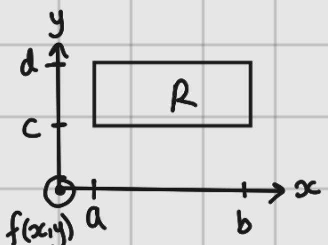
PX153 Maths for Physicists

Section 1 - Integration

1.1 Multiple Integrals

For a Continuous function $f(x,y)$ over rectangular domain R:

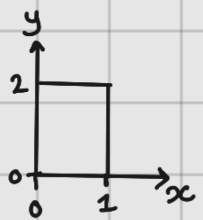


$$\boxed{I = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

→ Changing the order of integration should have no effect on the final result

Example:

Evaluate $I = \iint_R xy \; dx\,dy$ where R has corners $(0,0), (0,2), (1,0), (1,2)$

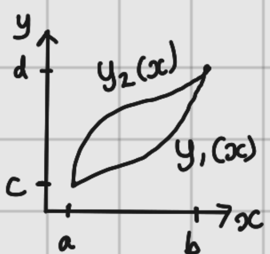


$$I = \int_0^2 \int_0^1 xy \; dx\,dy$$

$$= \int_0^2 \left[\tfrac{1}{2}x^2 y\right]_0^1 dy = \int_0^2 \tfrac{1}{2} y \; dy$$

$$= \left[\tfrac{1}{4}y^2\right]_0^2 = 1$$

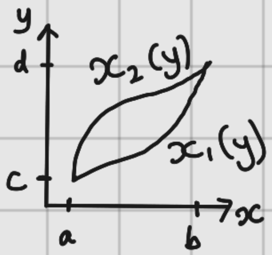
For a non-rectangular region of integration



→ Region R is bounded by 2 curves $y_1(x)$ and $y_2(x)$

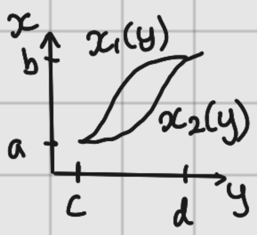
$$\boxed{I = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx}$$

→ Changing order of integration should have no effect on the final result, however method must be changed slightly



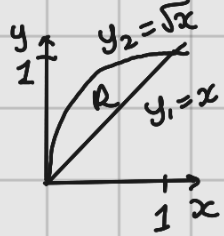
$$\boxed{I = \int_c^d \int_{x_2(y)}^{x_1(y)} f(x,y)\, dx\, dy}$$

↳ here the x_1 function is on top, as you have to integrate from top to bottom with x as the dependent variable



→ by drawing the domain with the x variable on the dependent axis, $x_1(y)$ is on top thus is on top of the integral

Example:



Evaluate $f(x,y) = xy^2$ over domain R

$$I = \int_0^1 \int_x^{x^{1/2}} xy^2\, dy\, dx$$

$$I = \int_0^1 \left[\frac{1}{3}xy^3\right]_x^{x^{1/2}} dx = \int_0^1 \frac{1}{3}x^{5/2} - \frac{1}{3}x^4\, dx$$

$$I = \left[\frac{2}{21}x^{7/2} - \frac{1}{15}x^5\right]_0^1 = \frac{2}{21} - \frac{1}{15} = \frac{1}{35}$$

1.2 Volume Integrals

$$\boxed{I = \iiint f(\vec{r})\, dV}$$

→ notated as such as to be used with any coordinate system

↳ using cartesian coordinate system would be

$$I = \iiint f(x,y,z)\, dx\, dy\, dz$$

↳ using spherical coordinate system would be

$$I = \iiint f(\phi,\theta,r)\; r^2 \sin\theta \; d\theta\, d\phi\, dr$$

↳ using cylindrical coordinate system would be

$$I = \iiint f(\theta, r, z)\; r\, d\theta\, dr\, dz$$

1.3 Coordinate Systems

Spherical Coordinates

(θ, ϕ, r)

θ – angle between vector and z axis

ϕ – angle to the x axis in the xy plane

$0 \leq \theta \leq \pi \qquad 0 \leq \phi \leq 2\pi$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

