Contents

1 Introduction

1.1 Celestial Coordinates

Celestial coordinates are given as angles from a sphere centered on earth, which conventionally has the radius of 1 AU, so that the Sun moves on the surface of this sphere.
The azimuthal angle is usually called the right ascension α and is measured from the position of the vernal equinox (when sun transverses the projection of the equator on the sphere in the spring). From the equator plane towards the north pole (negative for opposite direction). This angle is usually called declination δ.
But, the earth is not stable in space. First, it orbits around the Sun, so the Sun moves relatively counter-clokwise around the Earth on the celestial sphere with speed approximately $1''$ per day. Furthemore, earth is rotated by approximately 23.5 degrees, which sets the plane of apparent motion of the Sun as inclined by 23.5 degrees (upwards towards the North pole). The Earth also rotates around its axis every 24 hours (approximately), which is reason why right ascension is usually given in hours (as in the day), instead of degrees. The declination is stil given in degrees (from -90 to 90). Furthemore, the tilted axis of the Earth itself rotates (precession) due to the effects that Moon and Sun have, rotating the axis (and the direction of the north pole with respect to plane of earths orbit, but not its magnitude) with speed about $50''$ per year. Therefore, the astronimical coordinates have to have a correspondant year/epoch to refer to. Right now, we refer to how stars were positioned in year 2000.
There are also so called galactical coordinates, which use the centre of the Milky Way as origin, but these are not used very much in observational astronomy, as they are not very practical.

1.2 Trigonometric Parallax

Trigonometric parallax π of an object is an angle between line connecting an observed object to the Sun and the line connecting it to the Earth. Since the objects are usually much further away than Earth from Sun, in small angle approximation

$$\sin\pi \approx \pi = \frac{1\text{ AU}}{d}$$

Hence, the distance to the object from Earth d can be calculated as

$$d = \frac{1\text{ AU}}{\pi}$$

If π is given in arc-seconds, the distance to the object in parsecs is given as

$$d[\text{pc}] = \frac{1}{\pi['']}$$

If the star moves an angle α over half a year, this angle corresponds to twice the parallax.

1.3 Flux and Luminosity

Luminosity L is the total light radiation power of a star, integrated over all radiating wavelengths with respective luminosities L_λ

$$L = \int_0^\infty L_\lambda d\lambda$$

The flux is then defined as the intensity of the light radiation at distance d away from the star, as

$$f = \frac{L}{4\pi d^2}$$

and respectively

$$f_\lambda = \frac{L_\lambda}{4\pi d^2}$$

It should be noted that this is for spherically symmetrical sources, otherwise

$$f = \frac{dL}{dS}$$

where dL is the amount of light power going through the small area normal to the direction of travel of the light dS at distance d from the star.

We than clearly see that

$$L = \int_{sphere} f dS$$

One of the important fluxes is the solar constant - the power of light from the Sun at Earth, which is about 1370 W m^{-2}.
Usually, we measure flux over some filter with transmission coefficient $T(\lambda)$. Then, the band flux is

$$\bar{f}_\lambda = \frac{\int_0^\infty f_\lambda T(\lambda) d\lambda}{\int_0^\infty T(\lambda) d\lambda}$$

1.4 Magnitudes

Magnitudes are the measure of stars brightness on a logarithmic scale, which is historically connected to the functioning of the human eye. Relative magnitudes are denoted by m and are defined relatively as

$$m_1 - m_2 = -2.5 \log \left(\frac{f_1}{f_2} \right)$$

where log is the logarithm with base 10.
In terms of luminosities and distances

$$m_1 - m_2 = -2.5 \log \left(\frac{L_1}{L_2} \frac{d_2^2}{d_1^2} \right) = 5 \log \left(\frac{d_1}{d_2} \right) - 2.5 \log \left(\frac{L_1}{L_2} \right)$$

Therefore, if the stars are at the same distance, if star 1 is more luminous, the difference is negative and hence $m_2 > m_1$ - the scale is inverted in a sense that smaller magnitudes mean brighter stars.
However, these magnitudes are always relative to one another. To establish a scale, we need a zero point. In this case, it is decided that the star Vega has always relative magnitude zero, in all wavelengths (see discussion about colour later).
The absolute magnitude M of a star is given by the relative magnitude that the star would have if we were in a distance 10 pc away from it. Given the relative magnitude of the star m, the absolute magnitude is

$$M - m = 5 \log \left(\frac{10 \text{ pc}}{d} \right) - 2.5 \log \left(\frac{L}{L} \right) = 5 \log \left(\frac{10 \text{ pc}}{d} \right)$$

$$M = m + 5 \log \left(\frac{10 \text{ pc}}{d} \right)$$

Or, for d in parsecs

$$M = m + 5 \left(1 - \log \left(d[\text{pc}] \right) \right)$$

This means that Vega has absolute magnitude of about 0.58 (distance to Vega is about 25.05 ly), and is no longer a zero on this case. The absolute magnitude is more a measure of the stars actual luminosity, and hence more representative of the star.

