

PX153 - Maths for Physicists

Section 3 - Linear Algebra

3.1 Matrix Terminology

- A matrix is a rectangular array of numbers of size $m \times n$, with m rows and n columns

- An element can be referred to as a_{ij} - row i , column j

- A matrix can be represented as $A = (a_{ij})_{m \times n}$. a_{ij} can then be defined.
eg.

$$C = (C_{ij})_{2 \times 2} \quad C_{ij} = i - j \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- The transpose of a vector, denoted by A^T , is produced by swapping the rows and columns of A

Special Matrices

- $\theta_{m \times n}$ - all elements equal to zero
- $A_{n \times n}$ - Square matrix
- $A = \text{diag}(a_{11}, a_{22} \dots a_{nn})$ - diagonal matrix, $a_{ij} = \theta \quad \forall i \neq j$
- $I_n = \text{diag}(1, 1 \dots 1)$ - diagonal matrix with only 1s eg. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Triangular matrix - $a_{ij} = \theta$ if $i < j$ OR $i > j$

3.2 Matrix operations

- 2 Matrices can be added if they have the same size

$$A = (a_{ij})_{m \times n} \quad B = (b_{ij})_{p \times q} \quad \text{if } m=p \text{ and } n=q$$

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$

- A matrix can be multiplied by a scalar by multiplying each element by the scalar

$$A = (a_{ij})_{m \times n}$$

$$\lambda A = (\lambda a_{ij})_{m \times n}$$

- 2 Matrices can be multiplied if the number of columns of the first is the same as the number of rows on the second

- $A = (a_{ij})_{m \times n} \quad B = (b_{ij})_{p \times q}$

If $n=p$ AB is valid but BA is not (if $q \neq m$)

AB will have size $m \times q$

- The product of 2 matrices AB is defined at the dot product of the product of the i^{th} row of A with the j^{th} column of B

- $AB = C = (c_{ij})_{m \times q}$ where $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$

Example

$$A =$$

$$B = \begin{bmatrix} 5 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(5) + 2(0) & 1(-3) + 2(-2) \\ -1(5) + 3(0) & -1(-3) + 3(-2) \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -5 & -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5(1) - 3(-1) & 5(2) - 3(3) \\ 0(1) - 2(-1) & 0(2) - 2(3) \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 2 & -6 \end{bmatrix}$$

$$AB \neq BA$$

- The commutator of 2 matrices is defined as

$$[A, B] = AB - BA$$

- If you multiply a row vector by a column vector a dot product is produced

$$(A)_{1 \times n} \times (B)_{n \times 1} = \left(\sum_{k=1}^n a_k b_k \right)_{1 \times 1}$$

- If you multiply a column vector by a row vector an outer product is produced

$$(B)_{n \times 1} \times (A)_{1 \times n} = (M)_{n \times n}$$

3.3 Gaussian Elimination

- Gaussian elimination can be used to reduce a system of equations or a matrix
- There are 3 allowed operations:
 - Interchange 2 equations/rows
 - Replace an equation/row by itself + C times another equation
 - Multiply an equation/row by a non-zero constant
- This is done to produce equations that are easy to solve in row reduced echelon form
 - The first non-zero entry should be 1
 - The first non-zero entry should be to the right of the previous row
 - All other column entries that contain a leading 1 are zero

3.4 Trace and Determinants

↳ Both only defined for Square Matrices

- The trace of a matrix is the sum of diagonal elements

$$\text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

$$-\text{Tr}(A+B) = \text{Tr} A + \text{Tr} B$$

$$-\text{Tr}(AB) = \sum_{i=1}^n a_{ii} b_{ii}$$

- The determinant of a square matrix A is denoted as $|A|$ or $\det A$. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |A| = ad - bc$$

