1 Cosmological Equations

1.1 Key Observations

Cosmology as a discipline is relatively young. The idea of a dynamic universe is relatively new in physics, and is based on several key observations.

1.1.1 Night Sky

Observations show that the concentration of stars in the universe is relatively uniform on the very large scale. What does this mean for the intensity of observed light during night?
We will try to integrate the intensity of the light incoming towards us from the universe (either whole or one half, does not change the result). Each part of the universe produces intensity

$$\frac{dI}{dV} = \frac{I_0}{4\pi r^2}$$

where V is the volume of the part of the universe, I_0 is the average intensity per unit volume of the universe. We can write specifically, $I_0 = nI_s$, where I_s is the average intensity from a star and n is the number density of the stars, both of which we can assume to be approximately constant. The total intensity is then

$$I = \int_{universe} \frac{dI}{dV} dV = \int_0^\infty \int_0^\pi \int_0^\pi \frac{I_0}{4\pi r^2} r^2 \sin\theta d\theta d\phi dr = \frac{I_0}{2} \int_0^\infty dr \to \infty$$

Therefore, in this model, the night sky should be essentially infinitely bright.
To solve this dilema, many models have been proposed. Perhaps the most famous one is that the interstellar dust absorbs this radiation. But, since the dust does not have any energy to convert the radiation energy to, it would probably just heat up over time and reradiate the light, therefore it does not solve this problem. Valid solution is therefore that either the universe is not infinitely old, which is the model we now accept.

1.1.2 Galaxy Movement

When measuring red shift, and therefore velocity, of distant galaxies, they all seem to be moving away from us, with speeds increasing as the distance to the galaxy increases. This is explained by the idea of expanding space - space between any two objects is continuously increasing with certain density of increase - the further away an object is, the more space is created each second. The constant characterizing this expansion is called the Hubble constant H_0, and its value is, for close enough galaxies

$$H_0 = (68 - 73)\,\text{km s}^{-1}\text{Mpc}^{-1} \tag{1}$$

where H_0 is called the present Hubble parameter (called also the Hubble constant). Therefore, an object in distance 1 Mpc away is moving at speed approximately 70 km s^{-1} from us.
To do this more quantitatively, consider two points at time t_1, $\vec{r}_1(t_1)$ and $\vec{r}_2(t_1)$. Suppose that we now let time t pass. Due to expansion of space, the magnitude of the vector connecting these two points, $\vec{r}_{12}$, will increase as

$$|\vec{r}_{12}|(t) = |\vec{r}_{12}|(t_1)a(t)$$

where $a(t)$ is the so called scale factor. Hence, the speed at which the distance between the points increases (and hence relative speed of the points with respect to each other) is

$$v = \frac{d(|\vec{r}_{12}|)}{dt} = |\vec{r}_{12}|(t_1)\frac{da}{dt} = |\vec{r}_{12}|(t_1)\dot{a}(t)$$

Therefore, the speed at which points move away from each other per unit distance (the length density of space expansion) is

$$H = \frac{v}{|\vec{r}_{12}|(t)} = \frac{|\vec{r}_{12}|(t_1)\dot{a}(t)}{|\vec{r}_{12}|(t_1)a(t)} = \frac{\dot{a}}{a}$$

Hence

$$H = \frac{\dot{a}}{a} \tag{2}$$

is the general definition for the Hubble parameter.

This mathematical formalism requires for a to be increasing as time passes. This of course implies that going back in time, a had to be decreasing, which hints that at some point back in time t_0, $a(t_0) = 0$ and all points in space coincided in one singularity - the big bang. To figure out how long ago this was given that the expansion was at the same rate always, we need to figure out how long does it take for speed v to cover the distance between the objects at present time, i.e.

$$t - t_0 = \frac{|\vec{r}_{12}|(t)}{v(t)} = \frac{1}{H_0} \approx 13.6 \text{ Gyr}$$

1.1.3 Cosmological Principle

To some reason, this was already stated above, but it is important enough to restate - it seems that at large scales, the universe appears homogeneous and isotropic, meaning that it is invariant under translation and rotation. This works as an important constraint on the equations we can use to describe the universe.

1.2 General Relativity Consequences

The most complete theory of the universe on the big scale today is the general relativity. Here, we discuss a few consequences the general relativity has for the ideas we express in cosmology. Firstly, general relativity (GR) allows for objects to move away from each other faster than the speed of light. This means that objects can be out of causal contact as consequence of the expanding space between them. This gives rise to the idea of the observable universes - for every point in the universe, the observable universe is such subset of the universe that the light from this subset could have reached the original point in the lifetime of the universe. Therefore, observable universes are imagined as spheres centered on a given point. Importantly, if we have two points on the opposite ends of a observable universe of a third point, the two points can be outside of each others observable universe - they can be out of causal contact (they could not have influenced each others behaviour). This idea becomes very important constraint for explanation of cosmic microwave background.
Another idea is the equivalence principle, which states that because the inertial and gravitational masses of objects are indistinguishable, the gravitational force is just an inertial effect, i.e. occurs due to kinetically accelerated motion. In order to explain the behaviour of objects in this sense, we define that objects move along geodesics - paths of shortest distance - in spacetime. By presence of energy or mass in the spacetime, the spacetime curves, which causes changes in geodesics, which we observe as an action of a force. Critical for the description of the spacetime is the idea of curvature.
Curvature can be zero (for straight plane), positive (for a space which looks like a part of a sphere) and negative (for space which looks like a hyperboloid). These can be differentiated by the sum of the angles in a closed triangle. For zero curvature, Euclidian geometry applies and the sum of all angles in triangle is π radians. In positive curvature, the sum of the angles can be more than π, in negative curvature, it can be less than π. As illustration, consider a closed triangle on Earth's surface, with one side of length 1 m going from north pole to the south, second side of length 1m going to the east from the end point of first side and last side closing the triangle to the north again. Second and last side are at an angle $\frac{\pi}{2}$ to each other, same as the first and second side. Since the first and the last side are at a non-zero angle to each other, the sum of all angles is higher than π.

1.3 Newtonian Freedman Equation

If the movement under the influence of gravity is simply due to the shape of the spacetime, we need to somehow use equivalence principle to enforce the observed dependencies from Newtonian mechanics into our new theory. For example, we expect that the potential energy of objects will still be the expressable in the Newtonian terms.
Therefore, with only a limited amount of hand-waving, we are able to introduce relatively correct cosmological equations that will help us understand the behaviour of universe.
To start, we know that the total energy of any object is the sum of the kinetic and potential energy. In Newtonian mechanics, the total energy U for an object of mass m in distance r from object with mass M, moving radially, is

$$U = \frac{1}{2}m(\dot{r})^2 - G\frac{mM}{r}$$

where G is the Newtonian gravitational constant $G \approx 6.67 \times 10^{-10}\text{N kg}^{-2}\text{m}^2$ and $\dot{r} = \frac{dr}{dt}$. Consider now that object M represents a sphere that is a subset of universe containing more mass than a nearby surrounding universe. The mass contained in this sphere can be expressed as

$$M = \frac{4}{3}\pi r^3 \rho$$

where ρ is the average mass density inside the sphere. So, we have

$$U = \frac{1}{2}m(\dot{r})^2 - \frac{4}{3}\pi r^2 \rho G m$$

Now, consider that the distance r increases between the objects due to scale factor as discussed before

$$r(t) = r(t_1)a(t)$$

where $a(t_1) = 1$. This is essentially the same as expressing Eulerian coordinate $r(t)$ in terms of the Lagrangian coordinate $r(t_1)$ - the scale factor does not change the distance between each and every object differently, only scales them. Therefore, if we defined our unit distance as a distance between two specific objects, we would not find any change in ratios of the distances. So, we can write $r(t_1) = x$, where x is the Lagrangian coordinate. Then

$$\dot{r} = (\dot{a})x$$

and

$$U = \frac{1}{2}m(\dot{a})^2 x^2 - \frac{4}{3}\pi \rho G m a^2 x^2$$

Hence

$$\frac{2U}{mx^2 a^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}G\rho$$

Now, we need to rather arbitrarily define that the curvature of the spacetime will be k

$$kc^2 = -\frac{2U}{mx^2}$$

we can see that this is related to the energy present at given point of the spacetime, which makes sense. Therefore, we can write that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \tag{3}$$