1.5 Colours

Colours of a star refer to the magnitudes of the star in a certain filter. For example, in Johnson UVBR filters (which are the astronomical standard), for example m_U is

$$m_U = m_{Vega,U} - 2.5 \log \left(\frac{f_U}{f_{Vega,U}} \right)$$

where $f_U = \frac{\int_0^\infty f_\lambda U(\lambda) d\lambda}{\int_0^\lambda U(\lambda) d\lambda}$ is the flux integrated over the filter. By definition, we say $m_{Vega,X} = 0$ and hence

$$m_U = -2.5 \log \left(\frac{f_U}{f_{Vega,U}} \right)$$

The colour is then defined as

$$U - B = m_U - m_B$$

Since the star for which it is measured is at the same distance for m_U and m_B, we have

$$M_U - M_B = m_U + 5(1 - \log d) - m_B - 5(1 - \log d) = m_U - m_B$$

Furthermore

$$U - B = -2.5\left[\log\left(\frac{f_U}{f_{Vega,U}}\right) - \log\left(\frac{f_B}{f_{Vega,B}}\right)\right] = -2.5\left[\log\left(\frac{f_U}{f_B}\right) - \log\left(\frac{f_{Vega,U}}{f_{Vega,B}}\right)\right]$$

Hence the colours determine how much more flux is going through a certain wavelength range than from Vega. Colours are usually representative of the stars temperature, and more blue stars are usually the hotter stars.

1.6 Very Distant Stars

Very distant stars have not measurable parallax. Distance to them can be determined using so called standard candles - processes in stars that enable as to calculate the luminosity/absolute magnitude of the stars. From the measured flux, we can then deduce the distance to the stars.
Typical standard candles are supernovae or cepheid stars, whose luminosity fluctuates and follows empirical relationship

$$M = -2.76\log(P[\text{days}]) - 1.4$$

where P is the period of the oscillations.
For extremely distant stars, we can use the redshift due to expansion of space to determine their distance. The Hubble constant is $H_0 \approx 70\text{km s}^{-1}\text{Mpc}^{-1}$, hence the speed of a very distant star due to expansion of space is

$$v = H_0 d$$

1.7 Blackbody Radiation

Blackbody is a theoretical objects that does not reflect any light and all light incident is absorbed as thermal energy. The oscillations due to this thermal energy then produce specific radiation. Blackbody is clearly a very idealised object, but stars are surprisingly close to profile of the blackbody radiation, with specific temperature as the parameter of the spectrum (which is not necesseraly the real surface temperature of the star). For some stars, most of the spectrum is blackbody-like, while for others only some parts are blackbody-like (partially due to stellar atmospheres).
Radiance B_λ of the blackbody is the amount of radiation emmited by a small surface of the blackbody per wavelength per solid angle projected onto the direction plane of polar angle $\theta = 0$. This means that the total power emmited out of the blackbody is

$$L = \int_{blackbodysurface} dS \int_{halfsolidanglespace} d\omega \int_0^\infty d\lambda B_\lambda \cos(\theta) d\lambda$$

where the $\cos\theta$ term is the projection term.
For blackbody, the radiance is theoretically predicted as

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

with

$$\int_0^\infty B_\lambda d\lambda = \frac{\sigma T^4}{\pi}$$

where $\sigma = 5.67\times10^{-8}\,\text{W m}^{-2}\text{K}^{-4}$. Hence the luminosity of a star with radius R and effective temperature T is

$$L = \int_0^{2\pi}\int_0^\pi R^2\sin\theta d\theta d\phi \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\theta' d\theta' d\phi' \frac{\sigma T^4}{\pi}\cos\theta' =$$

$$= 4\pi\sigma T^4 R^2 \int_0^\pi \sin\theta d\theta \int_0^{\frac{\pi}{2}} \sin\theta'\cos\theta' d\theta = 4\pi R^2\sigma T^4[\cos\theta]_\pi^0[\frac{1}{2}\sin^2\theta']_0^{\frac{\pi}{2}} = 4\pi R^2\sigma T^4$$

Other property of the blackbody radiation is that it peaks at λ_{max} which obeys

$$\lambda_{max} = \frac{b}{T}$$

where $b = 2.898\times10^{-3}$ m K is constant. This is called the Wien's displacement law. It should be noted however that because the stars do not obey the blackbody radiation law perfectly, the peak effective temperature and total flux effective temperature might differ quite significantly.

2 Stellar Spectra

2.1 Spectral Classification

2.1.1 Harvard Classification

Stellar clasification is also historical way how to separate different types of stars dependent on their spectral characteristics. The Harvard classification uses approximate ordering by temperature, using also identification of specific features, such as spectral lines or molecular lines.
The categories are as follows

- O0 - O9
  - Very hot stars, usually peaking far in the blue spectrum
  - Highly ionised, with spectral lines from He II (ionized helium) at about 4700 Å.
  - Very weak hydrogen lines
  - Some emission lines can be present
- B0 - B9
  - Still peaking far in blue
  - No He II, but He I line very strong (neutral helium), at about 4200 Å.
  - Balmer series present
- A0 - A9
  - Peak and Balmer Series limit usually in blue, but not so far
  - Balmer lines very strong
  - Weak lines of ionized metals, such as Ca II, at about 3800 - 4000 Å.
- F0 - F9
  - Peak in middle of the spectrum
  - Weaker Balmer lines, but present
  - Lines of ionised metals (Ca II) and some neutral metals
- G0 - G9
  - Peak slowly shifting to the red end
  - Strong Ca II and strong neutral metals
- K0-K9
  - Peak in red
  - Weaker Ca II and strong neutral metals
- M0 - M9
  - Peak after red
  - Some neutral metal lines, molecular bands (TiO) present

2.1.2 Morgan-Keenan Classification

Morgan-Keenan clasification uses the evolution stadium of the star (or at least the state that is expected to be the evolution stadium of the star).