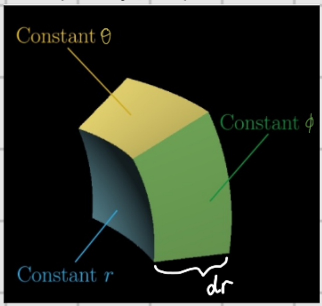
$\sin\theta$ gets component of r in xy plane

$\sin\phi$ or $\cos\phi$ takes x or y component

$$\boxed{dV = r^2 \sin\theta \, d\theta \, d\phi \, dr}$$

how is this derived

In multiple integrals, each individual integral treats all coordinates as constants, except one. The shape being integrated can be divided into infinitely small pieces with volume dV. Each face of the shape shown represents a constant value.

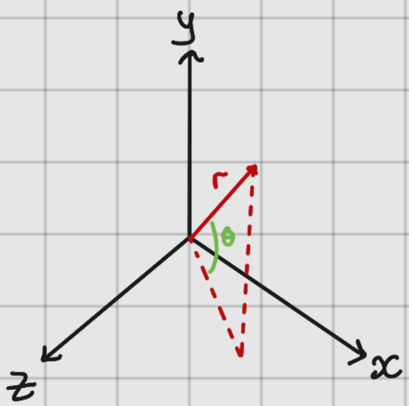


As the size of the block approaches zero, one edge represents a tiny change in length <u>dr</u>. As θ and ϕ are measured in radians, the other sides need to be calculated

- The edge representing change $d\theta$ has length <u>$r\,d\theta$</u> (using $l = r\theta$)
- The edge representing change $d\phi$ is the edge of a circle around the z axis with radius <u>$r\sin\theta$</u>

Putting this together $dV = (dr)(r\,d\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta \, dr\, d\theta\, d\phi$

Cylindrical Coordinates



(r, θ, z)

θ - angle between x axis and vector

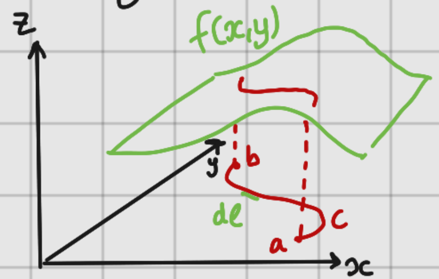
$x = r\cos\theta$ $\quad z = z$

$y = r\sin\theta$

$$dV = r \, d\theta \, dr \, dz$$

1.4 Line Integrals

Visualising Line integrals:



A path in the x-y plane can be parameterised by a variable t such that $x = g(t)$ $y = h(t)$ with endpoints a and b such that $a \leq t \leq b$. This can be denoted as C for curve

In the third dimension, a function of x and y gives value to each xy point.
- A line integral can be used to find the area between the path C and $f(x,y)$

The path is defined by $\vec{r} = x(t)\underline{\hat{i}} + y(t)\underline{\hat{j}}$.
Endpoints a and b can be substituted into $x(t)$ and $y(t)$ to find end values u_1 and u_2

Scalar line Integral $I = \int_C f(x,y,z)\,dl$

- An infintesimaly small change in length of the curve C can be defined as

$$dl = \sqrt{(dx)^2 + (dy)^2} \quad \text{(using pythagoras)}$$

- The area of the curve could then be defined as the sum of rectangular strips

$$\underbrace{f(x,y)}_{\text{strip height}} \cdot \underbrace{dl}_{\text{strip length}} \longrightarrow \underbrace{\int_{t=a}^{t=b} f(x,y)\,dl}_{\text{Sum of Strips}}$$

dl can be rewritten using:

$$dl = \frac{dt}{dt}\sqrt{(dx)^2 + (dy)^2}$$

$$dl = \sqrt{\frac{1}{(dt)^2}\left((dx)^2 + (dy)^2\right)}\;dt$$

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Using our parameterised path $\vec{r} = x(t)\hat{x} + y(t)\hat{y}$

$$dl = \left|\frac{d\vec{r}}{dt}\right| dt = dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)}$$

Therefore the Scalar line integral can be found:

$$I = \int_C f(x,y,z)\ dl = \int_{t_2}^{t_1} f(x,y,z)\left|\frac{d\vec{r}}{dt}\right| dt$$