This is called the Newtonian Friedmann equation. This equation highlights the importance of the curvature of the universe - very different potential fates of the universe depend on the value of the curvature. If we start with universe in one point (scale factor $a = 0$), positive curvature implies that universe will eventually return to $a = 0$, zero or negative curvature implies that universe will expand indefinitely over time.
The Friedmann equation can be also rearranged to find the curvature as

$$\frac{kc^2}{a^2} = \frac{8\pi}{3}G\rho - \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - H^2$$

where we noticed that the Hubble parameter is $H = \frac{\dot{a}}{a}$. This implies that there is a critical density ρ_c, for which the curvature turns to zero. For smaller densities, the curvature is negative, for higher densities, the curvature is positive. At the critical density

$$0 = \frac{8\pi}{3}G\rho_c - H^2$$

$$\rho_c = \frac{3H^2}{8\pi G}$$

1.4 Relativistic Friedmann Equation

Surprisingly, the transition to relativistic description of the universe is relatively simple. In the Newtonian Friedmann equation, we substitute the mass density by the more generall energy density as

$$\rho = \frac{\epsilon}{c^2}$$

where ϵ is the energy density. This energy density includes extra factors, for example energy of photons, which are massless. Furthermore, we express the curvature as

$$k = \frac{\kappa}{R_0^2}$$

where R_0 is the local radius of curvature. Therefore, κ now only expresses the sign of the curvature, i.e. $\kappa \in \{-1, 0, 1\}$.
Controversially, there is also an extra term called the cosmological constant Λ, which was initially added to the theory in order to enforce the posibility of a static universe. The full equation then takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon}{3c^2} - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} \tag{4}$$

The Λ term can be included in the energy density as

$$\frac{8\pi G\epsilon}{3c^2} - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} = \frac{8\pi G}{3c^2}\left(\epsilon + \frac{c^2}{8\pi G}\Lambda\right) - \frac{\kappa c^2}{R_0^2 a^2} = \frac{8\pi G}{3c^2}\epsilon' - \frac{\kappa c^2}{R_0^2 a^2}$$

Again, we can define the critical energy density for which the universe has zero curvature

$$\epsilon_c = \frac{3c^2 H^2}{8\pi G}$$

If the energy density is higher than the critical energy density, the universe has positive curvature, if it is lower, it has negative curvature.
We can put some meaning to this by considering the critical energy density at current time, which we can calculate by substituting $H = H_0$, which leads to

$$\epsilon_c \approx 8.8 \times 10^{-10} \text{ J m}^{-3} \approx 5.5 \text{ GeV m}^{-3}$$

which corresponds to about 5.5 stationary protons per cubic meter - very low energy density.
Since the relative size of the energy density with respect to critical energy density is the important factor in deciding the fate of the universe, we can define a parameter Ω

$$\Omega = \frac{\epsilon}{\epsilon_c} = \frac{\rho}{\rho_c}$$

that will help us quantify this size. For Ω greater than one, the energy density is greater than critical energy density, therefore curvature is positive and the universe will return to $a = 0$. For Ω less than one, the curvature is negative and the universe will expand forever. For Ω equal to one, the energy density is the critical energy density.
We can also rewrite the Friedmann equation as

$$H^2 = H^2\Omega - \frac{\kappa c^2}{R_0^2 a^2}$$

$$\frac{\kappa c^2}{R_0^2 a^2} = H^2(\Omega - 1) \tag{5}$$

in terms of the Ω factor. The question of course remains what is the actual energy density, and hence Ω in our universe? Currently, there are several sources of energy density we observe, and some other that we deduce.

1.4.1 Sources of Energy Density

First obvious source of energy are the stars and galaxies. We can estimate the mass due to these objects from the average luminosity of galaxies and the corresponding equivalent number of stars that would produce this luminosity, taking into account that the number of stars dN with mass M follows empirically

$$\frac{dN}{dM} \propto M^{-2.35}$$

But, the resultant energy density is nowhere near the critical energy density. This seems a little odd because the geometry of the universe suggests that we are in a nearly flat universe with observed $\Omega = 1.005 \pm 0.006$.

Therefore, we search for things that we overlooked during this calculation. Remembering that black holes, planets and neutron stars might not be observed, we are, under some reasonable assumptions, able to estimate the addition to Ω due to these objects, which we assume consist of baryons (type of particle) as

$$\Omega_b \approx 0.04$$

Therefore still not even close to 1. What are we missing? Put simply, we do not know, but we observe some phenomena that hints at what we are missing. For example, the rotation curves of galaxies suggest that as we move away from the centre of the galaxy, there is some additional mass inside the galaxy that is not observed - does not emit electromagnetic radiation. We call this mass the mass of some unknown state of matter - dark matter.
If we then predict the required mass of dark matter to satisfy the observed rotation curves, we can find out that the dark matter contribution to Ω is

$$\Omega_{DM} \approx 0.23$$

Again, not enough. Therefore, we are forced to assign the remaining 0.73 contribution to Ω to the Λ factor, and we call this the dark energy.

1.4.2 Curvature Measurements

Most precise curvature measurements on the scale of the universe are based on the cosmic microwave background (CMB) observations. At certain time in the early universe, as we will discuss later in greater depth, the recombination of protons and electrons into hydrogen atoms took place. At that point, the universe stopped being a hot plasma, and therefore became transparent to light. This means that all the EM radiation that was unable to propagate within the plasma could now escape and propagate through the universe. As the universe expanded, the initially UV photons were stretched all the way to the microwave spectrum. Right now, we observe CMB as nearly perfect spectrum of a black-body, with peak wavelength of about 1 mm, corresponding to the temperature $T \approx 2.725K$. The spectrum is remarkably isotropic, with small anisotropies restricting the curvature and value of Ω to about 1, as mentioned before.

1.4.3 Measuring Distances

Just to reiterate some parts of previous years, the distances on the galactic scale are very hard to determine, and usually build upon a certain hierarchy of standard candles. These are processes/objects which have specific properties which help us determine the distance to them without the need for geometrical measurements. Usually, we can determine the absolute luminosity of a standard candle, and by comparison with the observed luminosity, we can then determine the distance to the object.
For very close objects (up to about kpc), we are able to use parallax measurements to determine the distance. After that, cepheids can be a useful standard candle, and historically have been very important. These are stars oscillating with certain period corresponding to their size, which is related to the absolute luminosity. Further away, cepheids are not bright enough to be observed, and we have to rely on type Ia supernovae, which are less frequent, but much more luminous.
Importantly, all standard candles have to be standardized against methods lower in the hierarchy (starting with the parallax). Hence, any standard candle is only as good as the dataset for its standardization.
Finally, we should note that measurements of very distant type Ia supernovae indicate that the universe is in fact also accelerating the expansion. The reasons for this will be covered in following chapters.

2 Model Universes

We will now discuss few models of universe and try to predict some quantitative properties of these universes. But first, we will need to develop some more theory, as right now, the Friedmann equation contains too many unknowns ($H(t)$ and $\epsilon(t)$).