Cofactor C_{ij} $(-1)^{i+j}$

- The minor m_{ij} of a_{ij} for matrix A is defined as the determinant of $(n-1)(n-1)$ matrix obtained by removing row i and column j

$$-\text{The cofactor } C_{ij} = (-1)^{i+j} m_{ij}$$

- The determinant of any matrix can then be defined as

$$|A| = \sum_{j=1}^n a_{ij} C_{ij}$$

$$C_{ij} = (-1)^{i+j} m_{ij}$$

- For 3 vectors $\vec{e}, \vec{f}, \vec{g}$, their scalar triple product $\vec{e} \cdot (\vec{f} \times \vec{g})$ can be calculated from the determinant of a matrix of their components

$$\vec{e} \cdot (\vec{f} \times \vec{g}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

L It is also the volume of the parallelpiped formed by the 3 vectors

Properties of Determinants

- $|A| = |A^T|$
- $|AB| = |A||B|$
- $|I_n| = 1$
- $|\lambda A| = \lambda^n |A|$ where $A = (a_{ij})_{n \times n}$
- If 1 row of A is multiplied by λ , $|A|$ is multiplied by λ
- If B is obtained from A by interchanging 2 rows or columns,
 $|B| = -|A|$
- If a row i is replaced by $row i + \lambda row j$, $i \neq j$ then the determinant is unchanged
- If 2 rows are identical, $|A|=0$

→ The determinant of a matrix can be found from any row or column

3.5 Matrix Inversion

- The adjugate Adj. of a matrix $A = (a_{ij})_{m \times n}$ is defined as $(C_{ij})^T$

$$\text{Adj } A = (C_{ij})^T$$

- The inverse of a matrix is defined as

$$A^{-1} = \frac{1}{|A|} \text{Adj } A \quad \rightarrow \text{Only if } |A| \neq 0$$

Properties of Matrix Inverses

- If A and B have inverses ($|A|, |B| \neq 0$) then $(AB)^{-1} = B^{-1}A^{-1}$
- If $A = (a_{ij})_{n \times n}$ then $(A^T)^{-1} = (A^{-1})^T$

3.6 Solving $A\vec{x} = \vec{b}$

LU Decomposition

- A matrix A can be decomposed into a lower and upper triangular matrix $A = LU$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$= \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix}$$

↳ This can be used to solve $A\vec{x} = \vec{b}$ for \vec{x} by decomposing it to

$$U\vec{x} = \vec{y} \quad \text{and} \quad L\vec{y} = \vec{b}$$

↓
Solve Second ↓
Solve first

Matrix Inversion

- $A\vec{x} = \vec{b}$ Can be solved using matrix inversion as

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \rightarrow A^{-1}A = I \text{ and } IC = C$$

$$\vec{x} = A^{-1}\vec{b}$$

3.7 Special Matrices

Symmetric and AntiSymmetric Matrices

- If $A = A^T$, A is Symmetric
 - If $A = -A^T$, A is antiSymmetric
- Any $n \times n$ matrix is the sum of a Symmetric and antisymmetric matrix

Orthogonal Matrices

- Matrix A is orthogonal if $A^T = A^{-1}$
↳ eg.

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad A^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$
$$AA^T = \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \sin \theta \cos \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= I_2 \rightarrow \therefore A = A^T$$

Singular Matrices

- If $\det A = 0$, A has no inverse and is Singular

Hermitian Conjugate Matrices

- The hermitian conjugate of A is $A^H = (A^*)^T = (A^T)^*$
↳ $A^* = (a_{ij}^*)_{m \times n} \rightarrow$ Complex Conjugate of each element
- If $A = A^H$ then A is hermitian
- If $A = -A^H$ then A is anti-hermitian

- Any Square matrix is the sum of a hermitian and antihermitian matrix

Unitary Matrices

- Matrix A is unitary if $A^T = A^{-1}$

3.8 Matrix Operations on vectors

- A matrix A can be used to map a vector \vec{x} to another vector \vec{b}
 $\hookrightarrow A\vec{x} = \vec{b}$

