Contents

1 Maxwell Equations

Electromagnetic theory is essentaily a studium of Maxwell equations under different circumstances. As such, we first need to understand the easiest form of Maxwell equations and then get to the more complicated variants.
It should be said that the Maxwell equations are not a complete theory, but it is the most complete theory of electromagnetism that does not use quantum mechanics. It is compatible with special relativity and historically one of the most important theories developed. As such, many concepts from the theory are then inherited by other field theories.

1.1 Differential Form of Maxwell Equations

The differential form of Maxwell equations will be our starting point. The Maxwell equations are 4 linear first order partial differential equations for two vector fields, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$, where $\vec{r}$ is a position vector and t is the time. The equations read as

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

where ρ is the charge density of so free charges ("real" charges, this concept will be explained later) and ϵ_0 is the permitivity of vacuum. This equation is also called the Gauss's law.
The second equation is

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

The third equation is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

which is also called the Faraday's law.
The last equation is

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

which is also called the Ampere's law and where μ_0 is the permeability of vacuum and $\vec{j}$ is the vector current density of free currents.
These equations predict the forms of fields $\vec{B}$ and $\vec{E}$, but do not tell anything about mechanical behaviour of particles in such fields. To insert the mechanical behaviour inside this formalism, we use the Lorentz force formula

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{5}$$

where $\vec{F}$ is a force acting on a point particle of charge q, travelling with velocity $\vec{v}$.
These equations together will form a basis on which we build the electromagnetic theory. Obviously, this course does not explore much of the theory, as its reach is very big. However, we try to cover some fundamentals of the theory.

1.2 Integral Form of Maxwell Equations

The Maxwell equations can be integrated to obtain their integral form, instead of their differential form. This form is usually more useful when trying to determine the fields from specific source distributions (that is charge and current distributions), but is less useful when trying to understand behaviour of the fields in more detail.
We start by integrating the first equation. For this, we assume that the free charge density is bounded by some region V. Now suppose we do an integral over this region V of both sides of the first equation, i.e.

$$\int_V \nabla \cdot \vec{E} d^3 r = \int_V \frac{\rho}{\epsilon_0} d^3 r$$

where $d^3 r$ is the element of the volume V and we dropped the triple integral sign (the order of integral should be clear from the differential). Then, using vector Gauss's law, we have

$$\int_V \nabla \cdot \vec{E} d^3 r = \oint_{\partial V} \vec{E} \cdot d^2 \vec{r}$$

where ∂V is the boundary (surface) of the volume V and $d^2 \vec{r}$ is the element of this surface, pointing outwards from the volume. The integral is over a closed surface, since the volume V is enclosed by its boundary.

We can also usually assume that ϵ_0 is constant. Then

$$\int_V \frac{\rho}{\epsilon_0} d^3r = \frac{1}{\epsilon_0}\int_V \rho d^3r = \frac{Q}{\epsilon_0}$$

where Q is the total charge inside V. Thus, we have

$$\oint_{\partial V} \vec{E}\cdot d^2\vec{r} = \frac{Q}{\epsilon_0} \tag{6}$$

this means that the flux of the electrical field $\vec{E}$ out of a closed volume V depends only on the charge contained in the volume V.
For the second equation, we again do a integral, but this time over any volume V, to get

$$\int_V \nabla\cdot\vec{B} d^3r = 0$$

$$\oint_{\partial V} \vec{B}\cdot d^2\vec{r} = 0 \tag{7}$$

This means that the flux of a magnetic induction field $\vec{B}$ out of a closed volume is always zero. This can be also interpreted as inexistence of magnetic monopoles. For example, free charges are magnetic monopoles, and the flux of the electric field from a closed volume is dependent on the amount of electrical monopoles in this volume. The second Maxwell equation predicts inexistence of any analogue of charges for magnetism.
The third equation can be integrated over some open bounded surface S to get

$$\int_S (\nabla\times\vec{E})\cdot d^2\vec{r} = \int_S -\frac{\partial\vec{B}}{\partial t}\cdot d^2\vec{r}$$

Using Stokes' theorem

$$\int_S (\nabla\times\vec{E})\cdot d^2\vec{r} = \oint_{\partial S} \vec{E}\cdot d\vec{r}$$

Where ∂S is the curve creating the boundary of S. Since we integrate only with respect to position, the time derivative on the second side of previous equation can be taken in front of the integral. Therefore we have

$$\oint_{\partial S} \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d^2\vec{r} \tag{8}$$

This is the more familiar form of Faraday's law, as integral of the electric field over a closed curve must be the electromotive force, creating currents in the materials. We can therefore already see, that if the fields are changing in time, they are bound together - one cannot exist without the other.
The last Maxwell equation can be integrated the same as the previous one

$$\int_S (\nabla\times\vec{B})\cdot d^2\vec{r} = \int_S \mu_0\left(\vec{j}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d^2\vec{r}$$

$$\oint_{\partial S} \vec{B}\cdot d\vec{r} = \int_S \mu_0\left(\vec{j}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) d^2\vec{r} \tag{9}$$

We can see here that $\epsilon_0\frac{\partial\vec{E}}{\partial t}$ has dimensions of current density - it is usually called the displacement current density, and it is the part that was originally added to the Maxwell equations in order to make them more symmetrical and compatible with charge conservation.
These are the integral forms of Maxwell equations.

1.3 Continuity Equation

Consider now taking the divergence of the 4th Maxwell equation

$$\nabla\cdot(\nabla\times\vec{B}) = \nabla\cdot\left(\mu_0\vec{j}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

In cartesian coordinates, the divergence of a curl of a general vector field $\vec{v}$ is

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_x}{\partial z \partial y} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_y}{\partial x \partial z}$$

Since it does not matter in which order we take the partial derivatives, we arrive at

$$\nabla \cdot (\nabla \times \vec{v}) = 0 \tag{10}$$

Since this is a result irrespective of coordinate system, it applies always. Therefore, the divergence taken on 4th Maxwell equation leads to (assuming μ_0 and ϵ_0 are constants)

$$\nabla \cdot (\nabla \times \vec{B}) = 0 = \mu_0 \nabla \cdot \vec{j} + \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t}$$

Since we can exchange the partial derivatives order, we can write (dividing by μ_0 and shifting $\nabla \cdot \vec{j}$ to the other side of the equation)

$$\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} = -\nabla \cdot \vec{j}$$

Using 1st Maxwell equation

$$\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} = \epsilon_0 \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0} = \frac{\partial \rho}{\partial t}$$

since ϵ_0 is assumed to be constant. Therefore, we arrive at

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j} \tag{11}$$

which is a form of continuity equation. In this context, it states that the current density is the density of flux of charges, and that charges are always conserved. This can be better seen from the integrated variant of the continuity equation. Again, integrating over some volume V, we have

$$\int_V \frac{\partial \rho}{\partial t} d^3 r = -\int_V \nabla \cdot \vec{j} d^3 r$$

Using vector Gauss's law and taking the time derivative in front of the space integral

$$\frac{\partial}{\partial t} \int_V \rho d^3 r = -\oint_{\partial V} \vec{j} \cdot d^2 \vec{r}$$

And thus

$$\frac{\partial Q}{\partial t} = -\oint_{\partial V} \vec{j} \cdot d^2 \vec{r} \tag{12}$$

where Q is the charge within volume V. This is the integrated form of the continuity equation.