2.1 Additional equations

2.1.1 Fluid Equation

We can assume that all the processes inside the universe has to be adiabatic - the universe cannot output energy somewhere out of the universe. The first law of thermodynamics for adiabatic processes determines

$$0 = dQ = dE + pdV$$

Here, pressure and volume are both extensive quantities, and we should try to rewrite the equation in terms of intensive quantities instead.
The internal energy of the universe can be determined as

$$E(t) = V(t)\epsilon(t)$$

Hence

$$dE = Vd\epsilon + \epsilon dV$$

So we have

$$Vd\epsilon = -pdV - \epsilon dV = -(p+\epsilon)dV$$

Finally, we can express

$$dV = \frac{dV}{dt}dt = \frac{d}{dt}(\frac{4}{3}\pi r^3)dt = \frac{4}{3}\pi 3r^2\frac{dr}{dt}dt$$

Now, we describe r in terms of scale factor as $r(t) = a(t)x$, so that

$$dV = 3 \times \frac{4}{3}\pi a^2 x^3 \frac{da}{dt}dt = 3\frac{\dot{a}}{a}\frac{4}{3}\pi a^3 x^3 dt = 3\frac{\dot{a}}{a}Vdt$$

Hence

$$Vd\epsilon = -(p+\epsilon)3\frac{\dot{a}}{a}Vdt$$

and

$$\frac{d\epsilon}{dt} = -3(p+\epsilon)\frac{\dot{a}}{a} \tag{6}$$

This is called the fluid equation and it applies generally for all components producing energy density in the universe.

2.1.2 State Equation

Simply introducing the fluid equation however does not solve our problem - we introduced a new variable, p, which we do not know expression for. Fortunately, p can be determined for each component of the universe using a specific state equation. In general, we can write state equation as

$$p = \omega\epsilon \tag{7}$$

where ω is some dimensionless parameter.
For illustration, consider the case of matter. In first approximation, we can model matter as an ideal gas, obeying ideal gas equation

$$pV = Nk_BT$$

Hence

$$p = \frac{N}{V}k_BT = \frac{\mu N}{V}\frac{k_BT}{\mu} = \frac{\rho}{\mu}k_BT$$

where μ is the average mass per particle of matter. In classical limit, the thermal energy is related to the kinetic energy of the particles as

$$\frac{3}{2}k_BT = \frac{1}{2}\mu < v^2 >$$

where $< v^2 >$ is the mean square velocity. Therefore

$$k_BT = \frac{\mu}{3} < v^2 >$$

And therefore

$$p = \frac{1}{3} < v^2 > \rho$$

Finally, we can substitute in $\rho = \frac{\epsilon}{c^2}$ to get

$$p = \frac{1}{3}\frac{< v^2 >}{c^2}\epsilon \tag{8}$$

For massive matter, usually $< v^2 > \ll c^2$, and so $\omega \to 0$ and therefore $p \to 0$. For massless matter, which is usually the case of radiation, we conversely have $< v^2 >= c^2$, so $\omega = \frac{1}{3}$ and $p = \frac{1}{3}$. Without proof, I will also state that for the cosmological constant/dark energy state equation, $\omega = -1$, so that $p = -\epsilon$, which we now has to accept as a strange property of the dark energy.

2.1.3 Acceleration Equation

Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon}{3c^2} - \frac{\kappa c^2}{R_0^2 a^2}$$

Hence

$$(\dot{a})^2 = \frac{8\pi G}{3c^2}\epsilon a^2 - \frac{\kappa c^2}{R_0^2}$$

Since the overall curvature has to be constant in time, the whole second term is constant in time. Therefore, taking the time derivative leads to

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}\right)$$

$$\ddot{a} = \frac{4\pi G}{3c^2}\left(\dot{\epsilon}\frac{a^2}{\dot{a}} + 2\epsilon a\right)$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(2\epsilon + \frac{a}{\dot{a}}\dot{\epsilon}\right)$$

Now, we can substitute for $\dot{\epsilon}$ from the fluid equation (6) to get

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(2\epsilon + \frac{a}{\dot{a}}\left(-3\frac{\dot{a}}{a}(\epsilon + p)\right)\right) = \frac{4\pi G}{3c^2}\left(2\epsilon - 3\epsilon - 3p\right)$$

And therefore

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\epsilon + 3p\right) \tag{9}$$

This equation is called the acceleration equation, as it determines the second time derivative of the scale factor. It will be useful in the determination of fates of the universe.
By substituting in for the pressure from the state equation, we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(1 + 3\omega)\epsilon$$

Therefore, we see that for $\omega < \frac{-1}{3}$, the universe will accelerate its expansion. Clearly, in dark energy dominated universe, the expansion will be accelerated over time.

2.2 Integration of Friedmann Equation

We now have all the tools neccessary to carry out some integation of the Friedmann equation. Lets start with the fluid equation

$$\dot{\epsilon} = -3\frac{\dot{a}}{a}(\epsilon + p)$$

Substituting in from the state equation

$$\dot{\epsilon} = -3(1+\omega)\epsilon\frac{\dot{a}}{a} = -\alpha\epsilon\frac{\dot{a}}{a}$$

where $\alpha = 3(1+\omega)$. Now, we multiply the equation by a^α to get

$$a^\alpha\dot{\epsilon} + \alpha\epsilon a^{\alpha-1}\dot{a} = 0$$

We can spot that this is a full differential

$$a^\alpha\dot{\epsilon} + \alpha a^{\alpha-1}\dot{a}\epsilon = \frac{d}{dt}\left(a^\alpha\epsilon\right) = 0$$

which we can integrate to

$$a^\alpha\epsilon = C$$

$$\epsilon = \frac{C}{a^\alpha}$$

Consider now that we choose a reference time t_0 at which $a = a_0 = 1$ and energy density is $\epsilon = \epsilon_0$. Then

$$\epsilon = \epsilon_0 a^{-3(1+\omega)} \tag{10}$$

where I substituted back for alpha.
Consider now a flat universe (similar to our universe). The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon = \frac{8\pi G\epsilon_0}{3c^2}a^{-3(1+\omega)}$$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}a^{\frac{-3(1+\omega)}{2}}$$

$$a^{\frac{3(1+\omega)}{2}}a^{-1}\dot{a} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}$$

$$a^{\frac{1+3\omega}{2}}\dot{a} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}$$

For $\frac{1+3\omega}{2} \neq -1$, the equation has a special solution, as a^{-1} integrates to $\ln a$ and therefore

$$a = Ae^{\sqrt{\frac{8\pi G\epsilon_0}{3c^2}}t} \tag{11}$$

To make the condition more clear, it requires that

$$\frac{1+3\omega}{2} \neq -1$$

$$1 + 3\omega \neq -2$$

$$\omega \neq -1$$

Therefore, the special case is the case for the dark energy.
For other cases, a integrates as

$$\frac{2}{3(1+\omega)}a^{\frac{3(1+\omega)}{2}} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}t - B$$

where B is integration constant. If we have $\frac{3(1+\omega)}{2} > 0$, we can satisfy the requirement that at $t = 0$, $a = 0$ by setting $B = 0$ (notice that $t = 0 \neq t_0$). Then, we have

$$a = \left(\frac{6(1+\omega)^2\pi G\epsilon_0}{c^2}\right)^{\frac{1}{3(1+\omega)}} t^{\frac{2}{3(1+\omega)}} \tag{12}$$

