PX155 Classical Mechanics & Special Relativity

Section 1 - Newton's Laws and Forces

1.1 Newton's Laws

N1 - Every body continues in its state of rest or uniform motion unless acted on by an external force

↳ This only holds in inertial frames of reference – a FOR not undergoing acceleration or rotation

N2 - The rate of change of momentum of a body is equal to the total external force acting upon it

$$F = m\frac{dv}{dt} = ma$$

→ This only applies if the mass is content

N3 - To every action there is an equal and opposite reaction

1.2 Scalars and vectors

> Scalar – physical quantity that has magnitude only
eg. mass, time

> Vector – physical quantity that has magnitude and direction
eg. force, velocity, displacement, momentum, acceleration

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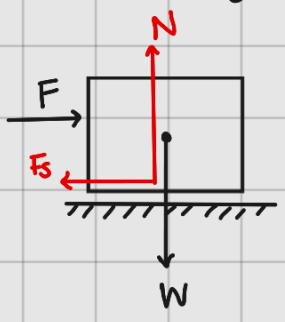
1.3 Contact Forces

Friction is the result of electrostatic forces between the atoms of two objects in contact

> Static friction – The friction force that arises between 2 bodies that are not moving relative to each other

Considering a block of weight W at rest on a horizontal surface



→ N is the normal reaction opposing W

→ F_s is the static frictional force opposing F

F_s provides a maximum force, which if exceeded by F, will cause the block to accelerate

$$\boxed{F_s \leq \mu_s N}$$

where μ_s is the coefficient of static friction

↳ maximum $F_s = \mu_s N$

Once motion occurs ($F >$ max F_s), the coefficient changes

$$\boxed{F_k = \mu_k N}$$

where μ_k is the coefficient of kinetic friction

In general $\mu_k < \mu_s$, so movement continues once it starts

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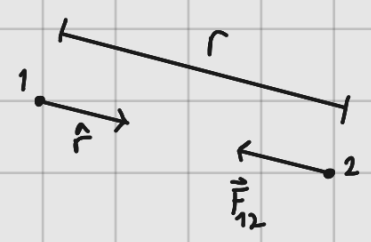
1.4 Gravitational Force

The gravitational force between 2 point masses m_1 and m_2 distance r apart is given by

$$\boxed{F = \frac{G m_1 m_2}{r^2}}$$

Where $G = 6.674 \times 10^{-11}\ Nm^2kg^{-2}$



The force exerted <u>by</u> mass 1 <u>on</u> mass 2 can be notated as

$$\boxed{\vec{F}_{12} = -\frac{G m_1 m_2}{r^2}\hat{r}}$$

where $\hat{r}$ is the unit vector pointing from 1 to 2

↳ The negative sign in the equation means it is an attractive force

Newton's shell theorem can be used to calculate gravitational forces on finite masses

Newton's shell theorem:

- For any particle outside the shell, the shell acts as if all its mass is concentrated in a point particle at the centre of the shell
- For any particle inside the shell, the grantational force due to the shell is zero

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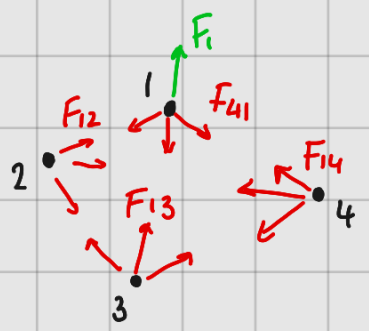
Section 2 - Systems of Particles and kinematics

2.1 Systems of Particles



A system of N particles at position vectors $\vec{r}_i$ Can have the force of particle j on particle i represented as $\vec{F}_{ji}$, and the external force on the particle $\vec{F}_i$

The total force on each particle is $\vec{F}_i + \sum_{j \neq i} \vec{F}_{ji} = m_i \vec{a}_i$

For all N particles summed, this is $\sum_i \vec{F}_i + \sum_i \sum_{j \neq i} \vec{F}_{ji} = \sum_i m_i \vec{a}_i$

Using N3, for every $\vec{F}_{ji}$ there is a cancelling $\vec{F}_{ij}$, leaving

$$\vec{F} = \sum_i \vec{F}_i = \sum_i m \vec{a}_i$$

By multiplying top and bottom by total mass $M = \sum_i m_i$ $\rightarrow$ $\vec{F} = \left(\sum_i m_i\right) \dfrac{\sum_i m_i \vec{a}_i}{\sum_i m_i}$

and substituting $\vec{a}_i = \dfrac{d^2 \vec{r}_i}{dt^2}$ $\rightarrow$ $\vec{F} = M \dfrac{d^2}{dt^2} \left(\dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \right)$

producing an equation for the centre of mass

$$\boxed{\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}}$$

which would have acceleration $\vec{a}_{cm} = \dfrac{\sum_i m_i \vec{a}_i}{\sum_i m_i}$

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So the total external force on the system of particles is

$$\boxed{\vec{F} = M\vec{a}_{cm}}$$

For continuous bodies where $N \to \infty$ and $m_i \to 0$

$$\boxed{\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{\int dm}}$$

where $\int \vec{r}\, dm = \int x\, dm + \int y\, dm + \int z\, dm$

So the x component of CoM is

$$x_{cm} = \frac{\int x\, dm}{\int dm}$$

Example: A bar of length L has mass per unit length $\mu = \alpha x$ where α is constant and x is the distance from one end. What is the position of the centre of mass

Using $x_{cm} = \dfrac{\int x\, dm}{\int dm}$

$$M = \int dm = \int_0^L \mu\, dx = \int_0^L \alpha x\, dx = \frac{1}{2}\alpha L^2$$

$$\int x\, dm = \int_0^L x\mu\, dx = \int_0^L \alpha x^2\, dx = \frac{1}{3}\alpha L^2$$

$$\therefore\ x_{cm} = \frac{\frac{1}{3}\alpha L^3}{\frac{1}{2}\alpha L^2} = \frac{2}{3}L$$

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2.2 Modelling Definitions

> Smooth – zero friction
> light – zero mass, no force needed to accelerate it
> rigid – does not bend
> flexible – completely bendy → strings are always flexible
> ideal – an ideal string is light, flexible and doesn't stretch
an ideal pulley is light, frictionless and rigid
> uniform – constant quantity throughout eg. density
> elastic collision – conserves kinetic energy
> inelastic collision – kinetic energy lost to heat, sound etc
> particle – point mass, size and rotation ignored

2.3 Equations of Motion

Constant Acceleration:

$$a = \frac{dv'}{dt'} \longrightarrow dv' = a\,dt'$$

integrating this gives

$$\int_u^v dv' = \int_0^t a\,dt' = a\int_0^t dt'$$

$$v - u = at \longrightarrow \boxed{v = u + at}$$

Similarly, $v = \frac{dx'}{dt'} \longrightarrow dx' = v\,dt'$

$$\int_{x_0}^{x} dx' = \int_0^t v\,dt' = \int_0^t (u + at')\,dt$$

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$$\boxed{s = x - x_0 = ut + \frac{1}{2}at^2}$$

where x_0 = original position
x = current position
s = distance travelled

by substituting $t = \frac{v-u}{a}$:

