting the packets moves, which is in a different direction than the packets themselves.

5.5.1 Standing Waves in Earth's Magnetic Field

Alfvén waves can propagate along the field lines of Earth's magnetic field in the ionosphere. At the ends of the ionosphere, the plasma ceases to exist - this is the same as setting boundary conditions for the waves along these fieldlines that at the ends of ionosphere, the oscillation amplitudes are zero. This means that standing Alfvén waves can form in the ionosphere, which can be used to derive information about the ionosphere. This is called the field line resonance, and on Earth, the periods of oscillations are in orders of seconds up to minutes.

5.6 Magnetoacoustic Waves

The other five equations have dispersion relation

$$\begin{pmatrix} -\frac{1}{\mu_0} k B_0 \cos\theta & -\rho_0 \omega & 0 & 0 & 0 \\ -\omega & -k B_0 \cos\theta & k B_0 \sin\theta & 0 & 0 \\ \frac{1}{\mu_0} k B_0 \sin\theta & 0 & -\rho_0 \omega & k & 0 \\ 0 & 0 & \rho_0 k & 0 & -\omega \\ 0 & 0 & 0 & \omega & -\omega\gamma\frac{p_0}{\rho_0} \end{pmatrix} \begin{pmatrix} B_{1x} \\ v_{1x} \\ v_{1z} \\ p_1 \\ \rho_1 \end{pmatrix} = 0$$

By using a little bit of dimensional analysis, we can figure out that the vector with the identical dimensions will be

$$\left(\frac{B_{1x}}{\sqrt{\mu_0}}, \sqrt{\rho_0} v_{1x}, \sqrt{\rho_0} v_{1z}, \frac{p_1}{\sqrt{p_0}}, \frac{\rho_1 c_A^2}{\sqrt{p_0}} \right)$$

Hence the dispersion relation becomes

$$\begin{pmatrix} -\frac{1}{\sqrt{\mu_0}} k B_0 \cos\theta & -\sqrt{\rho_0}\omega & 0 & 0 & 0 \\ -\omega\sqrt{\mu_0} & -\frac{1}{\sqrt{\rho_0}} k B_0 \cos\theta & \frac{1}{\sqrt{\rho_0}} k B_0 \sin\theta & 0 & 0 \\ \frac{1}{\sqrt{\mu_0}} k B_0 \sin\theta & 0 & -\sqrt{\rho_0}\omega & \sqrt{p_0} k & 0 \\ 0 & 0 & \sqrt{\rho_0} k & 0 & -\omega\frac{\sqrt{p_0}}{c_A^2} \\ 0 & 0 & 0 & \sqrt{p_0}\omega & -\omega\gamma\frac{p_0}{\rho_0}\frac{\sqrt{p_0}}{c_A^2} \end{pmatrix} \begin{pmatrix} \frac{B_{1x}}{\sqrt{\mu_0}} \\ \sqrt{\rho_0} v_{1x} \\ \sqrt{\rho_0} v_{1z} \\ \frac{p_1}{\sqrt{p_0}} \\ \frac{\rho_1 c_A^2}{\sqrt{p_0}} \end{pmatrix} = 0$$

I will make the equations to have dimensions of ω. Therefore, we have

$$\begin{pmatrix} \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & \omega & 0 & 0 & 0 \\ \omega & \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\cos\theta & -\frac{1}{\sqrt{\rho_0\mu_0}}kB_0\sin\theta & 0 & 0 \\ \frac{1}{\sqrt{\rho_0\mu_0}}kB_0\sin\theta & 0 & -\omega & \sqrt{\frac{p_0}{\rho_0}}k & 0 \\ 0 & 0 & \sqrt{\frac{\rho_0}{p_0}}c_A^2 k & 0 & -\omega \\ 0 & 0 & 0 & \omega & -\omega\gamma\frac{p_0}{\rho_0 c_A^2} \end{pmatrix}\begin{pmatrix} \frac{B_{1x}}{\sqrt{\mu_0}} \\ \sqrt{\rho_0}v_{1x} \\ \sqrt{\rho_0}v_{1z} \\ \frac{p_1}{\sqrt{p_0}} \\ \frac{\rho_1 c_A^2}{\sqrt{p_0}} \end{pmatrix} = 0$$