Vector Line Integral $I = \int \vec{F}\cdot d\vec{l}$

- A vector line integral is a dot product between a vector field $\vec{F}(x,y)$ and infinitesimal segment of C, $d\vec{l}$

The integral for a vector line integral is:

$$I_c = \int_C \vec{F}(x,y)\cdot d\vec{l} = \int_{t_2}^{t_1}\vec{F}(x(t),y(t))\cdot \frac{d\vec{r}}{dt}\ dt$$

Example: Find $\int_C \vec{F}\cdot d\vec{l}$ where $\vec{F}(x,y) = (x+y, x-y)$ along path C: $x = 2t^2+t+1$ $\quad y = t^2+1$ from $(1,1)$ to $(4,2)$

↳ path is already parameterised with t

1. Parameterise end points

At $(1,1)$: $2t^2+t+1=1$ $\qquad t^2+1=1$

$t(2t+1)=0 \qquad t^2=0$

$\underline{t=0} \quad t=-1/2 \qquad \underline{t=0}$ ← Only select values that satisfy both

At $(4,2)$ $2t^2+t+1=4 \qquad t^2+1=2$

$2t^2+t-3=0 \qquad t^2=1$

$(2t+3)(t-1)=0 \qquad \underline{t=1} \quad t=-1$

$\underline{t=1} \quad t=-2/3$

So $\quad 0 \leq t \leq 1$

2. Parameterise path and find $d\vec{r}/dt$

$$\vec{r} = x(t)\hat{x} + y(t)\hat{y}$$

&

$$\hat{r} = (2t^2+t+1)\hat{x} + (t^2+1)\hat{y}$$

$$\frac{d\hat{r}}{dt} = (4t+1)\hat{x} + (2t)\hat{y}$$

3. Parameterise $\vec{F}$ with t

$$\vec{F}(x,y) = (x+y)\hat{x} + (y-x)\hat{y}$$
$$\vec{F}(x(t),y(t)) = (2t^2+t+1+t^2+1)\hat{x} + (1+t^2-2t^2-t-1)\hat{y}$$
$$\vec{F}(x(t),y(t)) = (3t^2+t+2)\hat{x} + (-t^2-t)\hat{y}$$

4. Calculate Line Integral

$$I_c = \int \vec{F} \cdot d\vec{\ell} = \int_{u_1}^{u_2} \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

$$I = \int_{u=0}^{u=1} ((3t^2+t+2)\hat{x} + (-t^2-t)\hat{y}) \cdot ((4t+1)\hat{x} + (2t)\hat{y})$$

$$I = \int_0^1 (4t+1)(3t^2+t+2) + 2t(-t^2-t) \, dt$$

$$I = \int_0^1 10t^3 + 5t^2 + 9t + 2 \, dt$$

$$I = \left[\frac{10}{4}t^4 + \frac{5}{3}t^3 + \frac{9}{2}t^2 + 2t\right]_0^1 = \frac{32}{3}$$

<u>Conservative fields</u> do not show path dependence, and can be written as the gradient of a potential (scalar field)

$$\vec{v} = \vec{\nabla} f$$

- Conservative fields have integrals $\int_a^b \vec{v} \cdot d\vec{\ell}$ which depend only on end points a and b

→ Remember $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Example: Is $\vec{v} = 5y^2\hat{i} + 10xy\hat{j}$ conservative?

If $\vec{v}$ is conservative, there exists $\vec{v} = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$\frac{\partial f}{\partial x} = 5y^2 \rightarrow \int 5y^2\, dx = 5xy^2 + c$$

$$\frac{\partial f}{\partial y} = 10xy \rightarrow \int 10xy\, dy = 5xy^2 + c$$

$\therefore f(x,y) = 5xy^2 + c$, and $\vec{v}$ is Conservative

1.5 Surface Integrals

A Surface can be defined by 2 parameters, eg. $\vec{r}(\lambda, \mu)$
A small change in the surface area dS can be written as

$$dS = \left| \frac{\partial \vec{r}}{\partial \lambda} \times \frac{\partial \vec{r}}{\partial \mu} \right| d\lambda\, d\mu$$