2.2.1 Radiation Dominated Universe

For a radiation dominated universe, $\omega = \frac{1}{3}$ and

$$a = \left(\frac{32\pi G\epsilon_0}{3c^2}\right)^{\frac{1}{4}} t^{\frac{1}{2}}$$

The radiation dominated universe expands, but the rate of expansion decreases over time.
Lets say that we choose a certain reference time t_0 at which $a(t_0) = 1$. Then, we can simplify the expression as

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \tag{13}$$

The Hubble coefficient is

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \tag{14}$$

Finally, we can specify, based on (10), that the energy density in radiation dominated universe evolves as

$$\epsilon = \frac{\epsilon_0}{a^4} = \frac{\epsilon_0}{t^2}t_0^2 \tag{15}$$

2.2.2 Matter Dominated Universe

For matter, $\omega = 0$ and

$$a = \left(\frac{6\pi G\epsilon_0}{c^2}\right)^{\frac{1}{3}} t^{\frac{2}{3}}$$

Again, the universe does expand, but the expansion slows down over time. Setting $a(t_0) = 1$ leads to

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \tag{16}$$

and the Hubble parameter is

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \tag{17}$$

And similarly as before

$$\epsilon = \frac{\epsilon_0}{a^3} = \frac{\epsilon_0}{t^2}t_0^2 \tag{18}$$

2.2.3 Dark Matter Dominated Universe

As discussed before, $\omega = -1$ for dark matter and we have the special solution with

$$a = Ae^{\sqrt{\frac{8\pi G\epsilon_0}{3c^2}}t}$$

with exponentially expanding universe. Setting $a(t_0) = 1$

$$a = e^{\sqrt{\frac{8\pi G\epsilon_0}{3c^2}}(t-t_0)}$$

where we can notice that

$$\frac{8\pi G\epsilon_0}{3c^2} = H_0^2$$

where $H_0 = H(t_0)$ is the Hubble parameter at time t_0. Hence

$$a = e^{H_0(t-t_0)} \tag{19}$$

and Hubble parameter is

$$H = \frac{\dot{a}}{a} = H_0 \tag{20}$$

and

$$\epsilon = \epsilon_0 \tag{21}$$

2.2.4 Empty Curved Universe

A different approach to integration of the Friedmann equation could be taken by considering a curved but otherwise empty universe. The Friedmann equation is then

$$(\dot{a})^2 = -\frac{\kappa c^2}{R_0^2}$$

Clearly, we have no real solution for positive curvature, which would require $\dot{a}$ to be imaginary. However, for zero or negative curvature

$$(\dot{a})^2 = \frac{|\kappa|c^2}{R_0^2}$$

$$\dot{a} = \pm\frac{\sqrt{|\kappa|}c}{R_0}$$

$$a = A \pm \frac{\sqrt{|\kappa|}c}{R_0}t$$

which means that this universe is either stationary for zero curvature, or uniformly expanding or retracting for negative curvature.
Again, choosing reference time t_0 at which $a(t_0) = 1$, we have

$$a(t) = \frac{t}{t_0} \tag{22}$$

And Hubble coefficient is

$$H = \frac{\dot{a}}{a} = \frac{1}{t} \tag{23}$$

and $\epsilon = 0$ by definition (empty universe).

2.3 Multi-Component Universes

In order to understand the evolution of the universe, we need to write the Friedmann equation that combines the contributions to the energy density and scale factor from different sources in the universe. We understand that the energy density can be written as combination of energy densities from different sources, i.e.

$$\epsilon = \epsilon_r + \epsilon_m + \epsilon_\Lambda$$

where ϵ_r is the radiation contribution, ϵ_m is the matter contribution and ϵ_Λ is the dark energy contribution. We could also talk about the relative energy density compared to the critical energy density $\epsilon_c = \frac{3c^2H^2}{8\pi G}$. The relative energy density Ω is then given as

$$\Omega = \frac{\epsilon}{\epsilon_c} = \frac{\epsilon_r}{\epsilon_c} + \frac{\epsilon_m}{\epsilon_c} + \frac{\epsilon_\Lambda}{\epsilon_c} = \Omega_r + \Omega_m + \Omega_\Lambda$$

Specifically, at reference time t_0, we would write

$$\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$$

With reference to the time t_0, we can then express the energy densities from different types of sources using equations (15), (18) and (21) as

$$\epsilon = \frac{\epsilon_{r,0}}{a^4} + \frac{\epsilon_{m,0}}{a^3} + \epsilon_{\Lambda,0}$$

Hence

$$\Omega = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} \tag{24}$$

Now, we can remember the Friedmann equation expressed in terms of relative energy density (5)

$$\frac{\kappa c^2}{R_0^2 a^2} = H^2(\Omega - 1)$$

Hence

$$\frac{\kappa c^2}{R_0^2} = a^2 H^2(\Omega - 1)$$

Interestingly, in the last expression, the right hand side is a constant. If we evaluate it at time $t = t_0$, when $a = a(t_0) = a_0 = 1$, we have

$$\frac{\kappa c^2}{R_0^2} = H_0^2(\Omega_0 - 1)$$

Generally, we then have

$$\frac{H_0^2(\Omega_0 - 1)}{a^2} = H^2(\Omega - 1)$$

$$\frac{\Omega_0 - 1}{a^2} = \frac{H^2}{H_0^2}\Omega - \frac{H^2}{H_0^2}$$

Here, we can notice that

$$\frac{H^2}{H_0^2}\Omega = \frac{H^2}{H_0^2}\frac{\epsilon}{\epsilon_c} = \frac{H^2}{H_0^2}\frac{8\pi G}{3c^2H^2}\epsilon = \frac{8\pi G}{3c^2H_0^2}\epsilon = \frac{\epsilon}{\epsilon_{c,0}} = \frac{\epsilon_{r,0}}{a^4\epsilon_{c,0}} + \frac{\epsilon_{m,0}}{a^3\epsilon_{c,0}} + \frac{\epsilon_{\Lambda,0}}{\epsilon_{c,0}} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0}$$

And therefore we have

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} = \Omega + \frac{1-\Omega_0}{a^2} \tag{25}$$

This is the Friedmann equation for a multi-component universe. Generally, this is very hard to solve, but we can notice that depending on the value of a, some components become more or less important. Also, for nearly flat universe at time t_0, $(1-\Omega_0) \to 0$, and hence the last term is usually insignificant.

2.3.1 Universe Epochs

If we set t_0 to correspond to current time, we can use known measured and deduced values of $\Omega_{\Lambda,0} \approx 0.73$, $\Omega_{m,0} \approx 0.27$ and $\Omega_{r,0} \approx 10^{-4}$ to try determine the times when matter and radiation dominate the energy density. As a decreases, the relative importance of $\Omega_{m/r}$ rises.
For example, we are interested when the relative importance of Ω_m was the same as of Ω_Λ

$$1 = \frac{\Omega_m}{\Omega_\Lambda} = \frac{\Omega_{m,0}}{a(t)^3 \Omega_{\Lambda,0}}$$

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \approx 0.72$$

If at present day, $a(t_0) = 1$, we can approximate that this was in time when universe was the 0.72 times its current age, i.e. 9.8 Gyr. Before that, $a < 0.72$ and the matter was dominant source of energy density in the universe. Similarly, matter and radiation had equal importance when

$$1 = \frac{\Omega_r}{\Omega_m} = \frac{\Omega_{r,0}}{a(t)\Omega_{m,0}}$$

$$a(t) = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 4 \times 10^{-4}$$

By linear approximation (which might be not very useful in this case), we arrive at age of universe approximately 5 Myr - very early in the history of universe. In fact, this took place much earlier in the universe development, at about 50 000 years after big bang.