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

$$2as = 2u(v-u) + (v-u)^2 \rightarrow \boxed{v^2 = u^2 + 2as}$$
$$= 2uv - 2u^2 + v^2 - 2uv + u^2$$

Time Dependent acceleration

$$a = a(t') = \frac{dv'}{dt'} \longrightarrow dv' = a(t)\,dt'$$

$$\int_u^v dv' = \int_0^t a(t')\,dt \quad \text{then solve}$$

Position dependent acceleration:

$$a = a(x) = \frac{dv}{dt} \longrightarrow \int dv = \int a(x)\,dt'$$

↳ we don't know $x = x(t)$, so can rearrange using the chain rule

$$a(x) = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx} = v\frac{dv}{dx}$$

$\therefore \quad a\,dx = v\,dv$ ↓ integrate this

$$\boxed{\int_u^v v\,dv = \int_0^x a(x)\,dx} = \frac{1}{2}(v^2 - u^2)$$

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Velocity-dependent acceleration:

$$a = a(v) = \frac{dv}{dt} \longrightarrow \frac{dv}{a(v)} = dt$$

$$\boxed{\int_u^v \frac{1}{a(v)}\, dv = \int_0^t dt}$$

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Section 3 - Work and Energy

3.1 Work and kinetic Energy

The equation for position dependent acceleration $\int a\,dx = \int v\,dv$ can be modified so that the variable of integration is v^2 using:

$$\frac{d}{dv} v^2 = 2v \longrightarrow \frac{1}{2} dv^2 = v\,dv \longrightarrow \int v\,dv = \frac{1}{2}\int v^2 = \int a\,dx$$

If a particle of mass m is accelerating, using N2 finds:

$$\int ma\,dx = \int F\,dx = \frac{m}{2}\int d(v^2)$$

Kinetic energy can be defined as $\boxed{T = \frac{1}{2}mv^2}$ which can be used to give work done

$$\boxed{W = \int F\,dx = \Delta T}$$

$\rightarrow$ If force is constant $W = F\int dx = F(x_B - x_A)$

i.e $W = Fd$

For an infinitesimal step (so small F doesn't vary) $\boxed{dW = F\,dx = dT}$

This can be extended to 3D, giving:

$$dW = F_x\,dx + F_y\,dy + F_z\,dz = \vec{F}\cdot d\vec{r}$$

Where each term can be expressed as

$$F_x\,dx = m a_x\,dx = \frac{m}{2} d(v_x^2) = d\left(\frac{1}{2} m v_x^2\right)$$

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Thus $dW = d\left(\frac{1}{2}mv_x^2\right) + d\left(\frac{1}{2}mv_y^2\right) + d\left(\frac{1}{2}mv_z^2\right)$

generalising the definition of kinetic energy to

$$T = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) = \frac{1}{2}m\vec{v}\cdot\vec{v}$$

So in 3D $dW = \vec{F}\cdot d\vec{r} = dT$

$$\boxed{W = \int \vec{F}\cdot d\vec{r} = \Delta T}$$

3.2 Potential Energy

> Conservative force field – Work done is independent of the path taken

In a conservative field, the work done moving a particle from a fixed origin O to an arbitrary point P, $W(O \to P)$ only depends on P, defining a scalar function of P, potential energy, U.

$\underline{U = -W(O \to P)}$ $\longrightarrow$ negative sign so that work done moving from high to low potentials is positive

For a particle moving from A to B:

$$\begin{aligned} W(A \to B) &= W(A \to O) + W(O \to B) \\ &= -W(O \to A) + W(O \to B) \\ &= U(A) - U(B) = -\left(U(B) - U(A)\right) = -\Delta U \end{aligned}$$

$\Delta T = W$ always, and for conservative forces $\Delta T = -\Delta U$, so $\Delta(T+U) = 0$

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$$\boxed{T + U = \text{Constant} = E}$$ E = total energy

An equation for force can be derived from

$$U = -T = -\frac{1}{2}mv^2$$

$$\frac{dU}{dx} = -\frac{1}{2}m\frac{dv^2}{dx}$$ → From position dependent acceleration earlier remember $a\,dx = v\,dv = \frac{1}{2}dv^2$

↳ $\frac{1}{2}\frac{dv^2}{dx} = a$

$$\frac{dU}{dx} = -ma$$

$$\boxed{F = -\frac{dU}{dx}}$$

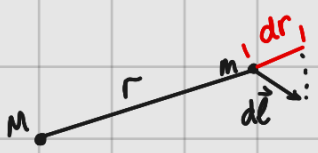
Which can be written in 3D as

$$\boxed{\vec{F} = -\vec{\nabla} U} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

When <u>you</u> do work to move a particle, you apply a force equal and opposite to the field, So you do work $W = \Delta U$

3.3 Gravitational Potential Energy

If mass m moves $d\vec{l}$ under gravitational attraction from mass M then



$$dW = \vec{F}\cdot d\vec{l} = -\frac{GMm}{r^2}\hat{r}\cdot d\vec{l} = -\frac{GMm}{r^2}dr$$

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Integrating gives

$$W = -GMm \int_A^B \frac{1}{r^2}\, dr = GMm\left(\frac{1}{r(B)} - \frac{1}{r(A)}\right)$$

↳ Gravity is conservative with potential energy

Assuming $U(\infty) = 0$, gpe can be defined as

$$\boxed{U(r) = -\frac{GMm}{r}}$$

"Gravitational potential" is used to mean potential energy per unit mass

$$u(r) = \frac{U(r)}{m} = -\frac{GM}{r}$$

3.4 Power

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$ → Power is a scalar

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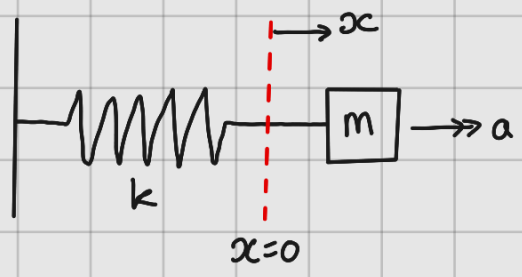
Section 4 - Simple Harmonic Motion

4.1 Mass on a Spring



The spring produces a force on the mass

$$\boxed{F = -kx}$$

Where k is the spring constant

Using N2:

$$-kx = ma = m\frac{d^2x}{dt^2} \longrightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

this can be written as

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0} \qquad \boxed{\omega_0^2 = \frac{k}{m}}$$

To Solve, try a Solution in the form $x(t) = Ae^{\lambda t}$ or $A\cos(\lambda t)$

$$\frac{dx}{dt} = \lambda A e^{\lambda t} = \lambda x \qquad \frac{d^2x}{dt^2} = \lambda^2 A e^{\lambda t} = \lambda^2 x$$