Remembering that $c_A = \frac{B_0}{\sqrt{\rho_0\mu_0}}$ and writing the acoustic speed of sound as $c_S = \sqrt{\gamma\frac{p_0}{\rho_0}}$, we can write the dispersion relation as

$$\begin{pmatrix} c_A k\cos\theta & \omega & 0 & 0 & 0 \\ \omega & c_A k\cos\theta & -c_A k\sin\theta & 0 & 0 \\ c_A k\sin\theta & 0 & -\omega & \frac{c_S}{\sqrt{\gamma}}k & 0 \\ 0 & 0 & \frac{\sqrt{\gamma}}{c_S}c_A^2 k & 0 & -\omega \\ 0 & 0 & 0 & \omega & -\omega\frac{c_S^2}{c_A^2} \end{pmatrix}\begin{pmatrix} \frac{B_{1x}}{\sqrt{\mu_0}} \\ \sqrt{\rho_0}v_{1x} \\ \sqrt{\rho_0}v_{1z} \\ \frac{p_1}{\sqrt{p_0}} \\ \frac{\rho_1 c_A^2}{\sqrt{p_0}} \end{pmatrix} = 0$$

By swapping some rows (which we are free to do) and multiplying some equations by -1

$$\begin{pmatrix} \omega & c_A k\cos\theta & -c_A k\sin\theta & 0 & 0 \\ c_A k\cos\theta & \omega & 0 & 0 & 0 \\ -c_A k\sin\theta & 0 & \omega & -\frac{c_S}{\sqrt{\gamma}}k & 0 \\ 0 & 0 & 0 & \omega & -\omega\frac{c_S^2}{c_A^2} \\ 0 & 0 & -\frac{\sqrt{\gamma}}{c_S}c_A^2 k & 0 & \omega \end{pmatrix}\begin{pmatrix} \frac{B_{1x}}{\sqrt{\mu_0}} \\ \sqrt{\rho_0}v_{1x} \\ \sqrt{\rho_0}v_{1z} \\ \frac{p_1}{\sqrt{p_0}} \\ \frac{\rho_1 c_A^2}{\sqrt{p_0}} \end{pmatrix} = 0$$

Now, we need to solve the secular equation

$$0 = \begin{vmatrix} \omega-\lambda & c_A k\cos\theta & -c_A k\sin\theta & 0 & 0 \\ c_A k\cos\theta & \omega-\lambda & 0 & 0 & 0 \\ -c_A k\sin\theta & 0 & \omega-\lambda & -\frac{c_S}{\sqrt{\gamma}}k & 0 \\ 0 & 0 & 0 & \omega-\lambda & -\omega\frac{c_S^2}{c_A^2} \\ 0 & 0 & -\frac{\sqrt{\gamma}}{c_S}c_A^2 k & 0 & \omega-\lambda \end{vmatrix} =$$

$$= (\omega-\lambda)\begin{vmatrix} \omega-\lambda & c_A k\cos\theta & -c_A k\sin\theta & 0 \\ c_A k\cos\theta & \omega-\lambda & 0 & 0 \\ -c_A k\sin\theta & 0 & \omega-\lambda & -\frac{c_S}{\sqrt{\gamma}}k \\ 0 & 0 & 0 & \omega-\lambda \end{vmatrix} - \frac{\sqrt{\gamma}}{c_S}c_A^2 k\begin{vmatrix} \omega-\lambda & c_A k\cos\theta & 0 & 0 \\ c_A k\cos\theta & \omega-\lambda & 0 & 0 \\ -c_A k\sin\theta & 0 & -\frac{c_S}{\sqrt{\gamma}}k & 0 \\ 0 & 0 & \omega-\lambda & -\omega\frac{c_S^2}{c_A^2} \end{vmatrix} =$$

$$= (\omega-\lambda)^2\begin{vmatrix} \omega-\lambda & c_A k\cos\theta & -c_A k\sin\theta \\ c_A k\cos\theta & \omega-\lambda & 0 \\ -c_A k\sin\theta & 0 & \omega-\lambda \end{vmatrix} + \sqrt{\gamma}c_S\omega k\begin{vmatrix} \omega-\lambda & c_A k\cos\theta & 0 \\ c_A k\cos\theta & \omega-\lambda & 0 \\ -c_A k\sin\theta & 0 & -\frac{c_S}{\sqrt{\gamma}}k \end{vmatrix} =$$

$$= (\omega-\lambda)^2\left((\omega-\lambda)\begin{vmatrix} \omega-\lambda & c_A k\cos\theta \\ c_A k\cos\theta & \omega-\lambda \end{vmatrix} - c_A k\sin\theta\begin{vmatrix} c_A k\cos\theta & -c_A k\sin\theta \\ \omega-\lambda & 0 \end{vmatrix}\right) -$$

$$-\omega c_S^2 k^2\begin{vmatrix} \omega-\lambda & c_A k\cos\theta \\ c_A k\cos\theta & \omega-\lambda \end{vmatrix} =$$

$$= (\omega-\lambda)\left[(\omega-\lambda)^2\left((\omega-\lambda)^2 - c_A^2k^2\cos^2\theta\right) - (\omega-\lambda)^2 c_A^2 k^2\sin^2\theta\right] - \omega c_S^2 k^2\left((\omega-\lambda)^2 - c_A^2k^2\cos^2\theta\right) = 0$$