1.4 Poynting Theorem

We now included the integral versions of all the equations, with the exception of the Lorentz force equation. But, even this equation can be integrated, although in a slightly different way.
We will start by looking at the element of work done by the Lorentz force, dW. The element of work is defined as

$$dW = \vec{F} \cdot d\vec{r}$$

where $\vec{r}$ is the change in position of the particle on which the force is applied. In our case, this change can be expressed in terms of velocity of the particle $\vec{v}$ as

$$d\vec{r} = \vec{v} dt$$

where dt is some element of time. Consider now a set of point charges dq, described by charge density ρ in the space. From definition of density, it follows that

$$dq = \rho d^3 r$$

Hence, the Lorentz force on this element of charge is

$$\vec{F} = dq(\vec{E} + \vec{v} \times \vec{B}) = \rho(\vec{E} + \vec{v} \times \vec{B})d^3r$$

So

$$dW = \vec{F} \cdot \vec{v}dt = \rho(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v}d^3rdt = \rho(\vec{E} \cdot \vec{v})dVdt + \rho((\vec{v} \times \vec{B}) \cdot \vec{v})d^3rdt$$

Since $\vec{v} \times \vec{B} \perp \vec{v}$ (from definition of vector product),

$$(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

And therefore, the element of power $dP = \frac{dW}{dt}$ due to a one charge ρd^3r is

$$dP = \frac{dW}{dt} = \rho(\vec{E} \cdot \vec{v})d^3r$$

we therefore see that the power and/or work is only done by the electric field - the magnetic field does not change the energy of particles.
Integrating over the whole volume of the charge density ρ, we get the total power due to the field

$$P = \int_V \rho(\vec{E} \cdot \vec{v})d^3r \tag{13}$$

But, we can explore further, if we consider that at every point, the current density is simply

$$\vec{j} = \rho\vec{v} \tag{14}$$

This will be a first of many so called constitution relations - relations that are more definitions than real laws, however they make solving Maxwell equations much easier. Then

$$P = \int_V \vec{E} \cdot \vec{j}d^3r$$

Using the 4th Maxwell equation to express $\vec{j}$ as

$$\vec{j} = \frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0\frac{\partial \vec{E}}{\partial t}$$

we have

$$P = \int_V \vec{E} \cdot \left(\frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0\frac{\partial \vec{E}}{\partial t}\right)d^3r = \int_V \frac{1}{\mu_0}\vec{E} \cdot (\nabla \times \vec{B})d^3r - \int_V \epsilon_0\vec{E} \cdot \frac{\partial \vec{E}}{\partial t}d^3r$$

We now need to establish two more vector indentities. First, consider two vectors $\vec{a}$ and $\vec{b}$. In cartesian coordinates

$$\frac{\partial}{\partial t}(\vec{a} \cdot \vec{b}) = \frac{\partial}{\partial t}(a_xb_x + a_yb_y + a_zb_z) = a_x\frac{\partial b_x}{\partial t} + \frac{\partial a_x}{\partial t}b_x + a_y\frac{\partial b_y}{\partial t} + \frac{\partial a_y}{\partial t}b_y + a_z\frac{\partial b_z}{\partial t} + \frac{\partial a_z}{\partial t}b_z$$

But, this can be reexpressed as

$$\frac{\partial}{\partial t}(\vec{a} \cdot \vec{b}) = \vec{a} \cdot \frac{\partial \vec{b}}{\partial t} + \frac{\partial \vec{a}}{\partial t} \cdot \vec{b} \tag{15}$$

For a special case when $\vec{a} = \vec{b}$ then this becomes

$$\frac{\partial}{\partial t}(\vec{a} \cdot \vec{a}) = \frac{\partial}{\partial t}|\vec{a}|^2 = 2\vec{a} \cdot \frac{\partial \vec{a}}{\partial t}$$

Therefore, we can say that

$$\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}|\vec{E}|^2$$

And thus

$$P = \int_V \frac{1}{\mu_0}\vec{E} \cdot (\nabla \times \vec{B})d^3r - \int_V \frac{\epsilon_0}{2}\frac{\partial |\vec{E}|^2}{\partial t}d^3r$$

The second vector identity we need to establish is the value of divergence of a vector product of two vectors, i.e. $\nabla \cdot (\vec{a} \times \vec{b})$. In cartesian coordinates

$$\nabla \cdot (\vec{a} \times \vec{b}) = \frac{\partial}{\partial x}(a_y b_z - a_z b_y) + \frac{\partial}{\partial y}(a_z b_x - a_x b_z) + \frac{\partial}{\partial z}(a_x b_y - a_y b_x) =$$

$$= b_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + b_y \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + b_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) -$$

$$-a_x \left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}\right) - a_y \left(\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x}\right) - a_z \left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right) =$$

$$= \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$$

Therefore

$$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) \tag{16}$$

Therefore, we can express

$$\vec{E} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{B})$$

From the 3rd Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

And thus

$$\vec{E} \cdot (\nabla \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{B})$$

Applying (15) to the first term, we arrive at

$$\vec{E} \cdot (\nabla \times \vec{B}) = -\frac{1}{2}\frac{\partial |\vec{B}|^2}{\partial t} - \nabla \cdot (\vec{E} \times \vec{B})$$

And therefore

$$P = -\int_V \frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B}) d^3 r - \int_V \left(\frac{1}{2\mu_0}\frac{\partial |\vec{B}|^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial |\vec{E}|^2}{\partial t}\right) d^3 r$$

Assuming that μ_0 and ϵ_0 are constants, we then can use vector Gauss's law to derive

$$P = -\frac{1}{\mu_0} \oint_{\partial V} (\vec{E} \times \vec{B}) \cdot d^2\vec{r} - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2\mu_0}|\vec{B}|^2 + \frac{\epsilon_0}{2}|\vec{E}|^2\right) d^3 r \tag{17}$$

where $d^2\vec{r}$ is a element of a closed surface ∂V pointing outwards. We can therefore see that the power on a particle from the field depends on two factors - first is the change in time of the energy densities of the fields themselves, which are clearly $\frac{\epsilon_0}{2}|\vec{E}|^2$ and $\frac{1}{2\mu_0}|\vec{B}|^2$. The other part is a directional energy transport. For this transport, we define the Poynting vector as

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

and this vector determines the direction of energy transport by the wave. The fact that the power is proportional to the negative integral over the volume simply states that the power transferred by the wave is proportional to the integral of component of $\vec{S}$ incident on the surface ∂V, pointing inwards.
Therefore, we can interpret the Poynting vector as an intensity vector of the electromagnetic field.