So a Surface integral is

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_S f(\vec{r}(\lambda, \mu)) \left| \frac{\partial \vec{r}}{\partial \lambda} \times \frac{\partial \vec{r}}{\partial \mu} \right| d\lambda\, d\mu}$$

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Section 2 - Fourier Series

6.1 Fourier Series and Coefficients

A Fourier Series can be used to represent a function

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}$$

where $-L \leq x < L$ and a_n and b_n are constants

For a function $f(x)$ to be represented by a fourier Series it must:

- Be periodic $\leftrightarrow$ L is the period
- Be single valued and Continuous, except possibly at a finite number of finite discontinuities
- have a finite number of maxima and minima within one period
- the integral over one period of $|f(x)|$ must converge

The coefficients a_0, a_n and b_n can be found using

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx}$$

Orthogonality Relations:

$(\alpha, \beta \in \mathbb{Z})$

$$\int_{-\pi}^{\pi} \sin(\alpha x)\cos(\beta x)\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin(\alpha x)\sin(\beta x)\,dx = \int_{-\pi}^{\pi} \cos(\alpha x)\cos(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \pi & \text{if } \alpha = \beta \end{cases}$$

6.2 Properties of Fourier Series

- The Fourier Series is periodic, so $f(x+2L) = f(x)$
- If a function is symmetric, $f(x) = -f(x)$, and $b_n = 0$ for all n
  - eg. $\cos(nx)$, $|x|$
- If a function is antisymmetric, $f(x) = -f(-x)$, and $a_n = 0$ for all n
  - eg. $\sin(nx)$, x

- A Fourier Series can be used to prove the result of a series

Example: show $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} (2n+1)^{-2}$ using the Fourier Series

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} \dots\right.$$

let $x=0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} \dots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} \dots$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)^2}\right)$$

6.3 Parseval's theorem

A function $f(x)$ has a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ then Parseval's theorem gives

$$\frac{1}{L}\int_{-L}^{L} (f(x))^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

6.4 Fourier Sine and Cosine Series

If you want to compute a fourier series for a function within a range $0 \le x < L$, a Sine or Cosine Series can be used:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

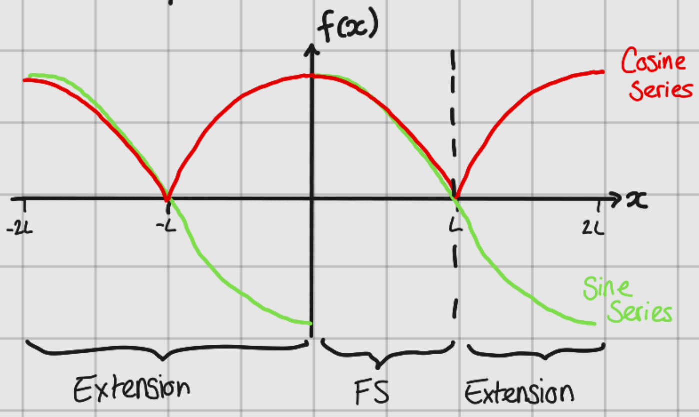
With Coefficients

$$a_0 = \frac{2}{L}\int_0^L f(x) \, dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The periodic extension of a Sine and Cosine Series for the same function $f(x)$ will differ as Cosine is symmetric and Sine is antisymmetric



6.5 Gibbs Phenomenon

- When a function is discontinuous at a value, the Fourier Series at that value will be halfway between the upper and lower value
- For discontinuity $f(x_d)$ at x_d, the Fourier Series will converge to

$$\boxed{\frac{f(x_d^-) + f(x_d^+)}{2}}$$

- At discontinuities, the Fourier Series will overshoot before the discontinuity. As the number of terms increases, the overshoot moves closer to the discontinuity

6.6 Convergence

Pointwise Convergence

- If $f(x)$ and $f'(x)$ are continuous over $[-L, L)$, except possibly at a finite number of points, the Fourier Series converges at every point

$$\boxed{\hat{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-) \right]}$$

where $f(x^+) = \lim_{\varepsilon \to 0} f(x+\varepsilon)$ and $f(x^-) = \lim_{\varepsilon \to 0} f(x-\varepsilon)$