It turns out that this is a 5th order equation that is not very easily solved. Therefore, I could not determine the eigenmodes properly here. However, we can make the dispersion relation alone simpler. We obtain the dispersion relation by setting $\lambda = 0$, which transforms the equation above into

$$\omega\left[\omega^2\left(\omega^2 - c_A^2k^2\cos^2\theta\right) - \omega^2 c_A^2k^2\sin^2\theta\right] - \omega c_S^2k^2\left(\omega^2 - c_A^2k^2\cos^2\theta\right) = 0$$

We can therefore factor out ω to get

$$\omega\left[\left(\omega^2 - c_S^2k^2\right)\left(\omega^2 - c_A^2k^2\cos^2\theta\right) - \omega^2 c_A^2k^2\sin^2\theta\right] = 0$$

We therefore have solution with $\omega = 0$ - a static translational solution. Other, more interesting solutions exist when

$$\left(\omega^2 - c_S^2 k^2\right)\left(\omega^2 - c_A^2 k^2 \cos^2\theta\right) - \omega^2 c_A^2 k^2 \sin^2\theta = 0 \tag{39}$$

This is the dispersion relation of so called magnetoacoustic waves. We can notice that it is bi-quadratic in ω^2, i.e. after multiplying through to get rid of all the brackets, we are left with

$$\omega^4 - \left(c_S^2 k^2 + c_A^2 k^2(\cos^2\theta + \sin^2\theta)\right)\omega^2 + c_S^2 c_A^2 k^4 \cos^2\theta = \omega^4 - \left(c_S^2 k^2 + c_A^2 k^2\right)\omega^2 + c_S^2 c_A^2 k^4 \cos^2\theta = 0$$

which can be solved as quadratic equation in ω^2. This leads to

$$\omega^2 = \frac{1}{2}\left(c_S^2 k^2 + c_A^2 k^2 \pm \sqrt{(c_S^2 k^2 + c_A^2 k^2)^2 - 4 c_S^2 c_A^2 k^4 \cos^2\theta}\right)$$

The minimum value of the expression under the square root occurs when $\cos^2\theta = 1$, and then, the expression becomes $(c_S^2 k^2 - c_A^2 k^2)^2$, which is always greater than or equal to zero - hence ω^2 is always real.
The maximum value of the expression under the square root occurs when $\cos^2\theta = 0$. Then, the smaller root is

$$\omega^2 = \frac{1}{2}\left(c_S^2 k^2 + c_A^2 k^2 - \sqrt{(c_S^2 k^2 + c_A^2 k^2)^2}\right) = 0$$

Hence, $\omega^2 \geq 0$, and therefore, two real frequencies are defined

$$\omega_F = \frac{k}{\sqrt{2}}\sqrt{c_S^2 + c_A^2 + \sqrt{(c_S^2 + c_A^2)^2 - 4 c_A^2 c_S^2 \cos^2\theta}} \tag{40}$$

$$\omega_S = \frac{k}{\sqrt{2}}\sqrt{c_S^2 + c_A^2 - \sqrt{(c_S^2 + c_A^2)^2 - 4 c_A^2 c_S^2 \cos^2\theta}} \tag{41}$$

Here, index F stands for "fast", index S for "slow". Magnetoacoustic waves are similar to acoustic waves in the sense that they are compressive, longtitudal and are driven by the pressure gradient forces. However, in plasma, the pressure gradient forces are modified by the magnetic pressure gradients, which creates more interesting behaviour.
Fast waves vary in speeds from maximum speed $c_F = \sqrt{c_S^2 + c_A^2}$ when travelling perpendicularly to the field ($\theta = \frac{\pi}{2}$) to speed zero when c_A when travelling along the field. But, in this case, the fast wave in fact degenerates to the Alfvén wave, and thus becomes incompressive. For fast wave, the the magnetic field and mass density oscillate in phase.
In the case when the slow wave propagates along the field with plasma $\beta < 1$, the slow wave travels at c_S and degenerates to standard acoustic wave. For the slow wave, the density and magnetic field oscillations are in anti-phase.
Since we could factor out the k dependence, both fast and slow wave are ellastic waves for a given direction of the field. The phase speed is therefore

$$v_{p,F/S} = \frac{1}{\sqrt{2}}\sqrt{c_S^2 + c_A^2 \pm \sqrt{(c_S^2 + c_A^2)^2 - 4 c_A^2 c_S^2 \cos^2\theta}}$$