Sub into ODE:

$$\lambda^2 x = -\omega_0^2 x$$
$$\lambda = \pm i\omega_0$$

So the general Solution is $\boxed{x(t) = Ae^{i\omega_0} + Be^{-i\omega_0}}$

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$$x(t) = A\cos(\omega_0 t) + Ai\sin(\omega_0 t) + B\cos(\omega_0 t) - Bi\sin(\omega_0 t)$$
$$= (A+B)\cos(\omega_0 t) + i(A-B)\sin(\omega_0 t)$$

$$\boxed{x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)}$$

$$C = A+B \qquad D = i(A-B)$$

Frequency and time period of SHM are given by:

$$\boxed{f = \frac{\omega}{2\pi}} \qquad \boxed{T = \frac{1}{f} = \frac{2\pi}{\omega}}$$

Motion is periodic, so $x(t+nT) = x(t)$ for $n \in \mathbb{Z}$
The general solution can alternatively be written as

$$\boxed{x(t) = A'\cos(\omega t + \phi)}$$

A' = amplitude
ϕ = phase angle

The shortest horizontal distance between the peak of a wave and $T=0$ is given by ϕ/ω

To determine constants in the solution, the initial conditions can be substituted into the general solution

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Example: A mass $m = 2kg$ is attached to a spring of strength $k = 8 Nm^{-1}$. At $t=0$, $x = 2m$ and $\dot{x} = 6 ms^{-1}$.

General solution $\quad x(t) = C \cos(\omega t) + D \sin(\omega t)$

$$\dot{x}(t) = -C\omega \sin(\omega t) + D\omega \cos(\omega t)$$

At $t=0$: $\quad x(0) = 2 = C(1) + D(0) =$

$$C = 2$$

$$\dot{x}(0) = 6 = -C\omega(0) + D\omega(1)$$

$$D\omega = 6$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2 \longrightarrow D = 3$$

So $x(t) = 2\cos 2t + 3 \sin 2t$

4.2 Energy and SHM

The work an external force needs to do <u>against</u> a spring force to stretch it is

$$W = \int_0^x kx' \, dx' = \frac{1}{2} kx^2$$

If $x=0$ is defined as zero potential, the potential energy of the spring is

$$\boxed{U = \frac{1}{2} kx^2}$$

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The sum of kinetic and potential energy for a mass oscillating with SHM is constant. If $x = A\cos(\omega t)$ then

$$U = \frac{1}{2} k A^2 \cos^2 \omega t \qquad T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m A^2 \omega^2 \sin^2 \omega t$$

$$T = \frac{1}{2} k^2 A^2 \sin^2 \omega t \qquad \left(\text{using } \omega^2 = \frac{k}{m}\right)$$

$$E = T + U = \frac{1}{2} k A^2 (\sin^2 \omega t + \cos^2 \omega t) = \frac{1}{2} k A^2$$

So the average of each energy is given by

$$\boxed{\langle T \rangle = \langle U \rangle = \frac{1}{4} k A^2}$$

4.2 Damped Oscillation

Another term can be added to the SHM Equation to provide a frictional force proportional to velocity

$$m \frac{d^2 x}{dt^2} = -kx - \gamma \frac{dx}{dt}$$ which can be rearranged to give

$$\boxed{\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2 x = 0}$$ where $\alpha = \frac{\gamma}{2m}$

or $\ddot{x} + 2\alpha \dot{x} + \omega^2 x = 0$

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Try solution $x(t) = Ae^{\lambda t}$:

$\dot{x} = \lambda A e^{\lambda t} = \lambda x$ $\qquad$ $\ddot{x} = \lambda^2 A e^{\lambda t} = \lambda^2 x$

Substituting into the ODE gives:

$$\lambda^2 x + 2\alpha\lambda x + \omega^2 x = 0$$
$$\lambda^2 + 2\alpha\lambda + \omega^2 = 0$$

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$

This can then be substituted into the trial solution to give the general solution

$$\boxed{\begin{aligned} x(t) &= Ce^{\lambda_1 t} + De^{\lambda_2 t} \\ &= e^{-\alpha t}\left(Ce^{t\sqrt{\alpha^2-\omega_0^2}} + De^{-t\sqrt{\alpha^2-\omega_0^2}}\right) \end{aligned}}$$

4.3 Light/Under Damping

$\alpha^2 < \omega^2$ produces a small damping force

Write $\omega' = \sqrt{\omega^2 - \alpha^2}$ so that the general solution becomes

$$x(t) = e^{-\alpha t}\left(Ce^{i\omega' t} + De^{-i\omega' t}\right)$$

Considering initial conditions $x(0) = 0$ $\quad v(0) = 0$ gives

$$\boxed{x(t) = e^{-\alpha t} x_0 \cos(\omega' t)}$$

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4.4 Heavy / Overdamping

$\alpha^2 > \omega^2$ produces a large damping force

let $K = \sqrt{\alpha^2 - \omega_0^2}$ So that $x(t) = e^{-\alpha t}\left(Ce^{Kt} + De^{-Kt}\right)$

Considering initial conditions $x(0) = 0$ $v(0) = 0$ gives

$$x(t) = e^{-\alpha t}\left(\frac{1}{2}x_0\left(e^{Kt} + e^{-Kt}\right)\right)$$

$$\boxed{x(t) = e^{-\alpha t} x_0 \cosh(Kt)}$$

4.5 Critical Damping

$\alpha^2 = \omega^2$

Using trial solution $x(t) = Ae^{\lambda t}$ gives $\lambda_1 = \lambda_2 = -\alpha$, so a second solution can be tried $x(t) = Dte^{\lambda t} = Dte^{-\alpha t}$

$$\frac{d}{dt}x(t) = -\alpha Dte^{-\alpha t} + De^{-\alpha t}$$
$$= De^{-\alpha t}(1 - \alpha t)$$
$$\frac{d^2}{dt^2}x(t) = -\alpha De^{-\alpha t} + (1 - \alpha t)(-\alpha De^{-\alpha t})$$
$$= \alpha De^{-\alpha t}(\alpha t - 2)$$

Substituting into the ODE gives

$$\alpha^2 t - 2\alpha + 2\alpha - 2\alpha^2 t + \omega_0 t = 0$$

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$$t(\omega_0^2 - \alpha^2) = 0$$

So $x(t) = Dt e^{-\alpha t}$ is a solution provided $\omega_0^2 = \alpha^2$
Giving a general solution of

$$\boxed{x(t) = Ce^{-\alpha t} + Dte^{-\alpha t}}$$

Considering initial conditions $x(0) = 0$ $v(0) = 0$ gives

$$x(t) = x_0 e^{-\alpha t}(1 + \alpha t)$$

4.6 Forced Oscillations

Adding a driving force to the damped SHM equation gives

$$\boxed{\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = A_0 \cos(\omega t)}$$

The solution to the complementary equation was found to be

$$x_c(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$$

where

$$\lambda_1 = -\alpha + i\omega' \qquad \omega' = \sqrt{\omega_0^2 - \alpha^2}$$
$$\lambda_2 = -\alpha - i\omega'$$

For the particular integral, $x_p(t) = a\cos(\omega t) + b\sin(\omega t)$ can be substituted into the ODE, then by equating coefficients can find

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$$a = \frac{A_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2} \qquad b = \frac{A_0 \; 2\alpha\omega}{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega_0)^2}$$