Uniform Convergence

- If $f(x)$ and $f'(x)$ exist and are continuous over $[-L, L)$ and $f(L) = f(-L)$ the Fourier Series converges uniformly to $f(x)$

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Section 3 - Linear Algebra

3.1 Matrix Terminology

- A matrix is a rectangular array of numbers of size $m \times n$, with m rows and n columns
- An element can be referred to as a_{ij} – row i, column j
- A matrix can be represented as $A = (a_{ij})_{m \times n}$. a_{ij} can then be defined.

  eg.

  $$C = (c_{ij})_{2\times2} \qquad c_{ij} = i-j \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- The transpose of a vector, notated by A^T, is produced by swapping the rows and columns of A

Special Matrices

- $0_{m \times n}$ – all elements equal to zero
- $A_{n \times n}$ – Square matrix
- $A = \text{diag}(a_{11}, a_{22} \dots a_{nn})$ – diagonal matrix, $a_{ij} = 0 \ \forall \ i \neq j$
- $I_n = \text{diag}(1, 1 \dots 1)$ – diagonal matrix with only 1s eg. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Triangular matrix – $a_{ij} = 0$ if $i < j$ OR $i > j$

3.2 Matrix operations

- 2 Matrices can be added if they have the same size

  $A = (a_{ij})_{m \times n} \qquad B = (b_{ij})_{p \times q} \qquad$ if $m = p$ and $n = q$

  $A + B = (a_{ij} + b_{ij})_{m \times n}$

- A matrix can be multiplied by a scalar by multiplying each element by the scalar

$A = (a_{ij})_{m \times n}$ $\qquad$ $\lambda A = (\lambda a_{ij})_{m \times n}$

- 2 Matrices can be multiplied if the number of columns of the first is the same as the number of rows on the second
  - $A = (a_{ij})_{m \times n}$ $\qquad$ $B = (b_{ij})_{p \times q}$

    If $n = p$ AB is valid but BA is not (if $q \neq m$)

    AB will have size $m \times q$

- The product of 2 matrices AB is defined at the dot product of the product of the i^{th} row of A with the j^{th} column of B
  - $AB = C = (c_{ij})_{m \times q}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Example

$$A = \qquad B = \begin{bmatrix} 5 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(5) + 2(0) & 1(-3) + 2(-2) \\ -1(5) + 3(0) & -1(-3) + 3(-2) \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -5 & -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5(1) - 3(-1) & 5(2) - 3(3) \\ 0(1) - 2(-1) & 0(2) - 2(3) \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 2 & -6 \end{bmatrix}$$

$$AB \neq BA$$

- The commutator of 2 matrices is defined as

$$\boxed{[A, B] = AB - BA}$$

- If you multiply a row vector by a column vector a dot product is produced

$$(A)_{1\times n} \times (B)_{n\times 1} = \left(\sum_{k=1}^{n} a_k b_k\right)_{1\times 1}$$

- If you multiply a column vector by a row vector an outer product is produced

$$(B)_{n\times 1} \times (A)_{1\times n} = (M)_{n\times n}$$

3.3 Gaussian Elimination

- Gaussian elimination can be used to reduce a system of equations or a matrix
- There are 3 allowed operations:
  - Interchange 2 equations/rows
  - Replace an equation/row by itself + c times another equation
  - Multiply an equation/row by a non-zero constant

- This is done to produce equations that are easy to solve in row reduced echelon form
  - The first non-zero entry should be 1
  - The first non-zero entry should be to the right of the previous row
  - All other column entries that contain a leading 1 are zero

3.4 Trace and Determinants

↳ Both <u>only</u> defined for square matrices

- The trace of a matrix is the sum of diagonal elements

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

- $Tr(A+B) = Tr A + Tr B$
- $Tr(AB) = \sum_{i=1}^{n} a_{ii} b_{ii}$

- The determinant of a square matrix A is notated as $|A|$ or $\det A$. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad |A| = ad - bc$$