- I - supergiants
- II - bright giants
- III - giants
- IV - subgiants
- V - main sequence (dwarfs)
- VI - sub-dwarfs
- D - white dwarfs

2.2 Stellar Atmospheres Description

2.2.1 Boltzmann and Saha equations

Assume that the electrons in an atom in a star are in thermal equilibrium with the rest of the of the star. Then, the electrons follow the Boltzmann distribution for occupation of different bound states in the atom, with

$$N_m = g_m \frac{e^{-\frac{(E_m+E_1)}{k_B T}}}{Z(T)}$$

where $Z(T)$ is the partition function, g_m is the degeneracy of the given occupacy state and E_m is the energy above the ground state energy E_1.
Hence

$$\frac{N_m}{N_1} = \frac{g_m}{g_1} e^{-\frac{E_m}{k_B T}}$$

The Saha equation determines the nnumber of electrons that have completly dissociated from their original atom and are now free electrons going through the star. The number of ionised atoms N_0^+ is give as

$$\frac{N_0^+ N_e}{N_0} = \frac{2g_0^+}{g_0} \left(\frac{2\pi m_e k_B T}{h^3}\right)^{\frac{3}{2}} e^{-\frac{W_i}{k_B T}}$$

where W_i is the ionisation energy, N_e is the electron density, N_0 is the original number of atoms in the ground state and g_0^+ and g_0 are the statistical weights for ground ionized - non-ionized states, which are determined from statistical mechanics. This can be further expanded using

$$N^+ = \frac{g_0^+}{Z^+(T)}$$

$$N = \frac{g_0}{Z(T)}$$

Hence

$$\frac{N^+ N_e}{N} = \frac{2Z^+(T)}{Z(T)} \left(\frac{2\pi m_e k_B T}{h^3}\right)^{\frac{3}{2}} e^{-\frac{W_i}{k_B T}}$$

Importantly, this equation does not feature electron speed v after it escapes, as the higher energies of electron after escape have lesser probability of occurance.
There are also absorbption lines due to transitions bewteen different momenta of free electrons, but these lines are so dense that they are usually seen as continuum. These are not discussed here further.

2.2.2 General Radiation Processes in Matter

The flux from a surface of an object out of this surface is, as already stated

$$\int_{half solidanglespace} I(\lambda)\cos\theta d\Omega = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} I_\lambda \cos\theta \sin\theta d\phi d\theta = 2\pi \int_0^{\frac{\pi}{2}} I(\lambda)\sin\theta\cos\theta d\theta$$

where $I(\lambda)$ is the radiance. We can use this definition to better understand following general processes that occur with radiation in matter.

Absorption Suppose radiation of radiance $I(\lambda)$ is incident on a small slab of width dz. The radiance outgoing at the end of the slab is reduced by absorption of the radiation by some amount $dI(\lambda)$. We then define the coefficient of absorption as

$$k_\lambda = \frac{-1}{I(\lambda)} \frac{dI}{dz}$$

So that

$$dI(\lambda) = -k_\lambda I(\lambda) dz$$

Scattering Scattering occurs if the radiation is not absorbed, but its direction is changed. Scattering coefficient σ_λ is related to the cross-section of the scattering and is defined similarly as absorption coefficient

$$dI(\lambda) = -\sigma_\lambda I(\lambda) dz$$

where $I(\lambda)$ is the incident radiance and $I(\lambda) + dI(\lambda)$ is the emergent radiance.

Emission Also, if the matter itself is excited, it can emmit radiation as well. Again

$$dI(\lambda) = \epsilon_\lambda dz$$

where ϵ_λ is the emmision coefficient. If the matter behaves like a black body and is in a thermal equilibrium, then we must have

$$dI(\lambda)_{emmision} = -dI(\lambda)_{absorption}$$

$$\epsilon_\lambda = k_\lambda I(\lambda)$$

where $I(\lambda)$ in this specific case will be the blackbody radiance from the matter piece.

2.3 Radiative Equilibrium

Radiative equilibrium occurs when the light emmited and absorbed by the matter is equal to the change of radiance from the matter, i.e.

$$dI(\lambda) = -k_\lambda I(\lambda)dz + \epsilon_\lambda dz$$

$$\frac{-1}{k_\lambda}\frac{dI}{dz} = I(\lambda) - \frac{\epsilon_\lambda}{k_\lambda} = I(\lambda) - S_\lambda$$

This is in the normal direction. In any other direction, we can define angle θ as in other radiance problems, so that radiation intensity in certain direction is multiplied by factor $\cos\theta$. We can further define the optical depth, which will determine how deep we can see throught the material at certain wavelength, as $d\tau = -k_\lambda dz \cos\theta$ Hence, the equilibrium condition becomes

$$\cos\theta \frac{dI}{d\tau} = I(\lambda, \mu) - S_\lambda$$

And for normal observation at direction $\cos 0 = 1$

$$\frac{dI}{d\tau} = I(\lambda) - S_\lambda$$

Hence

$$I(0, \lambda) = I(\tau, \lambda)e^{-\tau} + \int_0^\tau S_\lambda e^{-\tau} d\tau$$

For the blackbody radiation spectrum, this becomes

$$I(0, \lambda) = B(\lambda, T)(1 - e^{-\tau})$$

where $B(\lambda, T)$ is the blackbody radiance.