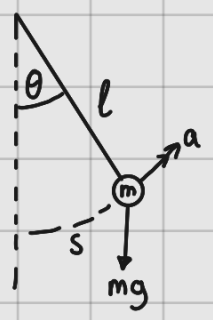
This does not produce an easy to read particular integral, So $x_p(t)$ can instead be written as

$$x_p(t) = A\sin(\omega t + \phi) = A\cos\phi\sin\omega t + A\sin\phi\cos\omega t$$

By comparing this to the particular integral, values can be found

$$\boxed{A = \frac{A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2}}} \qquad \boxed{\tan\phi = \frac{\omega_0^2 - \omega^2}{2\alpha\omega}}$$

4.7 Pendulums



For a mass m on string of length l, forces can be resolved

$$ma = -mg\sin\theta \qquad a = \frac{d^2s}{dt^2} \qquad s = l\theta$$

$$\frac{d^2\theta}{dt} = -\frac{g}{l}\sin\theta$$

For <u>Small θ</u>, the higher order terms in $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 \ldots$ can be neglected, leaving

$$\frac{d^2\theta}{dt^2} \approx \frac{g}{l}\theta$$

→ Try Solution $\theta(t) = Ae^{\omega t}$

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which can be used to find $\omega^2 = \frac{g}{\ell}$, leading to the equation for the period of a simple pendulum

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

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Section 5 - Circular Motion

5.1 Circular Motion

A particle travelling in a circle of radius r. By angle θ it has travelled distance $s = r\theta$. As r is constant:

$$v = |\vec{v}| = \frac{ds}{dt} = \frac{r\,d\theta}{dt}$$

With $\omega = d\theta/dt$ where ω is angular velocity

$$\boxed{v = \omega r}$$

$T = \frac{2\pi}{\omega}$ gives the time period for one complete revolution

> Centre of rotation - point around which the particle rotates
> plane of rotation - 2D plane which contains the circle described by the particle
> axis of rotation - line through the centre of rotation normal to the plane of rotation

The angular velocity vector, $\vec{\omega}$, can be defined as a vector of length ω pointing in a direction normal to the plane of rotation
The velocity vector $\vec{v}$ is always perpendicular to $\vec{r}$, the position vector pointing from the centre outwards

Using these, you can define

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

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The velocity vector moves at a constant speed but moves in a circle so there is a centripetal acceleration defined as

$$\boxed{a = \omega v = \omega^2 r = \frac{v^2}{r}}$$

and $\vec{a}$ is perpendicular to $\vec{v}$, pointing to the centre of rotation

5.2 Moments

Torque can be defined as the tendency of a force to cause a change in an objects rotational motion

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

The direction of τ can be determined by the right hand rule: $\vec{r}$ is thumb, $\vec{F}$ is index finger and $\vec{\tau}$ = middle finger

→ Torque must <u>always</u> be measured with respect to a specified point of reference

Using N2: $\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt}$

So

$$\tau = \vec{r} \times \vec{F} = m\vec{r} \times \frac{d\vec{v}}{dt}$$

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{= \vec{v} \times \vec{v} = 0} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{r} \times \frac{d\vec{v}}{dt}$$

So $\tau = m\vec{r} \times \frac{d\vec{v}}{dt} = m\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d}{dt}(\vec{r} \times \vec{p})$

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$\vec{r} \times \vec{p}$ is defined as angular momentum $\boxed{\vec{L} = \vec{r} \times \vec{p}}$

So $\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$

If a body is in equilibrium, its angular momentum does not change, so there is zero torque about any point

5.3 Systems of particles

For a system of particles with internal and external forces acting, the force on particle i is given by $\frac{d\vec{p}_i}{dt} = \vec{F}_i + \sum_{j \neq i} \vec{F}_{ji}$

So the torque around a point O is:

$$\vec{\tau} = \vec{r}_i \times \left(\vec{F}_i + \sum_{j \neq i} \vec{F}_{ji} \right) = \frac{d\vec{L}_i}{dt}$$

As internal forces between particles cancel out, the total torque from external forces equals the rate of change of total angular momentum

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i = \frac{d}{dt}\left(\sum_i \vec{L}_i \right) = \frac{d\vec{L}}{dt}$$

Consider a system of particles constituting a rigid body with common angular velocity $\vec{\omega}$ that rotate about a common axis. The origin O is a point lying on the axis.

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The angular momentum of a single particle is:

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times (m_i \vec{v}_i) = m_i \vec{r}_i \times \underbrace{(\vec{\omega} \times \vec{r}_i)}$$

this only holds if O is on axis of rotation

This vector triple product can be written as:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

So $\vec{L} = m_i \left(\vec{\omega}_i (\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i (\vec{r}_i \cdot \vec{\omega}_i) \right)$

By expanding this into vector components and simplified to be

$$\vec{L} = I \vec{\omega} \quad \text{where} \quad I = \sum_i m_i (x_i^2 + y_i^2)$$

for the case that the mass is symmetrically about the axis of rotation

5.4 Circular Angular Momentum

Generally, the angular momentum about an origin O of particle with momentum $\vec{p}$ and position vector $\vec{r}$, relative to O, is given by $\vec{L} = \vec{r} \times \vec{p}$

For the specific case of circular motion with angular velocity ω with O at the centre of rotation

As $\vec{p} = m\vec{v}$, $\vec{L}$ is parallel to $\vec{\omega}$ along the axis of rotation, and has magnitude

$$\boxed{L = mvr = mr^2\omega}$$

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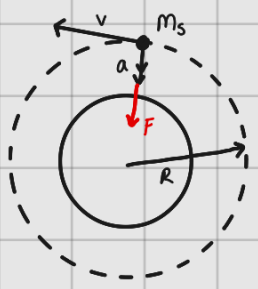
Moment of inertia can be defined as $\boxed{I = mr^2}$ and can be used to define:

$$\boxed{\vec{L} = I\vec{\omega}} \qquad \boxed{T = \frac{1}{2} I \omega^2}$$

↳ This only applies about the centre of rotation

5.5 Orbital Angular Momentum



A satellite orbiting the earth can be treated as a point mass with angular momentum about the Earth's centre with magnitude $L = I\omega = m_s \omega R^2$

A gravitational force acting provides centripetal acceleration

$$F = \frac{G M_E m_s}{R^2} = \frac{m_s v^2}{R}$$

The torque generated by this force is given by $\vec{\tau} = R \times \vec{F}$, however as the vectors act in opposite directions to each other, $\vec{\tau} = 0$

- Forces that act towards the centre of rotation generate no torque
  ↳ angular momentum is conserved

For circular orbits, orbital speed is conserved and is given by

$$\boxed{v = \sqrt{\frac{GM}{R}}}$$

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5.6 Continuous Rigid Bodies

Kinetic energy for rotation of rigid bodies is given by

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) = \frac{1}{2} I \omega^2$$

When $I = \sum_i m_i r_i^2 = \int r^2 dm = \int r^2 \rho \, dV$

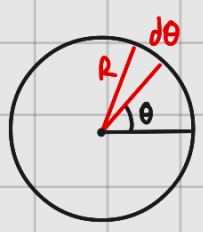
↳ r is the perpendicular distance between each mass element and the axis

For a 2D system $\vec{L} = I\vec{\omega}$

If $\vec{\tau}$ is parallel to $\vec{\omega}$, then $\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$

Example:

Calculate the moment of inertia of a thin ring of radius R, mass M about an axis through its centre, perpendicular to the plane of the ring:



← integrating around the ring

Assume a mass per unit length along the ring of $\sigma = \frac{m}{L}$

A small section $d\theta$ has mass $dm = \sigma \underbrace{R}_{\text{length}} d\theta$

So $I = \int r^2 dm = \sigma R^3 \int_0^{2\pi} d\theta = 2\pi\sigma R^3$

as $M = \underbrace{2\pi R}_{\text{circumference} \to \text{ring length}}\sigma$, we have $I = MR^2$

Example

Calculate the moment of inertia of a uniform circular disc of radius R mass M about an axis through its centre.