1.5 Electromagnetic Potentials

For a stationary magnetic field, the 3rd Maxwell equation leads to electric field being a conservative field, and therefore we can find a scalar potential that describes it. But, we of course want to be able to describe even changing fields. In order to create some form of potential that applies generally, consider the second Maxwell equation

$$\nabla \cdot \vec{B} = 0$$

As we have shown in (10), this applies automatically if

$$\vec{B} = \nabla \times \vec{A} \tag{18}$$

where $\vec{A}$ is some vector. This vector is called the vector potential, and can be found always for any magnetic field. Therefore, it is much more general than the scalar potential.
We will now try to find the electric field in terms of this vector potential. From the 3rd Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A}$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Therefore $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ is a conservative field, and therefore it is equal to the gradient of a scalar field. To see this, consider a gradient of a scalar field ∇f. Taking the curl in cartesian coordinates

$$\nabla \times (\nabla f) = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\hat{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\hat{k}$$

But because we can interchange the order of partial derivatives

$$\nabla \times (\nabla f) = \vec{0} \tag{19}$$

Usually, we choose rather function $\nabla(-\Phi) = -\nabla\Phi$, as Φ in this case has common interpretation as scalar potential for stationary $\vec{B}$ field. Therefore

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{20}$$

Now, we can redefine the theory of electromagnetism using only vector and scalar potential. To do this, consider the 1st Maxwell equation

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Substituting in for $\vec{E}$

$$\nabla \cdot \left(-\nabla\Phi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\Phi + \frac{\partial}{\partial t}\nabla \cdot \vec{A} = -\frac{\rho}{\epsilon_0} \tag{21}$$

The 4th Maxwell equation becomes

$$\nabla \times \left(\nabla \times \vec{A}\right) = \mu_0\vec{j} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial}{\partial t}\vec{A}\right)$$

Now, we need to prove yet another vector identity - we need to find expression for curl of a curl of a vector. To do this, consider the x component of vector $\vec{v}$

$$(\nabla \times (\nabla \times \vec{v}))_x = \left(\frac{\partial}{\partial y}(\nabla \times \vec{v})_z - \frac{\partial}{\partial z}(\nabla \times \vec{v})_y\right) = \left(\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right) =$$

$$= \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v_x =$$

$$= (\nabla(\nabla \cdot \vec{v}))_x - \nabla^2 v_x$$

Since this can be done for any component of the resultant vector, we have

$$\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2\vec{v} \tag{22}$$

Therefore, in the case above

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} = \mu_0\vec{j} + \mu_0\epsilon_0\left(-\nabla\frac{\partial\Phi}{\partial t} - \frac{\partial^2\vec{A}}{\partial t^2}\right)$$

Therefore, we have

$$\mu_0\epsilon_0\nabla\frac{\partial\Phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} + \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} = \mu_0\vec{j}$$

Which can be reordered as

$$\left(\nabla^2 - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\vec{A} - \nabla\left(\nabla \cdot \vec{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right) = -\mu_0\vec{j} \tag{23}$$

1.5.1 Gauge Transformations

In derivation of the form of electric field in terms of potentials, we used the fact that $\nabla \times (\nabla f) = 0$ for some scalar function f. But, this means that if we add gradient of some scalar function Λ to the vector potential $\vec{A}$, the magnetic field remains unchanged, as

$$\vec{B}_2 = \nabla \times \vec{A}_2 = \nabla \times (\vec{A}_1 + \nabla\Lambda) = \nabla \times \vec{A}_1 = \vec{B}_1$$

However, this transformation changes the electric field. To see this, we use (20)

$$\vec{E}_2 = -\nabla\Phi_2 - \frac{\partial \vec{A}_2}{\partial t} = -\nabla\Phi_2 - \frac{\partial \vec{A}_1}{\partial t} - \nabla\frac{\partial \Lambda}{\partial t} = -\nabla\left(\Phi_2 + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}_1}{\partial t}$$

If this is to be equal to

$$\vec{E}_1 = -\nabla\Phi_1 - \frac{\partial \vec{A}_1}{\partial t}$$

we clearly need

$$\Phi_2 = \Phi_1 - \frac{\partial \Lambda}{\partial t}$$

Therefore, we have a pair of transforms that do not change neither the electric nor the magnetic field

$$\vec{A}_2 = \vec{A}_1 + \nabla\Lambda \tag{24}$$

$$\Phi_2 = \Phi_1 - \frac{\partial \Lambda}{\partial t} \tag{25}$$

These trasforms are called the gauge transforms and Λ is called the gauge.

1.5.2 Coulomb Gauge

Suppose that we now choose stationary Λ. Then, $\Phi_2 = \Phi_1$, and we are only changing the $\vec{A}$ field. With reference to previous equations, we would achieve great simplification if we could choose Λ such that $\nabla \cdot \vec{A} = 0$. Consider now that we have some $\vec{A}_1$ for which $\nabla \cdot \vec{A}_1 \neq 0$. We than perform a gauge transformation with stationary gauge Λ so that $\nabla \cdot \vec{A}_2 = 0$. This means that

$$\nabla \cdot (\vec{A}_1 + \nabla\Lambda) = 0$$

$$\nabla^2\Lambda = -\nabla \cdot \vec{A}_1$$

This is a Poisson equation and it can be solved for every specific $\vec{A}_1$, but it is not attempted here. We are content with the possibility that there usually exists some gauge that enables us to set divergence of vector potential to zero.
The gauge that performs this is called the Coulomb gauge. Under the Coulomb gauge, the potential Maxwell equations become

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{26}$$

and

$$\nabla^2\vec{A} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \vec{A}}{\partial t}\right) = -\mu_0\vec{j} \tag{27}$$