2.4 Benchmark Model

Using equation (25), we are able to model the history of the universe with reference to data taken at current time. The model based in this is called the benchmark model of the universe.
In principle, any proposed history of the universe must follow the equation (25), which can be ensured by integral equation

$$\int_0^t H_0 dt = \int_0^{a(t)} \frac{1}{\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2\Omega_{\Lambda,0} + (1-\Omega_0)}} da$$

which follows from (25) by taking the square root

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}}$$

$$H = \frac{\dot{a}}{a} = H_0\sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}}$$

$$\dot{a} = H_0\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2\Omega_{\Lambda,0} + (1-\Omega_0)}$$

$$\int_0^a \frac{1}{\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2\Omega_{\Lambda,0} + (1-\Omega_0)}} = \int_0^t H_0 dt$$

This equation is usually solved numerically, unless a simplification is possible due to dominance of some particular source of energy.

2.4.1 Tests of Benchmark Model

One specific experiment that Benchmark model helps to explain is the apparent blue shift of certain objects in a specific range of distances away from us. This is happening because at certain time in the history of the universe, the expansion was actually decelerating - hence the redshift is not as apparent for these stars, which we interpret as extra blueshift.
To predict when this happened, we define the deceleration parameter q using the second derivative of a

$$a(t) \approx a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{\ddot{a}(t_0)}{2}(t - t_0)^2$$

$$\frac{a(t)}{a(t_0)} = 1 + \frac{\dot{a}(t_0)}{a(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}(t - t_0)^2 = 1 + H_0(t - t_0) - \frac{1}{2}q(t_0)H_0^2(t - t_0)^2$$

Which requires that

$$q(t_0) = -\frac{\ddot{a}(t_0)}{a(t_0)H_0} = -\frac{\ddot{a}}{aH^2}\Big|_{t=t_0}$$

And thus

$$q(t) = -\frac{\ddot{a}}{aH^2} \tag{26}$$

Now, if we remember the acceleration equation (9),

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3p) = -\frac{4\pi G}{3c^2}\sum_s \epsilon_s(1 + 3\omega_s)$$

where s are indices of the sources of energy, i.e. matter, radiation and dark energy. Multiplying the equation by $\frac{-1}{H^2}$

$$\frac{-\ddot{a}}{aH^2} = q = \frac{4\pi F}{3c^2H^2}\sum_s \epsilon_s(1 + 3\omega_s) = \frac{1}{2\epsilon_c}\sum_s \epsilon_s(1 + 3\omega_s) = \frac{1}{2}\sum_s \Omega_s(1 + 3\omega_s)$$

where ϵ_c is the critical energy density. For matter, radiation and dark energy, we have

$$q = \Omega_r + \frac{1}{2}\Omega_m - \Omega_\Lambda \tag{27}$$

This means that the universe was slowing down when (neglecting the radiation effect) $q > 0$, i.e.

$$\frac{1}{2}\Omega_m > \Omega_\Lambda$$

$$\frac{1}{2a^3}\Omega_{m,0} > \Omega_{\Lambda,0}$$

$$a < \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \approx 0.57$$

Therefore, universe started acceleration before the end of matter domination epoch. But, if we observe stars further away into the past then when the scale factor was 0.57 (which corresponds to redshift about 1), we observe the expected apparent blueshift of the stars due to the decelerating universe. This observation is very hard to predict by other theories besides the benchmark model.

2.4.2 Extreme Distances

If we are to make precise measurements for these redshifts, we need a good standard candles. However, as we move further into the young years of the universe, type Ia supernovae become very rare, as not enough stars could evolve into the supernovae at that stage of the universe yet (and those who could are only very massive stars, which are very rare).
Other good candidate for standard candle might be the gamma ray bursts, but this field is very new and the dataset for calibration is relatively small.
Overall, all these methods are based on shifts of the spectra produced due to the speed of the object. This includes time dilation effects, band stretching and band shifting, which are together named k-connections.

3 Dark Matter

The dark matter idea arises from essentially single viewpoint, which are the rotation curves of the galaxies. Lets assume that a galaxy is a uniform disc of stars with uniform mass planar density ρ such that $\rho\pi R^2 = M_0$, where M_0 is the mass of the galaxy and R is the visible radius of the galaxy. The objects orbiting in the galaxy should have velocity corresponding to the centripetal acceleration provided by the gravitational force of the galaxy, i.e. for an object of mass m in distance r from the centre of the galaxy

$$m\frac{v^2}{r} = \frac{GmM(r)}{r^2}$$

where $M(r)$ is the mass of the part of galaxy that is closer to the centre than the object with mass m. Hence

$$v = \sqrt{\frac{GM(r)}{r}}$$

For a uniform density, $M(r) = \rho\pi r^2 = M_0\frac{r^2}{R^2}$, hence

$$v = \sqrt{GM_0\frac{r}{R^2}}$$

as long as $r < R$, afterwards, we have simply

$$v = \sqrt{\frac{GM_0}{r}}$$

For galaxies, we could assume that most of the mass is concentrated around the core, so the velocities of the stars on the outskirts should gradually decrease. But, no such decrease is observed, instead, we often observe that the v for relatively big $r > R$ where R is now some relevant radius of part of the galaxy emmiting most light (close to core), the velocity is still at the same value as for $r = R$. This can be explained by additional mass present away from the core at density profile proportional to r - more mass is present further away from the centre, so that the velocity can remain the same.

3.1 Velocity Dispersion and Virial Theorem

For stable orbit in gravitational field, we have a very general theorem showing that

$$2T + V = 0$$

where T is the kinetic energy of an object and V is the potential energy of the object. For a single object in circular orbit, this can be easily shown. The object must have centripetal acceleration due to gravitational force, i.e.

$$m\frac{v^2}{r} = \frac{GmM}{r^2}$$

Hence

$$2 \times \frac{1}{2}mv^2 - \frac{GmM}{r} = 2T + V = 0$$

However, this, so called Virial theorem, applies even for other types of orbits and more particles.
For a collection of particles orbiting at radius R, we would write

$$T = \sum_i \frac{1}{2}m_i v_i^2 = \frac{1}{2}M < v^2 >$$

where we would call $< v^2 >$ the velocity dispersion. The potential energy could be determined as potential energy for uniform spherical distribution of mass

$$V = -G\int_0^M \frac{m(r)}{r}dm = -G\int_0^R \frac{4\pi}{3}\rho r^2 4\pi r^2 \rho dr - G\frac{16}{3}\pi^2\rho^2\int_0^R r^4 dr =$$

$$= -G\frac{16}{15}\pi^2\rho^2 R^5 = \frac{-3G}{5R}\left(\frac{4}{3}\pi R^3\rho\right)^2 = \frac{-3GM^2}{5R^2}$$

Hence, the virial theorem states that

$$M < v^2 > -\frac{3}{5}\frac{GM^2}{R} = 0$$

$$M = \frac{5 < v^2 > R}{3G}$$

Therefore, from determination of the velocity dispersion, we are able to determine the mass of the galaxy, and therefore also determine what is the expected mass of the dark matter in the galaxy.
Furthermore, by comparison with the luminosity, we can then determine the mass to luminosity ratios, which are of particular interest.

3.2 Weak Lensing

Other way how to detect dark matter is through so called weak gravitational lensing. This occurs when the light travelling from the galaxy gets distorted by intergallactic mass. This means that the image of the galaxy appears non-circular or otherwise deformed. The mass of the objects that cause the lensing can then be determined.