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← Integrating centre - outwards

→ Split into rings and integrate

Assume mass per unit area σ, then mass of a thin ring radius r, radial thickness dr is $dm = \underbrace{2\pi r}_{\text{Circumference}} \sigma \, dr$

It is r from all points, hence

$$I = \int r^2 \, dm = \int_0^R 2\pi\sigma r^3 \, dr = \frac{1}{2}\pi\sigma R^4$$

The mass of the disc $M = \underbrace{\pi R^2}_{\text{area}} \sigma$, so $I = \frac{1}{2} M R^2$

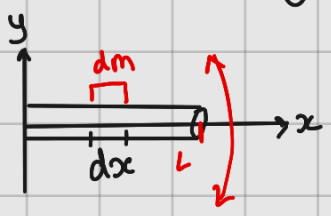
$M = \sigma A$

Moment of inertia examples:

- Hollow sphere about an axis through its centre $I = \frac{2}{3} M R^2$
- Uniform solid sphere about an axis through its centre $I = \frac{2}{5} M R^2$
- Thin rod about a perpendicular axis through its centre $I = \frac{1}{12} M L^2$
- Thin rod about a perpendicular axis through one end $I = \frac{1}{3} M L^2$

Derivations

→ Thin rod length L mass M about one end



→ Assume mass per unit length $\sigma = \frac{M}{L}$ $M = \sigma L$

$\therefore \; dm = \sigma \, dL$

as the rod lies along the x axis this becomes

$dm = \sigma \, dx$

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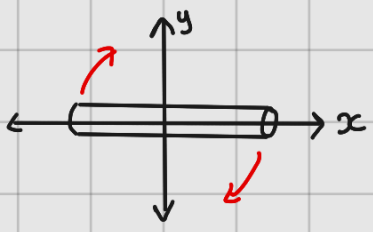
$$I = \int r^2\,dm = \int_0^L x^2 \sigma\,dx = \frac{1}{3}\sigma L^3$$

as $M = \sigma L$, this becomes $I = \frac{1}{3}ML^2$

→ Thin rod length L mass M about centre



Assume mass per unit length $\sigma = \frac{M}{L}$ $M = \sigma L$

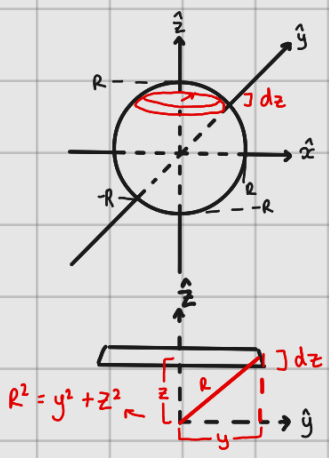
$dm = \sigma\,dl$ can be changed to $dm = \sigma\,dx$ as rod lies along x axis

$$I = \int r^2\,dm = \int_{-\frac{L}{2}}^{L/2} x^2 \sigma\,dx = \frac{1}{3}\sigma\left[x^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{3}\sigma\left(\frac{L^3}{8} + \frac{L^3}{8}\right)$$

$$I = \frac{1}{12}\sigma L^3$$

as $M = \sigma L$, $I = \frac{1}{12}ML^2$

→ Uniform solid sphere mass M radius R about its centre



- Assume mass per unit volume $\sigma = \frac{M}{V}$ → $M = \frac{4}{3}\pi R^2 \sigma$
- Sphere can be split up into infinitesimally thin disks

Each disk has radius $y = \sqrt{R^2 - z^2}$ So moment of inertia of $I_{disc} = \frac{1}{2}y^2\,dm$ where dm is the mass of the disk.

$dm = \sigma V = \sigma\pi y^2\,dz$ → $I_{disc} = \frac{1}{2}\sigma\pi y^4\,dz$

To get the moment of inertia of the whole sphere, integrate from bottom to top of sphere for all disks

$$I = \frac{1}{2}\sigma\pi\int_{-R}^{R} y^4\,dz = \frac{1}{2}\sigma\pi\int_{-R}^{R}(R^2 - z^2)^2\,dz = \frac{1}{2}\sigma\pi\left[R^4 z - \frac{2}{3}R^2 z^3 + \frac{1}{5}z^5\right]_{-R}^{R}$$

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$$I = \frac{1}{2}\sigma\pi\left(\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) - \left(-R^5 + \frac{2}{3}R^5 - \frac{1}{5}R^5\right)\right)$$

$$I = \frac{1}{2}\sigma\pi\left(\frac{16}{15}R^5\right) = \frac{8}{15}\sigma\pi R^5$$

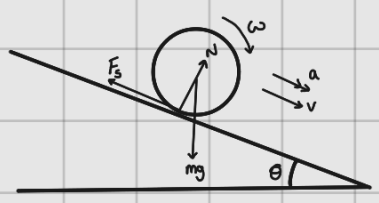
using $M = \frac{4}{3}\pi\sigma R^3$ $\quad I = \frac{8}{15}\pi R^5\left(\frac{3M}{4\pi R^3}\right) = \frac{2}{5}MR^2$

5.7 Parallel axis theorem

> The moment of inertia of an object of mass M around an axis that lies at a perpendicular distance d from the object's centre of mass equals the moment of inertia around a parallel axis through the centre of mass plus Md^2

$$\boxed{I\,(\text{any axis}) = I_{cm}(\parallel) + Md^2}$$

5.8 Cylinder Rolling down a slope



A cylinder with mass m, radius R and moment of inertia I rolls without slipping down a slope. What is its acceleration?

