PX448: Mathematical Methods for Physicists III Summary

2022/2023

Calculus of Variations

By considering a small deviation to some path $f(x, y, y')$, the integral over which is minimised, the Euler-Lagrange equation can be obtained:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0\,.$$

This can be simplified in the special cases where f does not *explicitly* depend on one of the variables. For example, if there is no explicit x dependence, we observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial x} = 0$ and thus $f - y'\frac{\partial f}{\partial y'}$ must be constant.

The Euler-Lagrange equations can be generalised to contain multiple independent variables (labelled x_i) and multiple dependent variables (labelled y_j):

$$\frac{\partial f}{\partial y_j} - \partial_i\left(\frac{\partial f}{\partial(\partial_i y_j)}\right) = 0\,.$$

To minimise a system with an additional constraint $\int g(x, y, y')\mathrm{d}x = k$ for constant k, Lagrange multipliers can be used such that $\mathcal{L} = f - \lambda g$. This Lagrangian is now the functional to be minimised and so satisfies the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0\,.$$

Complex Differentiation

A single-valued function $f(z)$ is differentiable if its derivative is unique (and so does not depend on how the limit $h \to 0$ is taken). The function is analytic in domain $\mathcal{D}$ if it is differentiable at every point in $\mathcal{D}$; $f(z)$ is analytic at a point z_0 if it is differentiable on an open disc about z_0. Singularities are points within the domain of analyticity at which $f(z)$ is *not* analytic. For example, $f(z) = 1/z$ is analytic everywhere apart from $z = 0$; this point is therefore a singularity.

For an analytic function $f(z) = u(x, y) + iv(x, y)$, we can take the derivative with h being purely real or imaginary as the limit $h \to 0$ is taken. Comparing the results yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\,.$$

These relate the real and imaginary parts of analytic functions to each other and can be used to verify whether a function is analytic or not. Another consequence of the Cauchy-Riemann equations is that analytic functions must satisfy the Laplace equation $\nabla^2 f = 0$.

The geometric interpretation of analytic functions is that the correspond to conformal maps. This means that the mapping from the z-plane (with $x - y$ axes) to the w-plane (with $u - v$ axes) has a local magnification which is independent of direction and preserves the sense of angles.

Contour Integration

Cauchy's theorem states that if $f(z)$ is analytic within and on a closed contour C, then

$$\oint_C f(z)\,\mathrm{d}z = 0\,.$$

This means that, provided the contour Γ does not pass outside $\mathcal{D}$, the integral

$$F(z) = \int_\Gamma f(z)\,\mathrm{d}z\,,$$

depends only on the end points of Γ. This can be demonstrated by setting $C = \Gamma_1 - \Gamma_2$ where $\Gamma_{1,2}$ are different paths between the same end points. By considering the definition of the derivative of $F(z)$, the integral can be shown to also be an analytic function.

Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,\mathrm{d}z\,,$$

where the point $z = z_0$ is a singularity. This can be extended to derivatives such that the nth derivative of $f(z)$ evaluated at $z = z_0$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z\,.$$

Liouville's theorem states that any analytic function which is everywhere bounded must be constant. This can be proven using Cauchy's integral formula to consider the difference between two points on a bounded function.

Power Series of Analytic Functions

All analytic functions can be written as power series in powers of $(z - z_0)$ about any $z_0 \in \mathcal{D}$. This is the Taylor series and is unique. The series is given by

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m\,,$$

which can be found by the binomial expansion of Cauchy's integral formula. Unlike real-valued functions, complex-valued functions are always equal to their Taylor series. The *circle of convergence* is the largest circle around z_0 within which $f(z)$ is analytic. If $f(z_0) = 0$ and the first nonzero derivative is the pth derivative, then $f(z)$ is said to have a *zero of order* p at $z = z_0$. If $p = 1$ it is said to have a *simple zero.* The Taylor series about $(z - z_0)$ of a function with a zero of order p at z_0 will begin with a term of order $(z - z_0)^p$.

This concept can be extended to cases where functions include singularities. If $f(z)$ is analytic within $\mathcal{D}$ enclosed by C apart from a singularity $z = z_0$, then

$$f(z) = \sum_{m=-\infty}^{\infty} F_m (z - z_0)^m\,, \quad \text{where } F_m = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{m+1}}\,\mathrm{d}z\,.$$

This is called a Laurent series and, as it is unique, does not depend on the method used to find it. If a Laurent series breaks off before $m = -\infty$, i.e. $f(z) = \sum_{m=-N}^{\infty} F_m (z - z_0)^m$, then $(z - z_0)^N f(z)$ must be analytic at (thus have a Taylor expansion about) $z = z_0$; the singularity is said to be *removable.*

If a circle can be drawn around a singularity within which $f(z)$ is analytic, the singularity is said to be *isolated.* If the Laurent series breaks off at order N, $f(z)$ is said to have a *pole of order* N at $z = z_0$ (and it is called a *simple pole* if $N = 1$). If the singularity is not removable and the Laurent series does not break off (i.e. $N = \infty$), it is said to be an *essential singularity.* In the case of non-isolated singularities, around which any finite circle necessarily encloses other singularities, the proof for the Laurent series breaks down and so the series may not exist.

If $f(z)$ does not return to its original value when traced around an arbitrarily small loop surrounding $z = z_0$, then z_0 is a *branch point* of f. In order to make $f(z)$ single-valued, a *branch cut* can be made which loops are not allowed to cross. As there will be a discontinuity across the branch cut, $f(z)$ is not analytic on the cut.

Calculus of Residues

The F_{-1} term in a Laurent series is called the *residue*. This is an important value as it can often be calculated using other methods and then used to solve the integral

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i F_{-1}\,.$$

In the case of the Laurent series expanded around a simple pole $z = a$,

$$\begin{aligned} f(z) &= \frac{1}{z-a} g(z)\,, \\ &= \frac{1}{z-a}\left[g(a) + g'(a)(z-a) + \frac{g''(a)}{2!}(z-a)^2 + \dots\right]\,, \end{aligned}$$

where $g(z)$ is also analytic and so can be expanded as a Taylor series. This means that the residue is simply $F_{-1} = g(a)$.

The residue theorem states that if $f(z)$ is analytic within and on the closed contour C except at a finite number of isolated singularities z_i, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_i F_{-1,i}\,.$$

If there is an integral of the form $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$, a semicircular contour can be created in the complex plane. As long as the value of the semicircular integral $\to 0$ as $R \to \infty$, the residue theorem can be used to evaluate the integral. Jordan's lemma does not require $f(z)$ to vanish quicker than $1/z$; it just needs to vanish. Integrals of the form $\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$ can be converted into contour integrals around the unit circle and simplified by recognising that

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right)\,, \quad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)\,.$$