The phase speeds of MHD waves and Afvén waves is compared in the polar graph in Fig. 5

5.7 MHD Instabilities

There exists a number of instabilities in plasma. We only briefly mention few of them. In general, the approach to find instabilities is to see whether small perturbations to a given state increase exponentially over time. This usually indicates instability.
The Rayleigh-Taylor instability occurs when a heavier fluid is sitting on top of a lighter fluid in some potential field. Any perturbation to the surface of the fluid increases exponentially. This can be seen from the dispersion relation of the surface waves, which goes something like

$$\omega^2 = g\frac{\rho_L - \rho_H}{\rho_L + \rho_H}k$$

where g is the acceleration of the fluids due to the potential field. Therefore, if the density of the fluid on top ρ_H is higher than the density of the fluid on bottom ρ_L, ω is imaginary and the state is unstable.

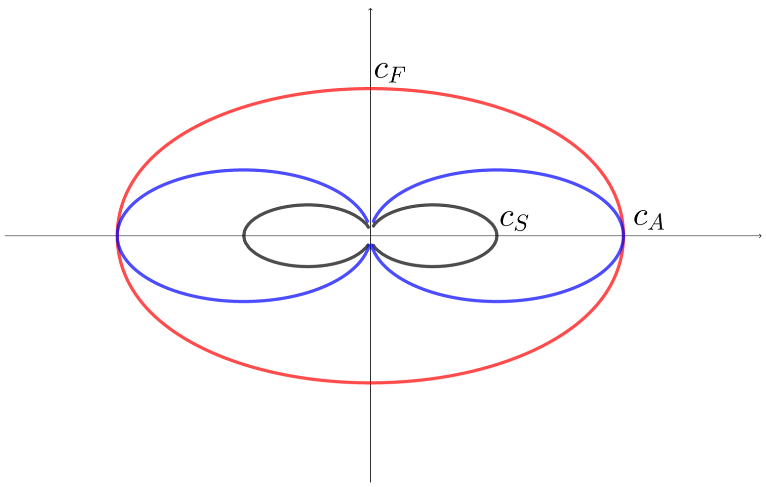

Figure 5: Polar graph of the sqaure of phase speed v_p^2 of different waves in plasma - the distance of the origin determines v_p^2, the angle subtended from the x axes determines θ - the angle between the direction of propagation and the external field $\vec{B}_0$. At the origin, black lines and blue lines should respectively connect. The blue lines represent the Alfvén waves, black lines represent the slow magnetoacoustic waves and red lines represent the fast magnetoacoustic waves. The maximum speeds c_F, c_A and c_S are marked.

Kink and sausage instabilities occur for a jet of plasma. They both correspond to instabilities due to magnetic tension gradients. Kink instability happens when a localised wavelike bend occurs on otherwise straight jet of plasma. The amplitude of this bend tends to increase over time. The sausage instability happens when the radius of the plasma jet decreases. The radius tends to increase without stopping until the jet is cut in half.
Kelvin-Helmholtz instability occurs when two plasma flows are shear flows next to each other - plasma tends to mix, i.e. perturbations to shear flow interfaces tend to increase over time.
There are many other instabilities, but these are not discussed here.

6 Summary of EM Wave Properties in Plasma

Name	**Dispersion Relation**	**Polarization**
	Cold Plasma Without External Field	
EP Oscillation	$\omega = \omega_{ep}$	Linear Longtitudal
EM Waves	$\omega^2 = \omega_{ep}^2 + k^2c^2$	Transverse Linear
	Cold Plasma With External Field, $k \parallel B_0$	
EP Oscillation	$\omega = \omega_{ep}$	Linear Longtitudal
LH Waves	$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{ep}^2}{\omega(\omega_{ec}+\omega)}}$	Transverse LH Circular
RH Waves and Whistler Waves	$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{ep}^2}{\omega(\omega_{ec}-\omega)}}$	Transverse RH Circular
	Cold Plasma With External Field, $k \perp B_0$	
Ordinary Waves	$\omega^2 = \omega_{ep}^2 + c^2k^2$	Transverse in direction of Field
Extraordinary Waves	$k = \frac{1}{c}\sqrt{\frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{\omega^2-\omega_H^2}}$	RH/LH Elliptical, Longt. + Trans.
	MHD Waves, $k \cdot B_0 = \cos\theta$	
Alfvén Waves	$\omega^2 = \frac{B_0^2}{\mu_0\rho_0}k^2\cos^2\theta$	Transverse linear
Magnetoacoustic Waves	$(\omega^2 - c_S^2k^2)(\omega^2 - c_A^2k^2\cos^2\theta) = \omega^2c_A^2k^2\sin^2\theta$	Lontitudal