Forces ⊥ to plane: $N = mg\cos\theta$

Forces ∥ to plane: $mg\sin\theta - F_s = ma$

$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$ $\quad \tau = rF_s = I\dot{\omega}$ → N and mg act through centre so add no torque

$L = mvr = I\omega$

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As there is no slip, $v = r\omega$, $a = r\dot{\omega}$

$$F_s = \frac{I\dot{\omega}}{r} = \frac{Ia}{r^2}$$

$$mg\sin\theta - \frac{Ia}{r^2} = ma \rightarrow a = \frac{g\sin\theta}{1 + I/mr^2}$$

5.9 The Compound Pendulum

Consider an object that pivots around an axis distance d from its centre of mass

$$\tau = -\overbrace{Mg}^{F}\ \overbrace{d\sin\theta}^{\perp r}$$

↳ Negative as torque acts clockwise and θ increases anticlockwise

$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt} \rightarrow I\ddot{\theta} = -Mgd\sin\theta$$

using $I = I_{cm} + Md^2$, for small θ, $\sin\theta \approx \theta$ so

$$(I_{cm} + Md^2)\ddot{\theta} + Mgd\theta = 0 \rightarrow \ddot{\theta} + \frac{Mgd}{I_{cm} + Md^2}\theta = 0$$

This has the standard SHM form

$$\boxed{\omega^2 = \frac{4\pi^2}{P^2} = \frac{1}{1 + (I_{cm}/Md^2)}\left(\frac{g}{d}\right)}$$

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Example

A circular disc of mass M and radius $R = 20\,cm$ is suspended from a pivot $d = 5\,cm$ from its centre. What is its period?

The disc has $I_{cm} = \frac{1}{2}MR^2$

$$\omega^2 = \left(1 + \frac{\frac{1}{2}MR^2}{Md^2}\right)^{-1}\left(\frac{g}{d}\right) = \left(1 + \frac{R^2}{2d^2}\right)^{-1}\left(\frac{g}{d}\right)$$

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(1 + \frac{R^2}{2d^2}\right)\left(\frac{d}{g}\right)}$$

$$P = 2\pi\sqrt{\left(1 + \frac{0.2^2}{2 \times 0.05^2}\right)\left(\frac{0.05}{9.81}\right)} = \underline{1.35\,s}$$

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Section 6 - Collisions

6.1 Collisions in 1D

Consider a particle mass m_1, speed u_1, which collides from the left with a particle mass m_2, speed $u_2 < u_1$. What are their final speeds v_1 and v_2?
↳ Speeds relative to a stationary lab frame

General assumptions:

- Total momentum p is conserved (No external force F)
- Potential energy is irrelevant

$$p_1 + p_2 = q_1 + q_2$$
$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Considering the speed of the centre of mass

$$\boxed{V_{cm} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{p_1 + p_2}{m_1 + m_2} = \frac{q_1 + q_2}{m_1 + m_2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}}$$

↳ V_{cm} is <u>unchanged</u> by the collision

If the speeds were measured in a different frame where the origin is the position of the centre of mass, quantities measured would be notated as v_1', u_2' etc. To find the speeds in the lab frame

$$u_1 = u_1' + V_{cm} \qquad u_1' = u_1 - V_{cm}$$
$$v_1 = v_1' + V_{cm} \qquad v_1' = v_1 - V_{cm}$$

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Conservation of momentum implies $p_1' + p_2' = q_1' + q_2' = 0$
So the particles momenta are equal and opposite

— Completely inelastic Collisions (All KE lost):

→ The two particles stick together after the collision, $q_1' = q_2' = 0$ and all KE is lost <u>in the COM frame</u>, final speed $= V_{cm}$

↳ All KE is <u>NOT</u> lost in the lab frame

— Elastic Collision (No KE Lost):

$$T_{Before} = \frac{(p_1')^2}{2m_1} + \frac{(p_2')^2}{2m_2} = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(p')^2$$

$$T_{After} = \frac{(q_1')^2}{2m_1} + \frac{(q_2')^2}{2m_2} = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(q')^2$$

Where $p' = |p_1'| = |p_2'|$ and $q' = |q_1'| = |q_2'|$

- If no energy is lost, $p' = q'$. Velocities reverse direction but keep the same magnitude in the COM frame → $V_1' = -U_1'$

Example:

A particle of mass $m_1 = 7kg$ travelling at $u_1 = 5ms^{-1}$ collides with one of $m_2 = 3kg$ travelling at $u_2 = -5ms^{-1}$. Calculate their final velocities assuming the collision is

a) Completely inelastic
b) elastic

$$V_{cm} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{7(5) + 3(-5)}{7+3} = 2ms^{-1}$$

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In the CoM frame $u_1' = u_1 - v_{cm} = 5 - 2 = +3\,ms^{-1}$

$u_2' = u_2 - v_{cm} = -5 - 2 = -7\,ms^{-1}$

a) Completely inelastic

→ particles join so $v_1 = v_2 = v_{cm} = 2\,ms^{-1}$

b) Elastic

$p' = q'$ so $|p_1'| = |q_1'|$ so $m_1|u_1'| = m_1|v_1'|$

$v_1' = -u_1' = -3\,ms^{-1}$

$v_2' = -u_2' = +7\,ms^{-1}$

6.2 Collisions in 3D

Momentum Conservation is $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2$

$$\vec{v}_{cm} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

For inelastic collisions, again $\vec{v}_1 = \vec{v}_2 = \vec{v}_{cm}$

For elastic collisions, the particles speed stays the same in CoM frame

Example: Particle $m_1 = 5kg$ travelling at $u_1 = 2ms^{-1}$ collides with initially stationary particle $m_2 = 7kg$. The first particle departs the collision with a new velocity vector at an angle $\theta_1 = 30°$ to its original path. Assuming the collision is elastic, find the angle θ_2 at which the second mass leaves the collision

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Choose axes so that u_1 moves only along the x axis and v_1 lies in the xy-plane

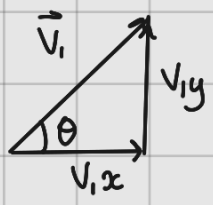
$$\vec{V}_{cm} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1+m_2} = \frac{5(2\hat{i}) + 7(0)}{5+7} = \frac{5}{6}\hat{i}$$

In the CoM frame:

$$\vec{u}_1' = \vec{u}_1 - V_{cm} = 2i - \frac{5}{6}i = \frac{7}{6}i$$
$$\vec{u}_2' = \vec{u}_2 - V_{cm} = 0 - \frac{5}{6}i = -\frac{5}{6}i$$

The speeds are unchanged in the elastic collision

$V_1' = \frac{7}{6}\,ms^{-1}$ $V_2' = \frac{5}{6}\,ms^{-1}$ and are in opposite directions



$$\vec{V}_1' = V_1'(\cos\theta_1'\,\hat{i} + \sin\theta_1'\,\hat{j})$$
$$\vec{V}_2' = -V_2'(\cos\theta_1'\,\hat{i} + \sin\theta_1'\,\hat{j}$$

Break down speeds into components

↳ Same θ as move in opposite directions in CoM frame

$$\vec{V}_1 = \vec{V}_1' + V_{cm} = (V_1'\cos\theta_1' + V_{cm})\hat{i} + V_1'\sin\theta_1'\,\hat{j}$$
$$\vec{V}_2 = \vec{V}_2' + V_{cm} = (V_2'\cos\theta_1' + V_{cm})\hat{i} - V_2'\sin\theta_1'\,\hat{j}$$

$$\tan\theta_1 = \tan\left(\frac{\pi}{6}\right) = \frac{V_{1y}}{V_{1x}} = \frac{V_1'\sin\theta_1'}{V_1'\cos\theta_1' + V_{cm}}$$

↳ $\theta_1' = 50.92°$

$$\tan\theta_2 = \frac{V_{2y}}{V_{2x}} = \frac{V_2'\sin\theta_1'}{V_2'\cos\theta_1' - V_{cm}}$$

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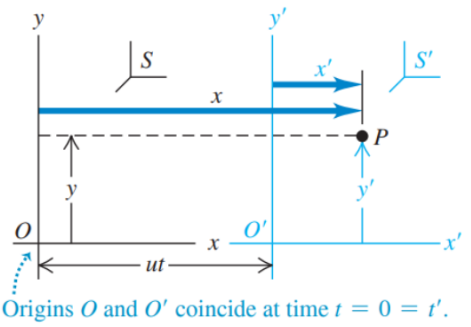
Section 7 - Special Relativity

7.1 Galilean Transformations

Galileio's principle of relativity – The laws of mechanics are the same in all uniformly moving reference frames

> Inertial frame – A uniformly moving reference frame

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.