Cofactor C_{ij} $(-1)^{i+j}$

- The minor m_{ij} of a_{ij} for matrix A is defined as the determinant of $(n-1)(n-1)$ matrix obtained by removing row i and column j
  - The cofactor $C_{ij} = (-1)^{i+j} m_{ij}$

- The determinant of any matrix can then be defined as

$$\boxed{|A| = \sum_{j=1}^{n} a_{ij} C_{ij}} \qquad \boxed{C_{ij} = (-1)^{i+j} m_{ij}}$$

- For 3 vectors $\vec{e}, \vec{f}, \vec{g}$, their scalar triple product $\vec{e} \cdot (\vec{f} \times \vec{g})$ can be calculated from the determinant of a matrix of their components

$$\vec{e} \cdot (\vec{f} \times \vec{g}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

↳ It is also the volume of the parallelpiped formed by the 3 vectors

Properties of Determinants

- $|A| = |A^T|$
- $|AB| = |A|\,|B|$
- $|I_n| = 1$
- $|\lambda A| = \lambda^n |A|$ where $A = (a_{ij})_{n\times n}$
- If 1 row of A is multiplied by λ, $|A|$ is multiplied by λ
- If B is obtained from A by interchanging 2 rows or columns, $|B| = -|A|$
- If a row i is replaced by row $i + \lambda$ row$j$, $i \neq j$ then the determinant is unchanged
- If 2 rows are identical, $|A| = 0$

→ The determinant of a matrix can be found from any row or column

3.5 Matrix Inversion

- The adjugate Adj. of a matrix $A = (a_{ij})_{m\times n}$ is defined as $(C_{ij})^T$

$$\boxed{\operatorname{Adj} A = (C_{ij})^T}$$

- The inverse of a matrix is defined as

$$\boxed{A^{-1} = \frac{1}{|A|} \operatorname{Adj} A}$$

→ Only if $|A| \neq 0$

Properties of Matrix Inverses

- If A and B have inverses $(|A|, |B| \neq 0)$ then $(AB)^{-1} = B^{-1}A^{-1}$
- If $A = (a_{ij})_{n \times n}$ then $(A^T)^{-1} = (A^{-1})^T$

3.6 Solving $A\vec{x} = \vec{b}$

LU Decomposition

- A matrix A can be decomposed into a lower and upper triangular matrix $A = LU$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$= \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix}$$

↳ This can be used to solve $A\vec{x} = \vec{b}$ for $\vec{x}$ by decomposing it to

$U\vec{x} = \vec{y}$ and $L\vec{y} = \vec{b}$

Solve Second   Solve first

Matrix Inversion

- $A\vec{x} = \vec{b}$ Can be solved using matrix inversion as

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b} \quad \rightarrow A^{-1}A = I \text{ and } IC = C$$
$$\vec{x} = A^{-1}\vec{b}$$

3.7 Special Matrices

Symmetric and Antisymmetric Matrices

- If $A = A^T$, A is symmetric
- If $A = -A^T$, A is antisymmetric

– Any nxn matrix is the sum of a symmetric and antisymmetric matrix

Orthogonal Matrices

- Matrix A is orthogonal if $A^T = A^{-1}$

↳ eg.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I_2 \rightarrow \therefore A = A^T$$

Singular Matrices

- If $\det A = 0$, A has no inverse and is singular

Hermitian Conjugate Matrices

- The hermitian conjugate of A is $A^\dagger = (A^*)^T = (A^T)^*$

↳ $A^* = (a^*_{ij})_{m \times n}$ → complex conjugate of each element

- If $A = A^\dagger$ then A is hermitian
- If $A = -A^\dagger$ then A is anti-hermitian

– Any square matrix is the sum of a hermitian and antihermitian matrix

Unitary Matrices

- Matrix A is unitary if $A^{\dagger} = A^{-1}$

3.8 Matrix Operations on vectors

- A matrix A can be used to map a vector $\vec{x}$ to another vector $\vec{b}$
  ↳ $A\vec{x} = \vec{b}$