Furthermore, for a stationary fields, this becomes system of four poisson equations

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\vec{A} = -\mu_0\vec{j}$$

Which has solution

$$\Phi(\vec{r}) = \int_V \frac{\rho(\vec{r'})}{4\pi\epsilon_0|\vec{r} - \vec{r'}|}d^3r'$$

$$\vec{A}(\vec{r}) = \int_V \frac{\mu_0\vec{j}(\vec{r'})}{4\pi|\vec{r} - \vec{r'}|}d^3r'$$

To check that this is indeed a solution, we take the laplacian

$$\nabla^2\Phi = \nabla^2 \int_V \frac{\rho(\vec{r'})}{4\pi\epsilon_0|\vec{r}-\vec{r'}|} d^3r'$$

Since the integration is with respect to $\vec{r'}$ and the laplacian with respect to $\vec{r}$, we can move the laplacian iside the integral

$$\nabla^2\Phi = \int_V \frac{\rho(\vec{r'})}{4\pi\epsilon_0}\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) d^3r'$$

We now need to determine the value of $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)$. While doing so, we must remember that for the point $\vec{r'} = \vec{r}$, the function $f = \frac{1}{|\vec{r}-\vec{r'}|}$ is not defined. Therefore, the behaviour of $\nabla^2 f$ at this point might be weird. For case when $\vec{r'} \neq \vec{r}$, in cartesian coordinates

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = \nabla^2\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right) =$$

$$= -\frac{\partial}{\partial x}\frac{(x-x')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}} - \frac{\partial}{\partial y}\frac{(y-y')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}} -$$

$$-\frac{\partial}{\partial z}\frac{(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}} =$$

$$= -\frac{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}} - (x-x')\frac{3}{2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}2(x-x')}{((x-x')^2+(y-y')^2+(z-z')^2)^3} -$$

$$-\frac{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}} - (y-y')\frac{3}{2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}2(y-y')}{((x-x')^2+(y-y')^2+(z-z')^2)^3} -$$

$$-\frac{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}} - (z-z')\frac{3}{2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}2(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^3} =$$

$$= \frac{3(x-x')^2 - [(x-x')^2+(y-y')^2+(z-z')^2]}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{5}{2}}} + \frac{3(y-y')^2 - [(x-x')^2+(y-y')^2+(z-z')^2]}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{5}{2}}} +$$

$$+\frac{3(z-z')^2 - [(x-x')^2+(y-y')^2+(z-z')^2]}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{5}{2}}} = 0$$

Therefore, $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = 0$ everywhere where $\vec{r'} \neq \vec{r}$. We will now try to use vector Gauss's law to determine the nature of this function at the point $\vec{r'} = \vec{r}$. We expect that the function will diverge, as $\frac{1}{|\vec{r}-\vec{r'}|}$ diverges at this point. Therefore, if the function is normalizable, it has identical behaviour as Dirac delta function - it diverges at one point, it is normalizable and it is zero everywhere. To check the normalizability

$$\int_{V,\vec{r}\in V} \nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) d^3r' < \infty$$

Since $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)$ is zero everywhere but at $\vec{r'} = \vec{r}$, we can choose an arbitrary volume of integration V, so we conviniently choose sphere centered on $\vec{r}$.
Using Gauss's law

$$\int_{V,\vec{r}\in V} \nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) d^3r' = \int_{V,\vec{r}\in V} \nabla\cdot\left[\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right] d^3r' = \oint_{\partial V}\left[\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right]\cdot d^2\vec{r'}$$

But, since we chose a sphere around $\vec{r}$, the integrand of the last integral is defined on the whole surface of integration. For this range

$$\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = -\frac{(x-x')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}}\hat{i} - \frac{(y-y')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}}\hat{j} -$$

$$-\frac{(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}}\hat{k}$$

So, in the range where it is defined

$$\nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = -\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} \tag{28}$$

Hence, in our case

$$\oint_{\partial V} \nabla\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\cdot d^2\vec{r'} = \oint_{\partial V}\left(-\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|}\right)\cdot d^2\vec{r'}$$

Since ∂V is a spherical surface centered around $\vec{r}$, we can now switch the coordinate system to spherical polar coordinates $\vec{r''}$ around origin at $\vec{r}$, which leads to

$$\oint_{\partial V}\left(-\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)\cdot d^2\vec{r'} = \oint_{\partial V}\left(-\frac{\vec{r''}}{|\vec{r''}|^3}\right)\cdot d^2\vec{r''} = -\int_0^{\pi}\int_0^{2\pi}\frac{r''\hat{e}_r}{(r'')^3}\cdot\hat{e}_r(r'')^2\sin\theta d\phi d\theta =$$

$$= -\int_0^{\pi}\int_0^{2\pi}\sin\theta d\phi d\theta = -2\pi\left[\cos\theta\right]_{\pi}^{0} = -4\pi$$

This means that

$$\forall\vec{r'}\neq\vec{r}:\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = 0$$

and

$$\forall V,\vec{r}\notin V:\int_V\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)d^3r' = 0$$

and

$$\forall V,\vec{r}\in V:\int_V\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)d^3r' = -4\pi$$

This means that we can establish identity

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = -4\pi\delta(\vec{r}-\vec{r'}) \tag{29}$$

where $\delta(\vec{r}-\vec{r'})$ is the Dirac delta function.
Hence our potential equation becomes

$$\nabla^2\Phi = \int_V\frac{\rho(\vec{r'})}{4\pi\epsilon_0}(-4\pi\delta(\vec{r}-\vec{r'}))d^3r' = -\frac{\rho(\vec{r})}{\epsilon_0}$$

which is our initial equation. The same can be done even for each component of the vector potential.

1.5.3 Lorenz Gauge

Lorenz gauge is a different gauge that sets

$$\nabla\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial\Phi}{\partial t}$$

This leads to Maxwell equations in form

$$\nabla^2\Phi - \mu_0\epsilon_0\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j}$$

These equations are naturally decoupled, which makes the calculations relatively easy. We however usually choose the Coulomb gauge, as for the static case, it is much easier to work with. We can also note that both potentials under this gauge are wavelike, but with sources. This leads to general solution for the potentials in form of

$$\Phi(\vec{r},t) = \int_V \frac{\rho(\vec{r'}, t - \frac{|\vec{r}-\vec{r'}|}{c})}{4\pi\epsilon_0} d^3r'$$

And

$$\vec{A}(\vec{r},t) = \int_V \frac{\mu_0\vec{j}(\vec{r'}, t - \frac{|\vec{r}-\vec{r'}|}{c})}{4\pi|\vec{r}-\vec{r'}|} d^3r'$$

But, if we are solving for a system of relatively small dimensions, we can approximate this by saying $t - \frac{|\vec{r}-\vec{r'}|}{c} \to t$, and thus we obtain the solution as for static case - we can therefore find approximate solutions for $\vec{A}$ in the same way as for static $\vec{E}$ in Coulomb gauge. This will be used extensively in following sections.