Origins O and O' coincide at time $t = 0 = t'$.

Consider 2 inertial frames S and S', oriented such that S' moves away from S in the $+x$ direction relative to S at speed u, with their relative axes aligned

- At a time $t = t' = 0$, the origins of each frame are aligned to the same position. After t seconds, frame S has moved away a distance ut
- A point P can be measured using coordinates in S, (x, y, z) and in S' (x', y', z')

↑ the image above shows how x and x' and ut can be related

The Galileian transforms are

$$t' = t \qquad t = t'$$
$$x' = x - ut \qquad x = x' + ut$$
$$y' = y \qquad y = y'$$
$$z' = z \qquad z = z'$$

→ x and x' are the "x-axis coordinates" of point p in frames S and S' respectively

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If an object moves at velocity $\vec{v}$ measured in frame S', its speed in S can be found to be $V_x = V_x' + u$

$$V_x = \frac{dx}{dt} = \frac{d}{dt}(x' + ut') = \frac{d}{dt'}(x' + ut')$$

$$= \frac{dx'}{dt'} + \frac{d(ut')}{dt'}$$

$$= V_x' + u$$

- Maxwell showed light travelled at a unique speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
  - According to Galileian velocity addition, the speed of light could be measured to be different in different frames
- The Michelson-Morley Experiment split a light beam and let it travel 2 different distances before joining again. As the light would be out of phase, interference would be observed.
  - This did not happen, proving light moves at c in all frames.

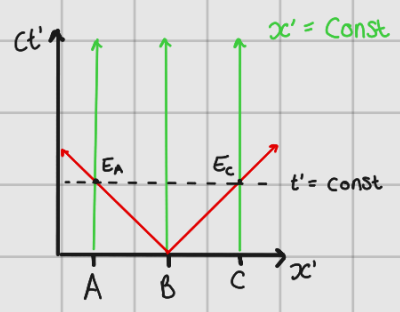
7.2 Postulates of Special Relativity

Einstein's 2 postulates:

1 - The laws of physics are the same in all inertial frames

2 - The speed of light in empty space is the same in all inertial frames

Consider 3 equally spaced observers A, B and C at rest in frame S'. B sends a pulse of light to A and C. Plotting a space-time diagram for S':



→ "World lines" through space time

→ Light travels at c, so $x' = \pm ct'$, so light rays are 45° lines that reach A and C simultaneously

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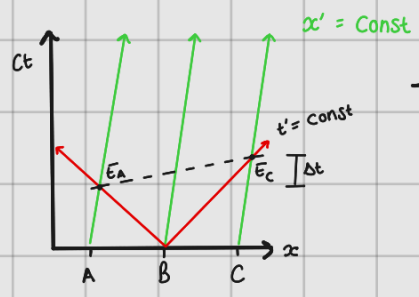
Considering the view in frame S, where A, B and C move at speed $+u$ in the $+x$ direction, so the worldlines are of the form $x = ut + \text{const}$



→ The light paths are still 45° to the new axes as $x = ct$, but events E_A and E_C are no longer simultaneous

↳ $t = t'$ is incorrect!

The Galileian transforms must be revised linearly

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Section 8 - Lorentz Transformations

8.1 Lorentz Transformations

The Lorentz transformations are:

$$t' = \gamma\left[t + \left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{x}{u}\right]$$ where γ is a constant

$$x' = \gamma(x - ut)$$
$$y' = y$$
$$z' = z$$

By using Einstein's Second postulate, a pulse of light emmitted from the origin at $t=t'=0$ in the $+x$ direction will travel a distance $x = ct$ and $x' = ct'$. By substituting equations for x' and t':

$$\gamma(x - ut) = c\gamma\left(t + \frac{1-\gamma^2}{\gamma^2 u}x\right)$$

Can be rearranged using $x=ct$ to give

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}}$$

This allows the Lorentz transformation for t' to be written as

$$t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

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8.2 Length Contraction

- A bar lying at rest in S' on the x-axis has ends at x'_A and x'_B giving it a length $L' = x'_B - x'_A$ in S'
- In S the bar is moving with speed u, so each end must be measured at the same time t in S. Transforming x to x' gives:

$$x'_A = \gamma(x_A - ut)$$
$$x'_B = \gamma(x_B - ut$$

This gives a length $x'_B - x'_A = \gamma(x_B - x_A)$
Using $L = x_B - x_A$, $L' = \gamma L$ for a bar stationary in S'

Doing the same for a bar stationary in S gives $L = \gamma L'$

A general equation can be written as

$$\boxed{L_u = \frac{L_0}{\gamma}}$$

Where L_0 is "proper length" measured in whichever frame the bar is stationary and L_u is measured in the frame moving at speed u

8.3 Time Dilation

A clock at rest in S' measures a time interval $\Delta t' = t'_A - t'_B$. Transforming to S:

$$t_A = \gamma(t'_A - ux'_A/c^2)$$
$$t_B = \gamma(t'_B - ux'_B/c^2)$$

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$x'_A = x'_B$ as the clock is at rest in S'. The time interval in S, Δt, is $\Delta t = \gamma \Delta t'$ for a clock stationary in S'

↳ In frame S, the time interval appears longer

The same can be done for a clock stationary in S
A general equation can be written as:

$$\boxed{\Delta t_u = \gamma \Delta t_0}$$

Where Δt_0 is "proper time" measured in the frame where the clock is stationary

Time dilation was observed in 1941 when muons produced by cosmic rays travelled into the atmosphere. 8% of muons were expected to survive due to their lifetime of $\tau = 2.2\,\mu s$, however 70% were observed.
→ This is due to the muons' lifetime being dilated in our frame

8.4 Relativistic Doppler Effect

Consider a light source L moving away from an observer A at speed u. L emits light with a "proper frequency" f_0.