Rotation in 3D

- A rotation anticlockwise around the z-axis transformed by matrix

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A rotation anticlockwise around the y-axis transformed by matrix

$$T = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- A rotation anticlockwise around the x-axis transformed by matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Stretching/Shrinking

- A vector can be enlarged by transformation matrix
  $T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

- Transformation matrix $T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ will unvert vector $\vec{r}$ to $-\vec{r}$

3.9 Eigenvalues and Eigenvectors

- If $A\vec{x} = \lambda\vec{x}$, $\vec{x}$ is the eigenvector of A and λ is the eigenvalue
  - $\lambda = 0$ is not an acceptable eigenvalue
  - For square matrices only

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda\vec{x} = 0$$
$$(A - \lambda I)\vec{x} = 0 \quad \rightarrow \text{ from this } |A - \lambda I| = 0$$

→ Using this, a quadratic of λ is formed, which can be solved for λs and $\vec{x}$s

- If the eigenvector found has infinite solutions (due to identical simultaneous equations, it can be normalised by setting $\vec{x}^T \cdot \vec{x} = 1$, and solving for the free parameter

Example: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3$$
$$0 = \lambda^2 - 6\lambda + 5$$
$$0 = (\lambda - 5)(\lambda - 1)$$
$$\lambda = 5 \quad \lambda = 1 \quad \rightarrow \text{Eigenvalues}$$

For $\lambda = 1$: $(A - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 4-1 & 1 \\ 3 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 + x_2 \\ 3x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \text{If } x_1 = t,\ x_2 = -3t$$

$\therefore \vec{x} = t\begin{pmatrix}1\\3\end{pmatrix}$ where t is a free parameter

$$\vec{x}^T \cdot \vec{x} = (t \quad -3t)\cdot\begin{pmatrix}t\\-3t\end{pmatrix} = t^2 + 9t^2 = 1$$

$$10t^2 = 1$$

$$t = 1/\sqrt{10}$$

$\therefore$ for $\lambda = 1$ $\quad \vec{x} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$

For $\lambda = 5$: $\quad (A - \lambda I) = \begin{pmatrix}-1 & 1\\3 & -3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\begin{pmatrix}-x_1 + x_2\\3x_1 - 3x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \rightarrow \text{If } x_1 = t,\ x_2 = t$$

$$\vec{x} = t\begin{pmatrix}1\\1\end{pmatrix}$$

$$\vec{x}^T \cdot \vec{x} = (t \quad t)\cdot\begin{pmatrix}t\\t\end{pmatrix} = t^2 + t^2 = 2t^2 = 1$$

$$\therefore t = 1/\sqrt{2}$$

$\therefore$ for $\lambda = 5$ $\quad \vec{x} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

- If $\vec{x}$ is an eigenvector, then $\alpha\vec{x}$ will also be one

3.10 Properties of Special Matrices

- Vectors are orthogonal and normalised if

  $\vec{x}^T \cdot \vec{x} = 1 \qquad \vec{y} \cdot \vec{x} = 0$

  - If vectors are complex, they are orthogonal if

    $\vec{x}^\dagger \cdot \vec{y} = 0$

- For Hermitian matrices $(A = A^{\dagger} = (A^*)^T$
  - Eigenvalues are always real
  - If $\lambda_i \neq \lambda_j$, eigenvectors are orthogonal

- For Unitary matrices $(U^{\dagger} = U^{-1})$
  - $|\lambda_i|^2 = 1 \rightarrow$

3.11 Basis Changes and Similarity transformations

- A Vector basis can be transformed to a new set of coordinates by multiplying by a transformation matrix S
  - If $\vec{x}' = S\vec{x}$, $\vec{x} = S^{-1}\vec{x}'$

- A Similarity transform is a mapping whose transformation matrix can be written in the form

$$A' = SAS^{-1}$$

  - A' and A are "similar matrices" – they have the Same Tr, det and λ_n
  - S is a nonsingular square matrix

- $I' = I$
- $|A'| = |A|$
- $|A' - \lambda I'| = |A - \lambda I|$
- $\mathrm{Tr}\, A' = \mathrm{Tr}\, A$

- S can be used to transform A to make A' diagonal if S is composed of the eigenvectors of A

Darcie Hurley 2023