1.5.4 Lorentz Force and Canonical Momentum

We can also reformulate the Loretz force formula in terms of potentials.

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) = q(-\nabla\Phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\nabla\times\vec{A}))$$

To progress, we need yet another vector identity. This time, we want to determine the value of $\vec{a}\times(\nabla\times\vec{b})$. Again, we will determine the value on one component, and argue that this can be generalized for all components. For the x component

$$(\vec{a}\times(\nabla\times\vec{b})) = a_y(\nabla\times\vec{b})_z - a_z(\nabla\times\vec{b})_y = a_y\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right) - a_z\left(\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x}\right) =$$

$$= a_y\frac{\partial}{\partial x}b_y + a_z\frac{\partial}{\partial x}b_z - \left(a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)b_x = a_x\frac{\partial}{\partial x}b_x + a_y\frac{\partial}{\partial x}b_y + a_z\frac{\partial}{\partial x}b_z - \left(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)b_x$$

We should also consider the sort of inverse relation

$$(\vec{b}\times(\nabla\times\vec{a}))_x = b_x\frac{\partial}{\partial x}a_x + b_y\frac{\partial}{\partial x}a_y + b_z\frac{\partial}{\partial x}a_z - \left(b_x\frac{\partial}{\partial x} + b_y\frac{\partial}{\partial y} + b_z\frac{\partial}{\partial z}\right)a_x$$

If we add these two together, we obtain

$$(\vec{a}\times(\nabla\times\vec{b}))_x + (\vec{b}\times(\nabla\times\vec{a}))_x = \frac{\partial}{\partial x}(\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\nabla)b_x - (\vec{b}\cdot\nabla)a_x =$$

$$= (\nabla(\vec{a}\cdot\vec{b}))_x - ((\vec{a}\cdot\nabla)\vec{b})_x - ((\vec{b}\cdot\nabla)\vec{a})_x$$

Therefore, we have vector identity

$$\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}) \quad (30)$$

Hence, in our case

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{v} - \vec{A}\times(\nabla\times\vec{v})$$

We will now assume that $\vec{v}$ does not depend explicitly on position (this does not mean that $\vec{v}$ cannot change with position, only that this change is describable in terms of other variables). Then

$$\vec{A}\times(\nabla\times\vec{v}) = \vec{0}$$

and

$$(\vec{A}\cdot\nabla)\vec{v} = \vec{0}$$

And therefore

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$$

Hence, the force equation becomes

$$\frac{d\vec{p}}{dt} = q\left(-\nabla\Phi - \frac{\partial\vec{A}}{\partial t} - (\vec{v}\cdot\nabla)\vec{A} + \nabla(\vec{v}\cdot\vec{A})\right)$$

But, since $\vec{A}$ only depends on t, x, y, z, we can recognize the convective derivative as

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \nabla)\vec{A}$$

And thus we have

$$\frac{d\vec{p}}{dt} = -q\frac{d\vec{A}}{dt} + q\nabla(\vec{v} \cdot \vec{A} - \Phi)$$

Since q is constant (charge conservation)

$$\frac{d}{dt}(\vec{p} + q\vec{A}) = -\nabla(q\Phi - q\vec{v} \cdot \vec{A}) \tag{31}$$

This is the canonical equation of motion.

1.5.5 Biot-Savart Law

We have seen that the magnetic induction field can be expressed as $\vec{B} = \nabla \times \vec{A}$ and that for fields in relatively small areas

$$\vec{A}(\vec{r}, t) \approx \int_V \frac{\mu_0 \vec{j}(\vec{r'}, t)}{4\pi|\vec{r} - \vec{r'}|} d^3r'$$

Hence, we can derive

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\int_V \frac{\mu_0 \vec{j}(\vec{r'})}{4\pi|\vec{r} - \vec{r'}|} d^3r' \right) = \int_V \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\vec{j}(\vec{r'})}{|\vec{r} - \vec{r'}|} \right) d^3r'$$

But, for any vector $\vec{v}$ and scalar function f

$$\nabla \times (f\vec{v}) = \left(\frac{\partial f v_z}{\partial y} - \frac{\partial f v_y}{\partial z} \right) \hat{i} + \left(\frac{\partial f v_x}{\partial z} - \frac{\partial f v_z}{\partial x} \right) \hat{j} + \left(\frac{\partial f v_y}{\partial x} - \frac{\partial f v_x}{\partial y} \right) \hat{k} = f(\nabla \times \vec{v}) + (\nabla f) \times \vec{v}$$

So

$$\nabla \times (f\vec{v}) = f(\nabla \times \vec{v}) + (\nabla f) \times \vec{v} \tag{32}$$

Hence, if $\vec{v}$ is constant with $\vec{r}$

$$\nabla \times (f\vec{v}) = (\nabla f) \times \vec{v}$$

In this case, since $\vec{j}(\vec{r'})$ only depends on $\vec{r'}$

$$\nabla \times \left(\frac{\vec{j}(\vec{r'})}{|\vec{r} - \vec{r'}|} \right) = \nabla \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) \times \vec{j}(\vec{r'})$$

Using (28)

$$\nabla \times \left(\frac{\vec{j}(\vec{r'})}{|\vec{r} - \vec{r'}|} \right) = -\left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right) \times \vec{j}(\vec{r'}) = \frac{\vec{j}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}$$

And thus

$$\vec{B} = \int_V \frac{\mu_0}{4\pi} \frac{\vec{j}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} d^3r' \tag{33}$$

which is known as the Biot-Savart law.