2.4 Line Broadening

The spectral lines are predicted to be a perfect delta distributions, having only one possible value. However, in experiment we see that the lines are broader distribution. We now discuss few sources of this uncertainty in the lines frequency/wavelength.
First reason is the fundamental broadening due to Heisenberg uncertainty principle, which predicts that

$$\Delta E \delta t \geq \hbar$$

So, the minimum uncertainty is

$$\Delta E = \frac{hc}{\lambda^2}\Delta\lambda = \frac{\hbar}{\Delta t}$$

For normal spectral lines, the emmision process takes about 10^{-8} s. So, for line in the middle of the spectrum

$$\Delta\lambda = \frac{\lambda^2}{2\pi c \Delta t} \approx 1.3 \times 10^{-14} \text{ m}$$

This is nowhere close to real distribution of the lines, so there must be other effects taking place.
Other possibility is that the frequencies are shifted due to Doppler effect of thermal movement of the particles. For Doppler effect, we have

$$\frac{\Delta\lambda}{\lambda} \approx \frac{v}{c}$$

The average thermal speed of the atoms is

$$v \approx \sqrt{\frac{2k_BT}{m_{atom}}}$$

which leads to

$$\Delta\lambda \approx \frac{\lambda}{c}\sqrt{\frac{2k_BT}{m_{atom}}} \approx 2 \times 10^{-11} \text{ m}$$

This is much more important than the fundamental uncertainty, but still not quite big enough.
Some extra effect can be derived from the fact that the stars are under immense pressure, which increases the speed of the reaction considerably, and thus increasing the uncertainty in wavelength, but these effects only important in extremaly pressurized stars, such as white dwarfs or supergiants.
Perhaps the most important broadening effect is due to the Doppler effect connected to the rotation of the star - different parts of the disk of the star are moving at different relative velocities. This causes that the lines are shifted at different places of the disk, and averaging together to a distribution.

2.4.1 Absorption Edges

The probability of absorbing a photon at specific wavelength given by atomic physics is biggest for energy of the photon approximately equal to the energy of the potential difference, and then it slowly decreases for higher energies. This leads to formation of absorption edges - borderline values of frequency after which the absorption occurs intensively - $k_\lambda - d\lambda$ is small if λ is the absorption edge and $k_\lambda + d\lambda$ is very big.

2.5 Modelling Stellar Atmospheres

Two more equations are needed to at least try modelling of the stellar atmosphere, and that is even with guessing the dependence of temperature.
First is the equation of hydrostatic equilibrium, which classically has form

$$-\nabla P \cdot d\vec{z} dS = -\frac{\partial P}{\partial z} dV = \rho dV g$$

i.e. the pressure gradient force exerted on an element of volume dV must be the same as the force of gravity downwards. Using optical depth in normal direction, this can be rewritten as

$$-\frac{\partial P}{\partial z} = k_\lambda \frac{\partial P}{\partial \tau} = \rho g$$

$$\frac{\partial P}{\partial \tau} = \frac{\rho g}{k_\lambda}$$

where τ still depends on chosen wavelength of observation (so that the model predicts directly what will be observed)
Other equation continues from Boltzmann and Saha equations, and simply represents the fact that for denser stars, the absorption coefficient is higher, i.e.

$$k_\lambda = \alpha_{\lambda,x} N_x$$

where N_x is the concentration of some atom/ion/electrons.
Furthermore, we usually assume that the flux is in equilibrium as well, so that at every point, the same flux goes through the same slab of matter, so

$$\frac{dF}{d\bar{\tau}} = 0$$

where $\bar{\tau}$ is again optical depth, but of the atmosphere now.
From models, we obtain that the top of the atmosphere is much hotter than the surface, with temperatures up to 20 000 K.

2.5.1 Limb Darkening

Many models can be tested on limb darkening phenomenon. If θ is the polar angle in spherical polars with axis pointing towards observer, the dependence of observed intensity $I(\lambda, \theta)$ goes as

$$I(\lambda, \theta) = I(\lambda, 0) \frac{1 + \beta_\lambda \cos\theta}{1 + \beta_\lambda}$$

The dependence is not just cosinoidal because at different angles, we see into different depths of the atmosphere.

2.6 Stellar Magnetic Fields

The most prominent magnetic phenomena in the stars (that we had observed) are the sunspots. Sunspots are regions where the temperature and particle concentration (which can be both determined from our models and observations) are much smaller than in the other regions. Since these are stable, something must be compensating for the lack of pressure. It has been determined that this is the magnetic field of the star, and that the pressure at the spot is given as

$$P = P^* + \frac{1}{2\mu_0}B^2$$

where P^* is classical gas pressure and B is the magnitude of the magnetic field. The pressure P^* can be approximated from the equation of state for an ideal gas as

$$P^* = N^* k_B T^*$$

where N^* is the concetration of the particles in the sunspot. From observations for a typical sunspot

$$\frac{N^*}{N} \approx 0.3$$

$$\frac{T^*}{T} \approx 0.6$$

Hence

$$\frac{P^*}{P} = \frac{N^*}{N}\frac{T^*}{T} \approx 0.18$$

For Sun, typically $P \approx 10^4$ Pa, and so

$$P - P^* = \frac{1}{2\mu_0}B^2$$

$$B = \sqrt{2\mu P\left(1 - \frac{P^*}{P}\right)} \approx 0.14T$$

Therefore, the field is not extreme, but since sunspots are rather large, it is still a markant phenomenon.

2.6.1 Foculae

Foculae are regions around sunspots where the magnetic field returns to the Sun, and therefore has opposite direction and tries to effectivelly push the matter into the sun. For this to be stable, the pressure from the gas has to be bigger, but since the particle concentration cannot change dramatically, it is the temperature that is holding the foculae, which are therefore brighter than the rest of the star.