- The time between 2 pulses is $\Delta t_0 = 1/f_0$. A observes this dilated as
$$\Delta t_u = \gamma \Delta t_0 = \frac{\gamma}{f_0}$$

- In A's frame, in the time taken to emit 2 pulses L moves away by $u \Delta t_u$, delaying the arrival of the second pulse by an additional $\frac{u \Delta t_u}{c}$
- A sees the pulses separated by
$$\Delta t_A = \Delta t_u + \frac{u}{c} \Delta t_u = \gamma \left(1 + \frac{u}{c}\right) \Delta t_0$$

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So A observes frequency $f = \dfrac{1}{\gamma(1 + u/c)\Delta t_0}$

hence

$$\boxed{f = \frac{f_0}{\gamma(1 + u/c)}}$$

for a source moving <u>away</u> from an observer

↳ If moving towards the observer, the + changes to −

8.5 The twin Paradox

- A and B are twins. A takes a journey to a star 30 light years away and back, travelling so fast that, to A, the trip only takes 1 year. When A returns, B is 60 years.
- From A's POV, it was B that went away, so B should be younger?

To solve the problem, identify 3 inertial frames:
- S: B's rest frame (Earth)
- S': A's rest frame, outbound, $+u$ relative to B
- S'': A's rest frame, incoming, $-u$ relative to B

Identify 3 key events:
1. A leaves Earth, $x = x' = x'' = 0$, $t = t' = t'' = 0$
2. A reaches star $x = L$ $t = L/u$
3. A returns to Earth $x = 0$, $t = 2L/u$

Computing t' and t'' remembering $\gamma(u) = \gamma(-u) = \gamma$

$$t' = \gamma\left(t - \frac{ux}{c^2}\right) \qquad t'' = \gamma\left(t + \frac{ux}{c^2}\right)$$

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to obtain:

Time	Event 1 $x=0$	Event 2 $x=L$	Event 3 $x=0$
t	0	L/u	$2L/u$
t'	0	$\gamma(L/u - uL/c^2)$	$\gamma(2L/u)$
t''	0	$\gamma(L/u + uL/c^2)$	$\gamma(2L/u)$

So total time for B: $\Delta t_B = (t_2 - t_1) + (t_3 - t_2) = \frac{2L}{u}$

Total time for A: $\Delta t_A = (t_2' - t_1') + (t_3'' - t_2'')$

$$\Delta t_A = \left[\gamma\left(1 - \frac{u^2}{c^2}\right)\frac{L}{u} - 0\right] + \left[\gamma\frac{2L}{u} - \gamma\left(1 + \frac{u^2}{c^2}\right)\frac{L}{u}\right]$$

$$= \frac{\gamma L}{u}\left(1 - \frac{u^2}{c^2}\right) + \frac{\gamma L}{u}\left(2 - 1 - \frac{u^2}{c^2}\right)$$

$$= \frac{2\gamma L}{u}\left(1 - \frac{u^2}{c^2}\right)$$

$$= \frac{1}{\gamma}\left(\frac{2L}{u}\right) = \frac{\Delta t_B}{\gamma}$$

So A ages less than B by a factor of γ

8.6 Relativistic Velocity Addition

Consider an object moving at speed v' relative to S'. At what speed v does it move in S?

In S, $v = \frac{dx}{dt}$ and $v' = \frac{dx'}{dt'}$. To express v in terms of v', we need to express x and t in terms of x' and t':

$$x = \gamma(x' + ut')$$

$$t = \gamma(t' + ux'/c^2)$$

therefore

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$$V = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{dx}{dt'}\left(\frac{dt}{dt'}\right)^{-1}$$

from the LTs:

$$\frac{dx}{dt'} = \gamma\left(\frac{dx'}{dt'} + u\right) = \gamma(v' + u)$$

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{u}{c^2}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{uv'}{c^2}\right)$$

So $V = \dfrac{\gamma(v'+u)}{\gamma(1+uv'/c^2)} \longrightarrow \boxed{V = \dfrac{v'+u}{1+uv'/c^2}}$ ← Einstein's relation for the addition of velocities

↳ for the inverse, swap $v \leftrightarrow v'$ and $u \leftrightarrow -u$

- The y and z components of the velocity $\vec{v} = (V_x, V_y, V_z)$ can be found to be

$$\boxed{\begin{aligned} V_y' &= \frac{V_y}{\gamma(1 - uV_x/c^2)} \\ V_z' &= \frac{V_z}{\gamma(1 - uV_x/c^2)} \end{aligned}}$$

8.7 How far could you go?

If you set off from Earth with a constant acceleration of 1g, how far could you go in 10 years measured on your clock

Let Earth be frame S, define S' to move at v, so you are instantaneously at rest.

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At time $\delta t'$ later in S', you have accelerated to $\delta v'$

$$\delta v = \frac{v + \delta v'}{1 + v\delta v / c^2} - v \approx \frac{\delta v}{\gamma}$$

from time dilation $\delta t = \gamma \delta t'$, so shrinking $\delta v \to dv$ and $\delta t \to dt$

$$a = \frac{dv}{dt} = \frac{dv'}{dt'} = \frac{1}{\gamma^3} a'$$

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PX155 Classical Mechanics & Special Relativity

Section 9 - Relativistic Mass, Energy and Momentum

9.1 Relativistic Momentum

- The classical approach to momentum conservation does not work with Special Relativity. If 2 particles collide in S', their momenta calculated in S using $p=mv$ is not conserved.
- Using the relativity principle, physical laws are frame independent.

Consider mass as a function of speed, $m=f(v)m_0$, where
- m_0 = rest mass
- $f(v) \to 1$ as $v \to 0$

$\therefore \quad p = mv = f(v)m_0 v$

It can be derived that $f(u) = \gamma(u)$, hence

$$\boxed{m = \gamma m_0} \qquad \boxed{\vec{p} = \gamma m_0 \vec{v}}$$

9.2 Causality and Light Cones

- Events that occurr in the same order in all frames are causal events
  - Can only occurr if there is enough time for light to travel between them

- In a space-time diagram, "cones" of light can be drawn at 45° out of each event ($x=ct$). Another event can only be caused by this if it falls within the light cone

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9.3 Kinetic Energy

Kinetic energy is the work done by a force

$$E_k = \int F\,dx = \int \frac{dp}{dt}\,dx = \int \frac{dp}{dt}\frac{dx}{dt}\,dt = \int \frac{dp}{dt}\,v\,dt$$

Integrating by parts gives $\quad E_k = vp - \int p\,dv \quad$ with $\quad p = \gamma m_0 v$

$$E_k = [vp]_0^v - \int_0^v \gamma m_0 v = vp - \int_0^v \frac{m_0 v}{(1-v^2/c^2)^{1/2}}\,dv$$

which can be solved to give

$$\boxed{E_k = (\gamma - 1)\,m_0 c^2}$$

The total energy of a particle is given by

$$\boxed{E = E_0 + E_k = m_0 c^2 + (\gamma - 1) m_0 c^2 = \gamma m_0 c^2}$$

9.4 Energy, Mass & Momentum

In Special Relativity, mass and energy combine into one conserved quantity

Using $\vec{p} = \gamma m_0 \vec{v}$ and $E = \gamma m_0 c^2$:

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m_0^2 (c^4 - c^2 v^2) \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4 \end{aligned}$$

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$$\boxed{E^2 = (pc)^2 + (m_0c^2)^2}$$

For massless particles: $\boxed{E = pc}$

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