1.6 Electromagnetic Waves in Vacuum

To find the wave equation hidden inside Maxwell equations, we need to first decouple the Maxwell equations. Start by taking the curl of the 4th Maxwell equation

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \nabla \times \vec{j} + \mu_0\epsilon_0 \frac{\partial}{\partial t} \nabla \times \vec{E}$$

Using 3rd Maxwell equation and (22)

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \nabla \times \vec{j} - \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Using the 2nd Maxwell equation and putting field terms on one side

$$\left(\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{B}=\mu_0\nabla\times\vec{j} \tag{34}$$

We could similarly start from the 3rd Maxwell equation and take curl

$$\nabla\times(\nabla\times\vec{E})=-\frac{\partial}{\partial t}\nabla\times\vec{B}$$

and using 4th Maxwell equation

$$\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\mu_0\frac{\partial\vec{j}}{\partial t}-\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Using the 1st Maxwell equation

$$\left(\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{E}=-\mu_0\frac{\partial\vec{j}}{\partial t}-\frac{1}{\epsilon_0}\nabla\rho \tag{35}$$

Importantly, in vacuum, $\vec{j}=\vec{0}$ and $\rho=0$. Therefore, we have two wave equations

$$\nabla^2\vec{E}=\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{B}=\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

The phase speed of this wave is common and is

$$c=\frac{1}{\sqrt{\mu_0\epsilon_0}}$$

which is the speed of light in vacuum. These equations are linear, and therefore we can construct an arbitrary wave from the simple solutions of these equations by superposition.
The simplest solution is the plane wave solution (start with electric field)

$$\vec{E}=\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

where $\vec{E}_0$ is some constant amplitude.
Plugging this solution in enables us to find the dispersion relation

$$\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t^2}$$

The laplacian operates as

$$\nabla^2\vec{E}=\vec{E}_0\nabla^2e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}(-k_x^2-k_y^2-k_z^2)=-|\vec{k}|^2\vec{E}$$

The double time differentiation produces

$$\frac{\partial^2\vec{E}}{\partial t^2}=-\omega^2\vec{E}$$

Thus

$$-|\vec{k}|^2=\frac{-\omega^2}{c^2}$$

$$\omega^2=c^2|\vec{k}|^2 \tag{36}$$

Similar result is valid for magnetic field.
We now want to derive how differential vector operators alter the wave fields $\vec{E}$ and $\vec{B}$. We start with divergence

$$\nabla\cdot\vec{E}=\nabla\cdot(\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)})$$

But, for any vector $\vec{v}$ and scalar f

$$\nabla\cdot(f\vec{v})=\frac{\partial}{\partial x}(fv_x)+\frac{\partial}{\partial y}(fv_y)+\frac{\partial}{\partial z}(fv_z)=f(\nabla\cdot\vec{v})+(\nabla f)\cdot\vec{v} \tag{37}$$

Hence, in our case for constant $\vec{v}$

$$\nabla \cdot \vec{E} = (\nabla e^{i(\vec{k}\cdot\vec{r}-\omega t)}) \cdot \vec{E}_0 = i(k_x\hat{i} + k_y\hat{j} + k_z\hat{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)} \cdot \vec{E}_0 = i\vec{k} \cdot \vec{E}$$

So

$$\nabla \cdot \vec{E} = i\vec{k} \cdot \vec{E} \tag{38}$$

Similarly, using (32)

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E} \tag{39}$$

and, as we have already shown

$$\nabla^2 \vec{E} = -(\vec{k} \cdot \vec{k})\vec{E} = -(\vec{k}^2)\vec{E} = -|\vec{k}|^2 \vec{E} \tag{40}$$

Finally

$$\frac{\partial}{\partial t}\vec{E} = -i\omega\vec{E} \tag{41}$$

Therefore, Maxwell equations for waves in vacuum become

$$\vec{k} \cdot \vec{E} = 0$$

$$\vec{k} \cdot \vec{B} = 0$$

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$\vec{k} \times \vec{B} = -\frac{1}{c^2}\omega\vec{E}$$

From the first two equations, it follows that $\vec{E} \perp \vec{k}$ and $\vec{B} \perp \vec{k}$. From the other two equations, it follows that $\vec{E} \perp \vec{B}$ and thus we have $\vec{E} \perp \vec{B} \perp \vec{k}$.

2 Static Fields in Matter

2.1 Field of Electric Dipole Moment

Consider two stationary point charges, one at position $\vec{r}_0$ with value $-q$ where q is positive charge, other at position $\vec{r}_0 + \vec{dl}$ with value q.
The scalar potential from these charges is

$$\Phi(\vec{r}) = \int_V \frac{(q\delta(\vec{r'} - (\vec{r}_0 + \vec{dl})) - q\delta(\vec{r'} - \vec{r}_0))}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r'}|} d^3r' = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r} - (\vec{r}_0 + \vec{dl})|} - \frac{1}{|\vec{r} - \vec{r}_0|}\right)$$

If we consider a potential in big distance $|\vec{r}| >> |\vec{dl}|$, this can be approximated as

$$\Phi(\vec{r}) \approx \frac{q}{4\pi\epsilon_0}\nabla'\left(\frac{1}{|\vec{r} - \vec{r'}|}\right)\Bigg|_{\vec{r'}=\vec{r}_0} \cdot \vec{dl}$$

Where ∇' is taken with respect to $\vec{r'}$ coordinates. Using symmetry of $|\vec{r} - \vec{r'}|$ and (28)

$$\nabla'\left(\frac{1}{|\vec{r} - \vec{r'}|}\right) = \nabla'\left(\frac{1}{|\vec{r'} - \vec{r}|}\right) = -\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} = \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}$$

Hence

$$\Phi(\vec{r}) = \frac{q\vec{dl} \cdot (\vec{r} - \vec{r}_0)}{4\pi\epsilon_0|\vec{r} - \vec{r}_0|^3}$$

We can now define the dipole moment of these two charges as

$$\vec{p} = q\vec{dl}$$

Then, we have

$$\Phi(\vec{r}) = \frac{\vec{p} \cdot (\vec{r} - \vec{r}_0)}{4\pi\epsilon_0|\vec{r} - \vec{r}_0|^3} \tag{42}$$

where $\vec{r}_0$ is the position of the dipole with moment $\vec{p}$.