2.6.2 Sunspot cycle

Sunspots tend to appear in cycles lasting about ten years. To explain this behaviour a Babcock model is used, which is now qualitatively explained. The situation starts with magnetic field lines connecting the north and the south magnetic pole of the Sun. We then assume that the star rotates faster on the equator than on the poles (angularly faster). This causes the magnetic lines to slowly stretch until they form independent loops. These loops afterwards approach the surface (especially at the poles) and become sunspots. Then, the sunspots slowly migrate towards the Sun equator, where they recombine and restore the connected magnetic field lines between the poles.

2.6.3 Zeeman effect

Zeeman effect occurs when the energies of orbitals of atoms/ions get distorted by the magnetic field - orbit in one direction has less energy than orbit in the opposite direction. The separation between the energy levels split by Zeeman effect is

$$\Delta E = \Delta m_l \frac{e\hbar}{2m_e}B$$

where Δm_l is the difference between the magnetic quantum number of the splitted levels (must have the other quantum numbers identical). But, usually only transitions by 0 or 1 m_l difference are allowed from higher orbitals, which has to be taken into account when searching for spectral lines in magnetic fields.
Zeeman effect for hydrogen has been observed at the Sunspots, which was quite a good accomplishment for quantum mechanics.

3 Stellar Masses and Radii

Generally, it is very hard to obtain absolute values of masses and radii of stars, especially if these stars are alone (with exception of Betelgeuse, which is so big and so close that its angular size can be measured directly). We now present a few methods of determining these.

3.1 Fresnel Diffraction

When object slowly goes behind a barier, the light emmited from it forms a specific diffraction pattern, which reflects the angular size of the object. In practice, we can use the eclipse of a moon before a star to determine its angular radius. To do so, we need very good temporal resolution of the camera, as the moon moves at speed approximately $54''$ per second, and we need to resolve angles of order of $10^{-3}\,''$.
When we know the angular size, we can determine the radius of the star if we know the distance to the star, which can be done either by parallax measurements for close stars or by standard candles.
The process of moon eclipsing the stars is also called the lunar occulation.

3.2 Visual Binaries

Visual binaries are binary stars for which we can clearly see the trajectories of both stars. For these to be stable, they must orbit while facing each other. Then, if we mark r_1 as distance from the centre of mass of star 1 and r_2 as distance from CoM of star 2, we have

$$\omega = \frac{v_1}{r_1} = \frac{v_2}{r_2}$$

For the centre of mass, we have

$$M_1 r_1 - M_2 r_2 = 0$$

Finally, the gravitational force is approximately the centripetal force, with (choosing for example star 1)

$$\frac{GM_1M_2}{(r_1+r_2)^2} = M_1 r_1 \omega^2$$

This expression is not symmetrical and we need to somehow restore the symmetry. Adding $M_2 r_1$ to the CoM equation

$$r_1(M_1+M_2) = M_2(r_1+r_2)$$

$$r_1 = M_2\frac{(r_1+r_2)}{(M_1+M_2)}$$

Restoring the symmetry by multiplying by M_1

$$M_1 r_1 = M_1 M_2 \frac{(r_1+r_2)}{(M_1+M_2)}$$

Therefore, substituting into the force equation

$$\frac{GM_1M_2}{(r_1+r_2)^2} = M_1M_2\frac{(r_1+r_2)}{(M_1+M_2)}\omega^2$$

$$\frac{G(M_1+M_2)}{(r_1+r_2)^3} = \omega^2$$

Hence the total mass can be obtained as

$$M_1 + M_2 = \frac{\omega^2 R^3}{G}$$

where $R = (r_1+r_2)$ is the distance between the stars.

3.3 Spectroscopic Binaries

Spectroscopic binaries are binary stars that have some component of the velocity in the direction of observation, and therefore the spectral lines from them are subject to Doppler effect.
To find the component of the velocity in the direction of observation, we define the inclination angle as the angle between the plane of orbit of the binary and the plane perpendicular to the observation direction. Then, the component of the velocity has amplitude $v_o = v \sin i$, where i is the inclination angle, and it oscillates as $\sin(\omega t)$, where ω is the angular frequency of binary orbit. Therefore, the wavelength of spectral lines is oscillating as

$$\delta\lambda = \lambda \frac{v}{c} \sin i \sin(\omega t) = \Delta\lambda \sin(\omega t)$$

Comparing the amplitudes for the same spectral line for the two stars

$$\frac{\Delta\lambda_2}{\Delta\lambda_1} = \frac{v_2}{v_1}$$

From the CoM equation and using the ω definition

$$\frac{M_2}{M_1} = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{\Delta\lambda_1}{\Delta\lambda_2}$$

Hence, we can determine the ratio of the masses as well, and thus if a binary is both spectroscopic and visual, we have

$$M_1 \left(1 + \frac{M_2}{M_1}\right) = \frac{\omega^2 R^3}{G}$$

$$M_1 = \frac{\omega^2 R^3}{G\left(1 + \frac{\Delta\lambda_1}{\Delta\lambda_2}\right)}$$

Similarly

$$M_2 = \frac{\omega^2 R^3}{G\left(1 + \frac{\Delta\lambda_2}{\Delta\lambda_1}\right)}$$