Rotation in 3D

- A rotation anticlockwise around the z -axis transformed by matrix

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A rotation anticlockwise around the y -axis transformed by matrix

$$T = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- A rotation anticlockwise around the x -axis transformed by matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Stretching/Shrinking

- A vector can be enlarged by transformation matrix

$$T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

- Transformation matrix $T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ will invert vector \vec{r} to $-\vec{r}$

3.9 Eigenvalues and Eigenvectors

- If $A\vec{x} = \lambda \vec{x}$, \vec{x} is the eigenvector of A and λ is the eigenvalue
 - $\lambda = 0$ is not an acceptable eigenvalue
 - For Square matrices only

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda \vec{x} = 0$$

$$(A - \lambda I)\vec{x} = 0 \rightarrow \text{from this } |A - \lambda I| = 0$$

→ Using this, a quadratic of λ is formed, which can be solved for λ s and \vec{x} s

- If the eigenvector found has infinite solutions (due to identical simultaneous equations), it can be normalised by setting $\vec{x}^T \cdot \vec{x} = 1$, and solving for the free parameter

Example: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3$$

$$0 = \lambda^2 - 6\lambda + 5$$

$$0 = (\lambda-5)(\lambda-1)$$

$$\lambda = 5 \quad \lambda = 1 \quad \rightarrow \text{Eigenvalues}$$

For $\lambda = 1$: $(A - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 4-1 & 1 \\ 3 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 + x_2 \\ 3x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \text{If } x_1 = t, x_2 = -3t$$

$\therefore \vec{x} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ where t is a free parameter

$$\vec{x}^T \cdot \vec{x} = (t \ -3t) \cdot \begin{pmatrix} t \\ -3t \end{pmatrix} = t^2 + 9t^2 = 1$$

$$10t^2 = 1$$

$$t = \frac{1}{\sqrt{10}}$$

$$\therefore \text{for } \lambda=1 \quad \vec{x} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\text{For } \lambda=5: \quad (A - \lambda I) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -x_1 + x_2 \\ 3x_1 - 3x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \text{if } x_1=t, x_2=t$$

$$\vec{x} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x}^T \cdot \vec{x} = (t \ t) \cdot \begin{pmatrix} t \\ t \end{pmatrix} = t^2 + t^2 = 2t^2 = 1$$

$$\therefore t = \frac{1}{\sqrt{2}}$$

$$\therefore \text{for } \lambda=5 \quad \vec{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- If \vec{x} is an eigenvector, then $\alpha \vec{x}$ will also be one

3.10 Properties of Special Matrices

- Vectors are orthogonal and normalised if

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$$\vec{x}^T \cdot \vec{x} = 1 \quad \vec{y} \cdot \vec{x} = 0$$

- If vectors are complex, they are orthogonal if

$$\vec{x}^T \cdot \vec{y} = 0$$

- For Hermitian matrices ($A = A^T = (A^*)^T$)
 - Eigenvalues are always real
 - If $\lambda_i \neq \lambda_j$, eigenvectors are orthogonal
- For Unitary matrices ($U^T = U^{-1}$)
 - $|\lambda_i|^2 = 1 \rightarrow$

3.11 Basis changes and Similarity transformations

- A Vector basis can be transformed to a new set of coordinates by multiplying by a transformation matrix S
 - If $\vec{x}' = S\vec{x}$, $\vec{x} = S^{-1}\vec{x}'$
- A Similarity transform is a mapping whose transformation matrix can be written in the form

$$A' = S A S^{-1}$$
 - A' and A are "similar matrices" - they have the same Tr, det and λ_n
 - S is a nonsingular square matrix
- $I' = I$
- $|A'| = |A|$
- $|A' - \lambda I'| = |A - \lambda I|$
- $\text{Tr } A' = \text{Tr } A$
- S can be used to transform A to make A' diagonal if S is composed of the eigenvectors of A