3.3 Mass-Luminosity Ratios

Usually, the scale we use for mass-luminosity ratios is derived from the Sun, with

$$\frac{M_\odot}{L_\odot} = 1$$

We can then observe other objects and from determinations of mass see if there is some non-stellar mass present in the objects. This does not neccessarily imply there is dark matter present - it might be just stars at different stage of their evolution - but for very high mass to luminosity ratios, we start to suspect there might be dark matter present.
Interestingly, only somewhat massive objects (starting from dwarf galaxies) seem to have dark matter present, with $M/L \approx 10 - 100$. Smaller structures, or galactic discs without halos, seem to have M/L about 3-5. This leads to the idea that there is some escape velocity of the dark matter - if the object is not massive enough, the dark matter is not bound to it. This can be expressed in the terms of the average velocity of the dark matter, which can be translated to temperature of dark matter.
Usually, dark matter is characterised as either hot or cold, with hot dark matter moving at relativistic velocities and cold dark matter moving at Newtonian velocities. As far as the benchmark model of the universe goes, the cold dark matter (CDM) is preffered (as in produces more plausible results in the model). Therefore, benchmark model is also called ΛCDM model.
But, this approach overpredicts the number of small structures - dwarf galaxies - that we should be observing by a factor of 10. This is called the substructure crysis.

3.4 Origins of Dark Matter

There are two widely accepted explanations for the origins of dark matter - particle physics suggest that dark matter might consist of weakly interacting massive particles (nicknamed WIMPS), while astrophysics comes with explanation by predicting existence of massive compact halo objects (MACHOS). These are now discussed further.

3.4.1 Massive Compact Halo Objects

The idea is that this is combined set of black holes, neutron stars, old white dwarfs and brown dwarfs. To observe these, a phenomena called microlensing (different than weak lensing) could be studied.
During microlensing, a transit of a MACHO across a line of sight to a luminous object (LO) causes light from the LO to bend around MACHO and therefore somewhat increase in intensity. We can detect this as a slight increase in the intensity from a source. We can distinguish this from other intensity fluctuations (solar flares etc.) by the fact that the increase occurs accross a wide range of wavelengths.
The microlensing can be used to estimate the mass of the MACHO. For this, we assume that only light that passes within so called Einstain radius from the MACHO gets affected. The Einstain radius is (in terms of angular size)

$$\theta_E = \left(\frac{4GM}{c^2 d} \times \frac{1-x}{x}\right)^{\frac{1}{2}}$$

where M is the mass of the MACHO, c is the speed of light, d is the distance from the source of light to the observer and $x = \frac{d_m}{d}$ is the ratio of the distance from the light source to the MACHO d_m and the distance from light source to the observer d. Numerically, this is, for stars in Magellanic cloud

$$\theta_E \approx 4 \times 10^{-4} \text{ arcsec} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \times \frac{d}{50 \text{ kpc}}$$

Hence, the time of the transit is (again, typically for Magellanic cloud)

$$\Delta t = \frac{\theta_E d}{2v} \approx 90 \text{ days} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \times \frac{200 \text{ km s}^{-1}}{v}$$

But, we can estimate v - the speed of MACHO, as a typical galactic velocity at given region. Then, the only unknown remaining is the mass of the MACHO M, which can be therefore determined.
Unfortunately, the measurements of mass and frequency of microlensing suggest that the MACHOS are not nearly enough to account for all dark matter in the galaxies. Therefore, we have to investigate the idea of weak particles as well.

3.4.2 Weakly Interacting Massive Particles

Common candidates for the weakly interacting massive particles are the neutral supersymmetric particles. We are interested in neutral particles as any charged particles could absorb EM radiation and eventually would start to radiate and become visible. The supersymmetry predicts that each particle in standard model has a more massive supersymmetric counterpart, with supersymmetric counterparts of bosons being fermions and vice versa. The lightest supersymmetric particle is called the neutralino, which is a mix of Z boson, Higgs boson and photon supersymmetric counterparts.
Neutralino $\tilde{\gamma}$ is a promising candidate as it is the lightest supersymmetric particle and so should be in range of energy to be created and also should be stable. Supersymmetric particles can decay via self interaction as

$$\tilde{\gamma} + \tilde{\gamma} \to \gamma + \gamma$$

where the end products are gamma ray photons. This reaction conserves in multiplication the so called R parity of the particles (1 for particles, -1 for supersymmetric particles, hence $(-1) \times (-1) = 1 \times 1$).
These gamma ray photons should have a very specific peak energy. However, as of yet, no indications of the peak energy from neutralino anihilation have been observed.
Therefore, particle physicists are trying their luck with ground based detectors, based on weak interaction. These are usually cryogenic and has to rely on some weakly interacting substrate for the detection. Amont some of the ideas there are detectors based on phonon interactions via the lattice nuclei, temperature changes in big reservoirs or Cherenkov radiation observations from created particles during interaction with nuclei. Unfortunately, no clear detection in these detectors took place (we are not exactly sure what to look for).
As a last resort, we are still trying to find the particles in particle colliders, although with the unknown rest-mass, the search is very hard (do not know what range of energies to focus on). No detection here either.

3.5 Modified Newtonian Dynamics

Alternative approach to explaining the rotation curves of galaxies is to postulate that Newtonian mechanics change form at very low acceleration regimes. It assumes that the Newton's second law takes form of

$$F = mau\left(\frac{a}{a_0}\right)$$

where $u(\alpha)$ is some function somewhat similar to tanh in sense that for small α, $u(\alpha) \approx \alpha$, while for big α, $u(\alpha) \approx 1$. Therefore, at big accelerations, we recover the standard Newtonian mechanics. However, at low accelerations,

$$F \approx \frac{ma^2}{a_0}$$

The a_0 here would be a new universal constant, of order 10^{-10} m s^{-2}.

This would predict that outside the main mass of the galaxy, where the acceleration is small, the force would be

$$m\frac{a^2}{a_0} = G\frac{mM_0}{r^2}$$

$$\frac{v^4}{r^2 a_0} = \frac{GM_0}{r^2}$$

Hence

$$v = \sqrt[4]{GM_0 a_0}$$

therefore constant, which is what we observe. However, there is no physical reasoning behind this theory and, perhaps more importantly, it implies an existence of an absolute inertial frame and therefore cannot be easily adapted for relativity. The Tensor Vector Scalar Gravity (TeVeS) is an attempt to do this.

4 Early Universe and Inflation

There are essentially three problems with the benchmark model of the universe alone. First is the so called horizon problem - we observe that the CMB radiation is that of a perfect blackbody everywhere on the sky, with the same temperature. But, clearly, opposite points on the sky are out of the causal contact, as they lie outside of each others observable universe. Why are they still in a thermal equilibrium then?
The second problem is called the flatness problem. We observe that the universe is flat, i.e. $1-\Omega \approx 10^{-4} \approx 0$. Remembering equation (5)

$$\frac{\kappa c^2}{R_0^2 a^2} = H^2(\Omega - 1)$$

At time t_0 when $a_0 = 1$

$$\frac{\kappa c^2}{R_0^2} = H_0^2(\Omega_0 - 1)$$

and therefore

$$\frac{1-\Omega}{1-\Omega_0} = \frac{H_0^2}{a^2 H^2} = \frac{1}{a^2 \frac{H^2}{H_0^2}}$$

remembering the equation (25) now

$$\frac{1-\Omega}{1-\Omega_0} = \frac{1}{a^2(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + (1-\Omega_0)a^{-2})}$$

For early universe, a is small, and so the matter and radiation terms will dominate, with

$$\frac{1-\Omega}{1-\Omega_0} \approx \frac{a^2}{\Omega_{m,0}a + \Omega_{r,0}}$$

$$1 - \Omega = \frac{a^2}{a\Omega_{m,0} + \Omega_{r,0}}(1-\Omega_0)$$

That means that in the very early universe, the universe must have been even flatter than it is now. Was this just a random chance? Or is it a consequence of some more complicated process.
The last problem is more of a particle physics problem - if we put no restrictions on the generation of magnetic monopoles, we would expect the monopole particles to be present in the early universe, but no such presence is detected.
Most of these problems can be solved by the idea of inflation period of the universe.