3.4 Eclipsing Binaries

When the smaller star eclipses over the disk of the bigger star, the flux from the bigger star momentarily increases. From the shape of the increase and duration of so called totality, we can determine the ratio of radii of the binary.
Assume that the star eclipses over the equator of the bigger star. Let t_e be the total time of the eclipse and t_t the time of totality - time when full area of the smaller star is in front of the bigger star. From simple geometry, the distance travelled during the whole eclipse by the smaller star is

$$d_e = 2R_1 + 2R_2 = 2(R_1 + R_2)$$

where R_1 is the radius of the smaller star and R_2 is the radius of the bigger star. The distance travelled during totality is

$$d_t = 2R_2 - 2R_1 = 2(R_2 - R_1)$$

The relative velocity of the smaller star is $v = v_1 + v_2$, which, applying the ω definition, is

$$v = (r_1 + r_2)\omega$$

where $r_{1/2}$ is the distance of star 1/2 from the CoM. Hence the time eclisping is

$$t_e = \frac{d_e}{v} = 2\frac{R_1 + R_2}{\omega(r_1 + r_2)}$$

the time in totality is

$$t_t = \frac{d_t}{v} = 2\frac{R_2 - R_1}{\omega(r_1 + r_2)}$$

Hence

$$t_e + t_t = \frac{4R_2}{\omega(r_1 + r_2)}$$

$$t_e - t_t = \frac{4R_1}{\omega(r_1 + r_2)}$$
$$\frac{t_e - t_t}{t_e + t_t} = \frac{R_1}{R_2}$$

Hence, we find the ratio of the distances from the times of eclipse.

4 Stellar Evolution

Big part of astrophysics is trying to understand the life of stars and thus explain the variety of star types that we observe. It should be noted however that lives of stars are very long, so any theories might be distorted because we only see a snipet of the life of each star (even though we see pretty far into the history of the universe).
The most common way of visualizing different stages of the life of a star is on an Hertzsprung-Russel diagram, H-R diagram for short.

4.1 H-R Diagrams

H-R diagrams plot stars' absolute luminosity M as a function of its colour, usually $B - V$. The magnitude axis has usually inversed scale, so that the brightest stars are at the top of the diagram.
Sometimes, for modelling and theory, the diagram rather plots $\log\left(\frac{L}{L_\odot}\right)$ vs T_{eff} of the star, where $L_\odot$ is the luminosity of the Sun. The temperature scale here is usually inversed and quite logarithmic. From general H-R diagram, we can observe several areas with increased concentration of stars.
First is the main sequence line, which is approximately a line going from 0 absolute magnitude and 0 colour to about 10 in absolute magnitude and 1.5 in colour - more red than Vega, hence colder than Vega.
Then we see a cluster of white dwarf stars, which is very small in colour (very hot) but also very small in absolute magnitude.
Finally, we have the giant branch, where stars are quite cold (very red colour), but also very luminous.

4.1.1 Constant Radii

On H-R diagram, contours of constant radii of the stars are straight lines. This can be seen easily from the expression for model H-R diagram

$$\log\left(\frac{L}{L_\odot}\right) = \log\left(\frac{4\pi R^2 \sigma T^4}{4\pi R_\odot \sigma T_\odot^4}\right) = 4\log\left(\frac{T}{T_\odot}\right) + \log\left(\frac{R}{R_\odot}\right)$$

Hence for constant radius, the expression is indeed linear with $\log(T)$.

4.2 Star Luminosity

There is a very important yet empirical relation that states that the luminosity of a star is proportional to the cube of its mass, i.e.

$$L \propto M^3$$

This is one of the relations that will help us understand the life of a star.

4.3 Open Clusters and Globular Clusters

Most information can be extracted from an H-R diagram if the stars that are in the diagram are all approximately the same age. This happens usually in star clusters, and we observe star clusters of two types - open and globular clusters.
Open clusters are relatively small sets of close young stars inside the galactic disk, while globular clusters are usually quite numerous sets of stars that are often found in the galactic halo. Their origin is unclear, but it is speculated that they are the remnants of cores of smaller galaxies that merged with the Milky way.

4.4 Stellar models

To understand how stars function, we need to try to understand the processes that occur inside of the stars, not just in their atmospheres. Again, we need to write down some general differential equations that will help us with modelling the star.
We assume that the star is radially symmetric. Then, the mass of a spherical shell is given as

$$dM_r = 4\pi r^2 \rho dr$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

The hydrostatic equilibrium must once again apply. But, the value of the gravitational attraction can be now expressed explicitly as

$$g = \frac{GM_r}{r^2}$$

And thus

$$-\nabla P \cdot d\vec{r} dS = \rho dV g$$

$$\frac{\partial P}{\partial r} = \frac{G\rho M_r}{r^2}$$

Then, we need an equation that describes the energy conservation in a star. Energy generated by the star mass in a spherical shell of radius r is

$$dL = 4\pi r^2 \rho dr \epsilon_n$$

where ϵ_n is the mass energy generation coefficient. The energy in a star is generated using the nuclear fusion, which will be explored later.
Hence

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon_n$$

Then, there are the heat transport equations. Basic equation that always applies is the radiative equation

$$\frac{dT}{dr} = \frac{-3k\rho}{16ac\pi r^2 T^3} L$$

where k is the absorption coefficient (integrated for all wavelengths) and a is a constant (radiation density constant). Also, convection can appear in stars that satisfy the Schwarzschild condition

$$\left(\frac{dP}{d\rho}\right)_{star} \geq \left(\frac{dP}{d\rho}\right)_{adiabatic}$$

If the condition is satisfied, then if some random bubble appears in the star with momentarily higher temperature or pressure, as it rises, it does not cool fast enough to stop rising, and thus the star convects.
As the star is not solid, we assume that there is not a lot of effective conduction taking place.
Therefore we have 4 equations and 1 condition and search for four functions, L, M_r, P and T. We have common boundary conditions

$$L(r = 0) = 0, M_r(r = 0) = 0, P(R) = 0, T(R) = T_{eff}$$

Furthermore we can relate P and rho using state equation, and derive k and ϵ_n from atomic physics.