2.2 Polarization Vector

Now, assume that we have a medium which is made of positive and negative charges. When we apply an external field, these charges separate slightly from each other, so a dipole moment between each pair of the charges occurs. To describe a continuum of dipoles, we introduce polarization vector. The polarization vector is effectively a dipole moment density - it gives an overall electric dipole moment per some small volume dV. Therefore, polarization vector $\vec{P}$ is defined as

$$\vec{P} = \frac{d\vec{p}}{dV} \tag{43}$$

where $d\vec{p}$ is the overall electric dipole moment, created by adding together all dipole moments in small volume dV.
Now, assume we have some finite continuum of volume V described by polarization field $\vec{P}(\vec{r})$, and for simplicity assume that there is no free charge present - electrical field $\vec{E}$ is only created by the dipoles, not by any extra charges.
What is this electric field? We will derive it from the scalar potential. The potential from all the dipoles is (because Maxwell equations and potential equations are all linear)

$$\Phi(\vec{r}) = \int_V d\Phi(\vec{r}, \vec{r'})$$

where $d\Phi(\vec{r}, \vec{r'})$ is the potential from the dipole field at position $\vec{r'}$. Using (42)

$$\Phi(\vec{r}) = \int_V \frac{d\vec{p}(\vec{r'}) \cdot (\vec{r} - \vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|^3}$$

And using definition of polarization

$$\Phi(\vec{r}) = \int_V \frac{(\vec{P}(\vec{r'})dV) \cdot (\vec{r} - \vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|^3} = \int_V \frac{\vec{P}(\vec{r'}) \cdot (\vec{r} - \vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|^3} d^3r' \tag{44}$$

We can now try to find out what effect will this polarization field have on Maxwell equations. Assuming a static case, the electric field is simply

$$\vec{E} = -\nabla\Phi$$

Hence the divergence of this field is

$$\nabla \cdot \vec{E} = -\nabla^2\Phi = -\nabla^2 \int_V \frac{\vec{P}(\vec{r}) \cdot (\vec{r} - \vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|^3} d^3r'$$

Since we integrate with respect to different variable, we can move the laplacian inside the integral

$$\nabla \cdot \vec{E} = -\int_V \frac{\vec{P}(\vec{r'})}{4\pi\epsilon_0} \cdot \nabla^2 \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right) d^3r' = -\int_V \frac{\vec{P}(\vec{r'})}{4\pi\epsilon_0} \nabla^2 \left(\nabla' \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) \right) d^3r'$$

Since we can exchange the order of partial derivatives

$$\nabla \cdot \vec{E} = -\int_V \frac{\vec{P}(\vec{r'})}{4\pi\epsilon_0} \nabla' \left(\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) \right) d^3r'$$

Using (29)

$$\nabla \cdot \vec{E} = \int_V \frac{\vec{P}(\vec{r'})}{\epsilon_0} \cdot \nabla'(\delta(\vec{r} - \vec{r'}))d^3r' =$$

$$= \int_V \frac{P_x(\vec{r'})}{\epsilon_0} \frac{\partial}{\partial x'} \delta(\vec{r} - \vec{r'})d^3r' + \int_V \frac{P_y(\vec{r'})}{\epsilon_0} \frac{\partial}{\partial y'} \delta(\vec{r} - \vec{r'})d^3r' + \int_V \frac{P_z(\vec{r'})}{\epsilon_0} \frac{\partial}{\partial z'} \delta(\vec{r} - \vec{r'})d^3r'$$

These integrals are essentially the same, only with respect to different elements. The 3D Dirac delta function is just the product of 3 1D Dirac delta functions, so

$$\delta(\vec{r} - \vec{r'}) = \delta(x - x')\delta(y - y')\delta(z - z')$$

To solve them, consider a integral

$$I = \int_V f(\vec{r'}) \frac{\partial}{\partial x'} \delta(\vec{r} - \vec{r'}) d^3 r' = \int_V f(x', y', z') \delta(y - y') \delta(z - z') \frac{\partial}{\partial x'} \delta(x - x') dx' dy' dz'$$

We can clearly factorize the integral in y' and z', so we have (where X' is the range of x', and similarly for other components)

$$I = \int_{X'} dx' \int_{Y'} dy' \int_{Z'} dz' f(x', y', z') \delta(y - y') \delta(z - z') \frac{\partial}{\partial x'} \delta(x - x') = \int_{X'} f(x', y, z) \frac{\partial}{\partial x'} \delta(x - x') dx'$$

Using substitution $u = x - x'$ ($x' = x - u$)

$$du = -dx'$$

$$\frac{\partial}{\partial x'} = \frac{\partial u}{\partial x'} \frac{\partial}{\partial u} = -\frac{\partial}{\partial u}$$

So, the integral becomes

$$I = \int_{X'} f(x', y, z) \frac{\partial}{\partial x'} \delta(x - x') dx' = \int_U f(x - u, y, z) \frac{\partial}{\partial u} \delta(u) du$$

Using integration by parts

$$I = \int_U f(x - u, y, z) \frac{\partial}{\partial u} \delta(u) du = [f(x - u, y, z) \delta(u)]_U - \int_U \frac{\partial f(x - u, y, z)}{\partial u} \delta(u) du$$

where $[g]_U = g(u_{max}) - g(u_{min})$ Substituting in back for $u = x - x'$

$$I = [f(x', y, z) \delta(x - x')]_{x_{min}}^{x_{max}} - \int_{X'} \frac{\partial f(x', y, z)}{\partial x'} \delta(x - x') dx'$$

if $x_{min} \neq x$ and $x_{max} \neq x$, then the expression inside the bracket is zero (this corresponds to divergence of the field at the point of position of point charge - we will simply assume that this bracket is always zero). Therefore

$$I = -\int_{X'} \frac{\partial f(x', y, z)}{\partial x'} \delta(x - x') dx' = -\left. \frac{\partial f(x', y, z)}{\partial x'} \right|_{x'=x} = -\frac{\partial f(x, y, z)}{\partial x}$$

So, we have shown that

$$\int_X f(x) \frac{\partial}{\partial z} \delta(x - z) dx = -\frac{\partial f(z)}{\partial z} \tag{45}$$

Hence

$$\nabla \cdot \vec{E} = -\frac{\partial}{\partial x} \frac{P_x}{\epsilon_0} - \frac{\partial}{\partial y} \frac{P_y}{\epsilon_0} - \frac{\partial}{\partial z} \frac{P_z}{\epsilon_0} = -\frac{\nabla \cdot \vec{P}}{\epsilon_0}$$

We can superpose this with the Gauss's law for free charges ρ, and thus obtain expression for divergence of $\vec{E}$ in a form of

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} - \frac{\nabla \cdot \vec{P}}{\epsilon_0} \tag{46}$$

We could also transfer the polarization term to the second side and multiply by ϵ_0 to get

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho$$

We now define a new variable, called electric displacement field, as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{47}$$