4.1 Inflation

During the inflation, we suspect a scalar field of some unknown inflaton particles emerges, with properties simmilar to dark energy, which after some time t_i again disappears. During the inflation time, we assume that the universe behaves like de Sitter universe - dark energy dominated universe, and therefore expands exponentially. To somewhat quantify this, consider a case when until time t_1, the universe is radiation dominated, so at t_1, the scale factor is

$$a_1 = \left(\frac{t_1}{t_0}\right)^{\frac{1}{2}}$$

After that, the universe expands exponentially as

$$a = a_1 e^{H_i(t-t_1)}$$

Until it stops inflating at t_2, $t_2 = t_1 + t_i$, so that the final scale factor a_2 is

$$a_2 = a_1 e^{H_i(t_2-t_1)}$$

Hence, we have

$$\frac{a_2}{a_1} e^{H_i t_i}$$

Usually, we model inflation with $H_i \approx 10^{36}$ s^{-1} and $t_i \approx 10^{-34}$ s. Therefore, we expect the universe to increase in size by a factor of $e^{100} \approx 10^{43}$. We should notice that although this expansion seems very rapid, it actually took about 100 Hubble times (age of universes at that time), so compared to previous processes, it was relatively slow process, however, much faster than could ever be caused by radiation or matter.
This idea is promising because it solves most of the problems of the benchmark model. If the universe was very small before the inflation, opposite ends of the observed sky were in causal contact and in thermal equilibrium. The inflation then was fast enough for the opposite ends to stay in equilibrium, although they escaped the causal contact.
The flatness problem is also solved. During the inflation, the $1 - \Omega$ evolves as

$$1 - \Omega = \frac{(1-\Omega_0)H_0^2}{a^2 H^2} = \frac{(1-\Omega_0)H_0^2}{H_i^2} a_1 e^{-2H_i(t-t_1)}$$

Hence

$$\frac{1-\Omega_2}{1-\Omega_1} = e^{-2H_i(t_2-t_1)} = e^{-2H_i t_i}$$

Therefore, the $1 - \Omega$ reduces during the inflation by a factor of e^{-200}. This means that $1 - \Omega$ might have been very far away from 0, but during the inflation, the universe was flattened out.
The monopole problem is solved as well - the calculations show that in the post-inflation universe, the monopoles could not form anymore. Relating back to expected density of monopoles in pre-inflation universe, the density of magnetic monopoles should be about 1 monopole particle in observable universe - hence explained why we do not observe them.
Furthermore, during the inflation, the quantum fluctuations in energy get enlarged on macroscopic proportions, which means that after the inflation, the energy density on the macroscopic scale is not uniform, and gravitational force can create structure in the universe - galaxies and galactic clusters.

4.2 Causes of Inflation

In particle physics, we predict that just after the big bang, energy scale of the universe was big enough that all the forces (gravity, strong, weak and electromagnetic force) acted as if mediated by a single field. As time progressed, these forces separated, usually via mechanism of phase transitions (specifically spontaneous symmetry breaking transitions). During these phase transitions, we usually see an emergence of the scalar field in the transitioning field. When strong force decouples from the electroweak force (after gravity decouples from the grand unified force), we predict that the scalar field is the inflaton field.
We can see why the scalar field is a useful model by considering the pressure and energy due to a scalar field. It can be shown that for scalar field ϕ, the energy density and pressure follow

$$\epsilon = \frac{1}{2}(\dot{\phi})^2 + V(\phi)$$

and

$$p = \frac{1}{2}(\dot{\phi})^2 - V(\phi)$$

where $\frac{1}{2}(\dot{\phi})^2$ is the kinetic term of the energy density and $V(\phi)$ is the potential term of the energy density. Specifically, for a static field

$$\epsilon = V(\phi) = -(-V(\phi)) = -p$$

which means that the state equation of the static scalar field is $p = -\epsilon$, and thus the static scalar field has the same properties as dark energy.

4.3 Baryogenesis

After the inflation, the universe returns to the radiation dominated era. It then expands, containing photons which can freely create baryon-antibaryon pairs, until the temperature drops enough so that the photons do not have enough energy anymore (this happens much later). However, for reasons not very well understood, some of the photons or some of the baryon products do not conserve the matter-antimatter symmetry, and over time, an excess of matter in the universe builds up.
At the stage where this symmetry breaking starts, the universe is less than 1 s old and the temperature is still about 10^{12} K, and the number of photons is about billion times the number of baryons in the universe. The expected probability of the anti-symmetric decay of the photon is estimated as

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} \approx 10^{-9}$$

where n_γ is the number of decaying photons, n_b is the number of created baryons and $n_{\bar{b}}$ is the number of created anti-baryons.
We are still not sure what causes this symmetry breaking, but we have so called Sacharov conditions on any candidate process

1. Baryon number is not conserved in this process
2. Charge and charge-parity violation occurs
3. The interactions of the particles and photons creating them must be out of thermal equilibrium

4.4 Nucleosyntesis

As the temperature continues to drop, the baryons start to get locked up in nuclei of atoms. To understand this process properly, consider that at high temperature, protons and neutrons can be converted to each other via β-type reactions

$$n + e^+ \leftrightarrow p + \bar{\nu}_e$$

$$n + \nu_e \leftrightarrow p + e^-$$

Classically, we would expect the number of protons and neutrons to be in thermal equilibrium and follow Maxwell-Boltzmann distribution, with

$$n = g\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}} e^{-\frac{mc^2}{k_BT}}$$

where n is the number concentration of certain species of particle, m is the mass of the particle and g is the degeneracy of reactions it can be created by. For both neutrons and protons, $g \approx 2$, so we could expect

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} e^{-\frac{(m_n - m_p)c^2}{k_BT}} \approx e^{-\frac{Q_n}{k_BT}}$$

where I used the fact that $m_n \approx m_p$ and labeled $Q_n = (m_n - m_p)c^2$ the energy difference between protons and neutrons. However, these reactions do not occur always - they are mediated by weak interaction and need presence of neutrinos. As the universe expands, the number density of neutrinos gets lower and the reactions eventually stop - the ratio of neutrons and protons stops changing. We call this the freeze out.
The rate of conversion reactions is $\Gamma = n_\nu \sigma_W c$, where n_ν is the number density of neutrinos, σ_W is the cross-section of the reaction, $\sigma_W \approx 10^{-47}\ \mathrm{m}^2 \frac{k_B^2 T^2}{(1\ \mathrm{MeV})^2}$ and we assume that the neutrinos move approximately at the speed of light. In radiation dominated universe,

$$n_\nu \propto a^{-3} \propto t^{-\frac{3}{2}}$$

$$\sigma_W \propto T^2$$

We can assume that $T \propto a^{-1} \propto t^{-\frac{1}{2}}$ and therefore

$$\Gamma \propto t^{-\frac{5}{2}}$$

The freeze-out then occurs when the rate of reaction reaches one reaction per Hubble time - age of universe at that time, $\Gamma = H$, which happens at about 1s at $T \approx 10^{10}$ K and the neutron-proton ratio is about 0.2.