4.5 Nuclear Reactions

There is number of processes that occur during nuclear fusion in a star. Basic setup is that the hydrogen is being burned into helium, which is cycle that works during most of the life of a star. Interestingly enough, the thermal energy of the hydrogen nuclei themselves is not sufficient to overcome the Coulomb barrier in most stars, and the fusion can only be realized via the quantum tunneling effect (the thermal energy is about one thousandth of the peak Coulomb repulsion energy of the nuclei).
To have an idea of order of energies involved, creation of one helium nucleus out of 2 neutrons and 2 protons create about 4×10^{-14} J of energy.
First set of reactions occuring in stars is the PP chain (proton-proton), which consists of three independent reaction chains. First reaction chain (PP I) is

$${}^1_1H + {}^1_1H \to {}^2_1H + e^+ + \nu_e$$

$$^2_1H + ^1_1H \rightarrow ^3_2He + \gamma$$

$$2^3_2He \rightarrow 2^1_1H + ^4_2He$$

This is usually the chain that occurs first in all stars, as it does not require any other element than hydrogen. As the star ages, other elements become present, which can catalyze other types of reaction, such as PP II chain

$$^3_2He + ^4_2He \rightarrow ^7_4Be + \gamma$$

$$^7_4Be + e^- \rightarrow ^7_3Li + \bar{\nu}_e$$

$$^7_3Li + ^1_1H \rightarrow 2^4_2He$$

The last PP chain, PP III chain is

$$^7_4Be + ^1_1H \rightarrow ^8_5B + \gamma$$

$$^8_5B \rightarrow ^8_4B + e^+ + \nu_e$$

$$^8_4Be \rightarrow 2^4_2He$$

The energy generated by the PP chain than climbs up to 4.3×10^{-12} J.
Other type of reactions is the so called CNO cycle, which is catalysed by relatively heavy elements carbon and nitrogen, and creates oxygen during the proces. The C part of the cycle is

$$^{12}_6C + ^1_1H \rightarrow ^{13}_7N + \gamma$$

$$^{13}_7N \rightarrow ^{13}_6C + e^+ + \nu_e$$

$$^{13}_6C + ^1_1H \rightarrow ^{14}_7N + \gamma$$

$$^{14}_7N + ^1_1H \rightarrow ^{15}_8O + \gamma$$

$$^{15}_8O \rightarrow ^{15}_7N + e^+ + \nu_e$$

$$^{15}_7N + ^1_1H \rightarrow ^{12}_6C + ^4_2He$$

The N part of the cycle is

$$^{14}_7N + ^1_1H \rightarrow ^{15}_8O + \gamma$$

$$^{15}_8O \rightarrow ^{15}_7N + e^+ + \nu_e$$

$$^{15}_7N + ^1_1H \rightarrow ^{16}_8O + \gamma$$

$$^{16}_8O + ^1_1H \rightarrow ^{17}_9F + \gamma$$

$$^{17}_9F \rightarrow ^{17}_8O + e^+ + \nu_e$$

$$^{17}_8O + ^1_1H \rightarrow ^{14}_7N + ^4_2He$$

But, these catalysts must appear from somewhere. One potential source of $^{12}_6C$ is the so called triple alfa process which can occur at high concentrations of helim. Then

$$2^4_2He \rightarrow ^8_4Be + ^4_2He \rightarrow ^{12}_6C$$

In the most massive stars, we can have even carbon and oxygen burning, but this produces a number of different products and is not discussed here.

4.6 Before Main Sequence

Before ignition, stars are formed by collapses of molecular/atomic clouds. For a cloud to collapse, its overall energy must be negative (a bound state). The overall energy is the gravitational potential energy plus the thermal kinetic energy

$$E_T = E_k + E_p = \frac{3}{2}Nk_BT - \frac{3}{5}\frac{GM^2}{r} < 0$$

where N is the number of particles in the cloud, M is the mass of the cloud, T is the temperature of the cloud and r is the size of the cloud (assume that it is approximately spherical). For $\bar{m}$ the average mass of a particle, we have

$$\frac{3}{2\bar{m}}Mk_BT < \frac{3}{5}\frac{GM^2}{r}$$

$$M > \frac{5k_BTr}{2\bar{m}G}$$

This condition however involves the M and r on different sides of the equation, which can be eliminated by using the average density

$$\bar{\rho} = \frac{3M}{4\pi r^3}$$

Hence

$$\frac{4}{3}\pi r^3\bar{\rho} > \frac{5k_BT}{2\bar{m}G}r$$

$$r > \sqrt{\frac{15k_BT}{8\pi\bar{m}\bar{\rho}G}}$$

Or, eliminating r from original equation

$$M > \frac{5k_BT}{2\bar{m}G}\sqrt[3]{\frac{3M}{4\pi\bar{\rho}}}$$

$$M > \left(\frac{5k_BT}{2\bar{m}G}\right)^{\frac{3}{2}}\sqrt{\frac{3}{4\pi\bar{\rho}}}$$

Or, taking one step further from this

$$\bar{\rho} > \left(\frac{5k_BT}{2\bar{m}G}\right)^3\frac{3}{4\pi M^2}$$

These last three conditions are conditions for so called Jeans radius/mass/density.
As the cloud collapses, it can simply reach a stable state before ignition, which means it stops the collapse until it looses temperature by radiation enough so that it can collapse. Usually, the clouds are not perfectly symmetrical nor homogeneous, so local areas of higher density can appear, which creates smaller protostars inside a big cloud. If one of this regions gains enough density, a proto-stellar disc starts to form - the collapsing subcloud must still conserve angular momentum, so the particles remain in one plane after many collisions. This proto-stellar discs sometimes eject jets from the centre perpendicular to the plane of the particles, as such jets have no angular momentum.
On H-R diagram, the cloud starts as very cold object of some luminosity, depending on the temperature and the size of the cloud. Afterwards, as the cloud collapses, for initially luminous clouds, the luminosity stays approximately constant, while for less luminous clouds it slightly decreases. After the ignition, the luminosity jumps a bit up and afterwards falls again, as secondary processes take place - adjusting the structure of the star and the core. Finally, the star arrives at the main sequence state and remains here for the most of its life.

4.7 After Main Sequence

After the main sequence, the behaviour of a star depends mainly on its mass. For stars like Sun ($M \leq 2M_\odot$), the star usually first depletes the hydrogen in a core, stops fusion and contracts, which enables fusion in the shell, which leads to expansion of the star - the star becomes the red giant. As a consequence of this, the nucleus can get further compressed, which initiates core helium burning. After some time, the pressure usually does so high that the triple alpha process is initiated, and so called helium flash occurs - the star momentarily peaks in luminosity and then quickly ejects the envelope, leaving behind a planetary nebulae with white dwarf in the centre. For more massive stars, the first helium flash might not eject the envelope,

but rather just mix the envelope of the star so that new hydrogen enters the shell/core, which then can burn for some time. This process is called a thermal pulse, and can occur several times.
For much more massive stars than a Sun ($\approx 25M_\odot$), the cycle repeats for higher elememnts until reaching iron, which is the last element (almost) that creates energy during its creation. Such star then usually either undergoes gravitational collapse to either a neutron star or a black hole.

5 Exoplanets

Search for exoplanets is always hard, as they are very small and relatively dark objects near very bright and massive objects. However, there is a number of techniques that are used and are quite succesful (if correct). The caution should however be exercised, because the field is really young and thus there might be mistakes.

5.1 Radial Velocity Surveys

Planet and the Star form effectively a binary system, and quite usually can behave like spectroscopic binaries (only sometimes with more planets). The main complication is that we do not know the Doppler shift from the planet. But, we still have

$$\Delta\lambda = \lambda\frac{v}{c}\sin i$$

where $\Delta\lambda$ is the amplitude of the oscillations in Doppler shift, λ is the wavelength of the spectral line of the star and v is the speed of the star.
Because the CoM and gravitational equations still apply, we can express

$$v = \omega r = \omega M_P\frac{R}{M_P + M} \approx \omega M_P\frac{R}{M}$$

where R is the distance between the planet and the star, M_P is the mass of the planet and M is the mass of the star (and mass of the star is usually much higher than the mass of the planet). Hence

$$\Delta\lambda = \lambda\frac{\sin i}{c}\omega M_P\frac{R}{M}$$

Hence

$$M_P = \frac{\Delta\lambda}{\lambda}\frac{Mc}{R\omega\sin i}$$

However, we do not know R here, which needs to be reexpressed from the velocity of the star, which makes the expression much more complicated. The important information to take away is that we can only detect $M_P\sin i$ from this observation, which is not a lot of information. But, this technique can be used to detect relatively small planets, which cannot be detected by other techniques, such as transits.

5.2 Transiting Exoplanets

The techniques used are the same as for transiting binaries, but the change in the luminosity is much smaller. Again, limb darkening can help us determine $\sin i$, so we can determine the radii from the totality and eclipse times and $\sin i$ from limb darkening.
Best information about planet can be then established if we have both transit measurement and radial velocity survey for a planter-star system, as we can determine both mass and radius, and thus the density of the planet.
The problem with transiting exoplanets is the noise, and for smaller exoplanets, stellar activity such as sunspots can render measurements for smaller planets completly impractical.
For some planets, further information about their size can be derived from the Rossiter-McLaughlin effect. As the planet moves in front of the star, the spectral lines, which are normaly broadend symmetrically by star rotation become unsymmetrically broadened, as the planet removes a certain component frequencies from the average distribution.

5.3 Discovered planets

Many of the planets discovered so far are so called hot jupiters - very big planets very close to their sun. This can be either very common in space or be due to observer bias - bigger planets closer to their sun are much easier to detect.
Usually, we talk about the habitable zone radius, which is a distance of the planet from its Sun at which the water could potentially exist. The habitable zone is established as

$$(1-A)\pi R_p^2 \frac{L}{4\pi d_p^2} = 4\pi R_p^2 \sigma T_p^4$$

where A is the reflection coefficient of the planet, R_p is the radius of the planet, L is the luminosity of the star, d_p is the distance of the planet from its sun and T_p is the temperature of the planet. Hence

$$d_p^2 = \frac{(1-A)L}{16\pi\sigma T_p^4}$$

But, we do not know the effect of possible atmosphere on the planet, so it is very hard to say whether some water actually exists in the liquid state on the planet.
Other important think to consider is that planets can be in so called tidal lock - the tidal forces from the star prevent the planet from rotating around its axis. This creates very hostile environments, with one side of the planet very hot and the other very cold.

5.3.1 Proxima Centauri B

Discovered in 2016, this is probably the closest exoplanet to us and is a candidate to direct atmosphere imagining spectroscopy, which could actually tell us about the atmosphere.
But, only radial search has been done for this planet, so we are in fact not sure whether it is a planet at all.

5.3.2 Trappist-1 System

System of 7 planets very close to a small (M9) star. Probably most of the habitable zone planets are in tidal lock (there is 3 total planets in the habitable zone).
The trappist 1 star is usually producing quite large flairs, so that definitely has an influence on the planets as well.