The resulting protons and neutrons can react via several reactions to form deuterium and helium nuclei. Reactions to form higher number nuclei are restricted, as they all have to go via the ^{8}Be path, which requires triple α reactions, and there is not enough time for a significant amount of these nuclei to occur, although some do occur. We can therefore approximate that nearly all neutrons are locked up in the helium nuclei. Therefore, the number of helium nuclei is

$$n_{He} = \frac{1}{2} n_n$$

The number of hydrogen nuclei is given by the remainder of protons

$$n_H = n_p - 2n_{He} = n_p - n_n$$

and therefore

$$\frac{n_{He}}{n_H} = \frac{n_n}{2(n_p - n_n)} = \frac{\frac{n_n}{n_p}}{2\left(1 - \frac{n_n}{n_p}\right)} \approx \frac{0.2}{2(1-0.2)} = \frac{1}{8}$$

The mass ratio of the helium in the universe is then

$$\varphi = \frac{m_{He}}{m_{He} + m_H} = \frac{1}{1 + \frac{m_H}{m_{He}}} \approx \frac{1}{1 + \frac{n_H}{4n_{He}}} = \frac{1}{3}$$

where I used that the mass of one hydrogen nucleus is approximately a quarter of the mass of helium nucleus, as it contains quarter of the nucleons. In reality, we observe $\varphi \approx 0.24$, as some of the neutrons are bound in higher elements and some have decayed before they could react.

4.5 Structure Formation

After the nucleosynthesis, the gravity starts to act on the energy density fluctuations caused by the inflation. The dynamic timescale for a collapse of a matter of excess density ρ can be determined to be

$$t_{Dyn} = \frac{1}{\sqrt{4\pi G\rho}}$$

To sketch how this is derived, consider a sphere of excess density ρ. The force due to gravity acting on the edge of the sphere is

$$F = \frac{G(\rho V)^2}{r^2}$$

where V is the volume of the sphere. The acceleration of the edge of the sphere is approximately

$$a \approx \ddot{r} = \frac{F}{\rho V} = \frac{G\rho V}{r^2} = \frac{4\pi G\rho r}{3}$$

where I used $V = \frac{4}{3}\pi r^3$. This is a differential equation with solution

$$r = r_0 e^{-\sqrt{\frac{4\pi G\rho}{3}}t}$$

Up to the factor of $\sqrt{3}$, we can see that this is the form of the dynamic timescale.
However, the collapse does not continue arbitrarily, but is opposed by the build up of pressure. The pressure can build up if the timescale of the collapsing is about the same as the speed of sound in the material (otherwise the collapse happens as a shockwave). Hence, we define the pressure timescale as

$$t_p = \frac{r}{c_s}$$

where r is the radius of the cloud and c_s is the speed of sound. From kinetic theory

$$c_s = \sqrt{\frac{\partial p}{\partial \rho}} = c\sqrt{\frac{\partial p}{\partial \epsilon}} = c\sqrt{\omega}$$

This means that the size of the structure formed is determined by equality

$$t_p = t_{Dyn}$$

if $t_p > t_{Dyn}$, the collapse cannot be counteracted by pressure. This means

$$\frac{R}{c_s} = \frac{1}{\sqrt{4\pi G\rho}}$$

$$R = \frac{c_s c}{\sqrt{4\pi G\epsilon}}$$

A proper calculation leads to

$$\lambda_J = c_s c\sqrt{\frac{\pi}{G\epsilon}}$$

where λ_J is the so called Jeans length. Alternatively, we could determine Jeans mass

$$M_J = \frac{4}{3}\pi\rho\lambda_J^3$$

These are parameters of the structure that could form from density fluctuations ρ/ϵ.

4.5.1 Structure in Cosmological Terms

We could notice that

$$t_{Dyn} = \frac{1}{\sqrt{4\pi G\rho}} = \sqrt{\frac{3c^2}{8\pi G\epsilon}}\sqrt{\frac{2}{3}} = \frac{1}{H}\sqrt{\frac{2}{3}}$$

where I used Friedmann equation for flat universe. Since $\frac{1}{H} = t_H$, we have

$$t_{Dyn} = \sqrt{\frac{2}{3}}t_H$$

Hence, the Jeans condition becomes

$$\sqrt{\frac{2}{3}}\frac{1}{H} = \frac{R}{c\sqrt{\omega}}$$

and

$$\lambda_J = 2\pi c\sqrt{\omega}\sqrt{\frac{2}{3}}\frac{1}{H}$$

Importantly, in radiation dominated universe, this means that $\lambda_J \approx \frac{c}{H}$, which means that the fluctuation would have to be over the size of the universe, which cannot happen. Thus, in radiation dominated universe, the structure cannot form.
It is only once the universe enters the matter dominated stage that the structure can form. At that point, $\omega = \frac{k_B T}{\mu c^2}$ where μ is the average mass of a particle, and

$$\lambda_J = 2\pi\sqrt{\frac{2}{3}}\sqrt{\frac{k_B T}{\mu}}\frac{1}{H} = 2\pi\sqrt{\frac{2}{3}}\frac{c_s}{H}$$

which leads to $M_J \approx 10^5 M_\odot$, which is a mass of observed dwarf galaxies - structure can therefore form.
The reason why we observe much bigger structures is probably the dark matter - as it does not react with radiation, it can independently collapse even in radiation dominated universe, creating primordial blackholes that serve as seeds of galaxies.
If we want to be more precise, we should mention that the collapse happens when the matter cannot transition to radiation anymore, which happens already in the matter dominated stage of the universe.

4.5.2 First Generation Stars

Stars that form from this first collapse are composed mainly of hydrogen and helium and usually are larger then later generation stars.

4.6 The Big Bang

Before the inflation and any separation of forces, the universe was essentially in a single point. We call this event the Big Bang, and it is a very hard to describe state of the universe, as the scale of events is so small that it requires a quantum treatment, but the gravity definitely plays an important role. Therefore, we need a quantum theory of gravity to explain what is going on, and we do not have one. Some candidates are the string theory, various M-theories, loop quantum gravity and others.
String theory tries to model every particle as 1D objects in an attempt to solve the problem of non-renormalizability of gravity. As such, string theory is also one of the candidates for a theory of everything. The quantum loop gravity only tries to quantise spacetime via so called spin networks. It is not a theory of everything - only tries to create quantum theory of general relativity.

4.6.1 Timescale of Big Bang

We usually talk about so called Planck time, and Planck length, which is a combination of quantum and gravitational constants that has dimensions of time/length, i.e.

$$t_p = \sqrt{\frac{G\hbar}{c^5}} \approx 10^{-44}\ \mathrm{s}$$

$$l_p = t_p c \approx 10^{-35}\ \mathrm{m}$$

At this size and time in the universe history, general relativity probably breaks down and we need the new, quantized theory of gravity.

4.6.2 Causes of Big Bang

It is not clear why the Big Bang took place at all. It might have been just a fluctuation in a vaccuum energy - given a long enough time, the Heisenbergs uncertainty principle predicts that arbitrarily large energy fluctuation is possible.
Other possibility, suggested by the string theories, is the idea of ekpyrotic universe - the universe is described as a surface of a higher dimensional plane. The Big Bang could then be caused by collision of two of these higher dimensional planes.

5 Summary of Single Component Universes

Dominating Energy Source	**Energy Density Scaling**	**Expansion Equation**	**Hubble Parameter**
Radiation	$\frac{\epsilon_0}{a^4}$	$a = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}$	$\frac{1}{2t}$
Matter	$\frac{\epsilon_0}{a^3}$	$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$	$\frac{2}{3t}$
Dark Energy	ϵ_0	$a = e^{H_0(t-t_0)}$	H_0
Curvature	0	$a = \frac{t}{t_0}$	$\frac{1}{t}$