Then, we obtain a variant of Gauss's law for polarizable media

$$\nabla \cdot \vec{D} = \rho \tag{48}$$

2.3 Magnetic Dipole Field

Consider a small current loop, consisting of a current I flowing in circle of radius dr. Lets choose a cylindrical coordinate system with origin at the centre of the circle so that the circle is inside the xy plane. The current density can be then expressed as

$$\vec{j}(\vec{r}) = \vec{j}(r, \phi, z) = I\hat{e}_\phi \delta(r - dr)\delta(z)$$

for current in $\hat{e}_\phi$ (anti-clockwise) direction.
The magnetic potential from this loop is

$$\vec{A}(\vec{r}) = \int_V \frac{\mu_0 \vec{j}(\vec{r'})}{4\pi|\vec{r} - \vec{r'}|} d^3r' = \int_0^{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{\mu_0 I \hat{e}_{\phi'} \delta(r' - dr)\delta(z')}{4\pi|\vec{r} - \vec{r'}|} r' dz' dr' d\phi'$$

the integral in z is trivial, so

$$\vec{A}(\vec{r}) = \int_0^{2\pi} \int_0^\infty \frac{\mu_0 I \hat{e}_{\phi'} \delta(r' - dr)}{4\pi|\vec{r} - \vec{r'}|} r' dr' d\phi'$$

Since $\hat{e}_{\phi'} = -\sin\phi' \hat{i} + \cos\phi' \hat{j}$ only depends on ϕ'

$$\vec{A}(\vec{r}) = \int_0^{2\pi} \frac{\mu_0 I \hat{e}_{\phi'}}{4\pi|\vec{r} - dr\hat{e}_{r'}|} dr d\phi'$$

Decomposing $\hat{e}_{\phi'}$ and $\hat{e}_{r'}$ to cartesian components

$$\vec{A}(\vec{r}) = \int_0^{2\pi} \frac{\mu_0 I(-\sin\phi' \hat{i} + \cos\phi' \hat{j})}{4\pi\sqrt{(x - dr\cos\phi')^2 + (y - dr\sin\phi')^2 + z^2}} dr d\phi'$$

The fraction can be simplified as follows

$$f = \frac{1}{\sqrt{(x - dr\cos\phi')^2 + (y - dr\sin\phi')^2 + z^2}} = \frac{1}{\sqrt{x^2 + y^2 + z^2}\sqrt{1 - 2\frac{dr(x\cos\phi' + y\sin\phi')}{x^2+y^2+z^2} + \frac{dr^2}{x^2+y^2+z^2}}}$$

Marking $|\vec{r}| = r$, we then have

$$f = \frac{1}{r\sqrt{1 - \frac{2dr(x\cos\phi' + y\sin\phi')}{r^2} + \frac{dr^2}{r^2}}}$$

For small $dr << r$, this can be approximated as

$$f \approx \frac{1}{r\left(1 - \frac{dr(x\cos\phi' + y\sin\phi')}{r^2}\right)} \approx \frac{1}{r}\left(1 + \frac{dr}{r^2}(x\cos\phi' + y\sin\phi')\right)$$

Hence

$$\vec{A}(\vec{r}) = \frac{\mu_0 I dr}{4\pi r} \int_0^{2\pi} \left(-\sin\phi' \hat{i} + \cos\phi' \hat{j}\right) \left(1 + \frac{dr}{r^2}(x\cos\phi' + y\sin\phi')\right) d\phi' =$$

$$= \frac{\mu_0 I dr}{4\pi r} \left[\int_0^{2\pi} -\sin\phi' \hat{i} d\phi' + \int_0^{2\pi} \cos\phi' \hat{j} d\phi' + \int_0^{2\pi} (-\sin\phi' \hat{i} + \cos\phi' \hat{j}) \frac{dr}{r^2}(x\cos\phi' + y\sin\phi') d\phi'\right]$$

The first two integrals go to zero. Therefore, we are left with

$$\vec{A}(\vec{r}) = \frac{\mu_0 I dr^2}{4\pi r^3} \int_0^{2\pi} (-\sin\phi' \hat{i} + \cos\phi' \hat{j})(y\sin\phi' + x\cos\phi') d\phi' =$$

$$= \frac{\mu_0 I dr^2}{4\pi r^3} \left[-\int_0^{2\pi} y\hat{i}\sin^2\phi' d\phi' - \int_0^{2\pi} x\hat{i}\sin\phi'\cos\phi' d\phi' + \int_0^{2\pi} y\hat{j}\cos\phi'\sin\phi' d\phi' + \int_0^{2\pi} x\hat{j}\cos^2\phi' d\phi'\right]$$

The middle two integrals again go to zero. The other integrals are standard

$$\int_0^{2\pi} y\hat{i}\sin^2\phi' d\phi' = y\hat{i}\int_0^{2\pi} \frac{1 - \cos(2\phi')}{2} d\phi' = \frac{y\hat{i}}{2}\left(2\pi - \left[\frac{\sin(2\phi')}{2}\right]_0^{2\pi}\right) = y\pi\hat{i}$$

And similarly

$$\int_0^{2\pi} x\hat{j}\cos^2\phi' d\phi' = \frac{x\hat{j}}{2}\left(\int_0^{2\pi}(1+\cos(2\phi'))d\phi'\right) = x\pi\hat{j}$$

Therefore we have

$$\vec{A}(\vec{r}) = \frac{\mu_0 I\pi dr^2}{4\pi r^3}(-y\hat{i} + x\hat{j})$$

If this formula is to be useful, we need to transform this result into the vector language. We can notice that

$$-y\hat{i} - y\hat{j} = \hat{k} \times \vec{r}$$

So, also using $r = |\vec{r}|$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I\pi dr^2}{4\pi|\vec{r}|^3}(\hat{k} \times \vec{r})$$

We now define the magnetic dipole moment as

$$\vec{\mu} = I\vec{S}$$

where I is the current in the current loop and $\vec{S}$ is the vector of the area that is bounded by the loop, in direction of right hand rule. In our case

$$\vec{\mu} = I\pi dr^2\hat{k}$$

Therefore, we have

$$\vec{A}(\vec{r}) = \frac{\mu_0(\vec{\mu} \times \vec{r})}{4\pi|\vec{r}|^3}$$

This was chosen in a special coordinate system centered on the dipole. If the dipole is instead at the position $\vec{r'}$, we have

$$\vec{A}(\vec{r}) = \frac{\mu_0(\vec{\mu} \times (\vec{r} - \vec{r'}))}{4\pi|\vec{r} - \vec{r'}|^3} \tag{49}$$

2.4 Magnetization Vector

Similarly as for polarization, in a medium in which current loops are present, these current loops together create a magnetic moment in the medium. So, we define a magnetic moment density - magnetization - as

$$\vec{M} = \frac{d\vec{\mu}}{dV} \tag{50}$$

where $d\vec{\mu}$ is the overall magnetic moment of all the current loops inside small volume dV. Therefore, the vector potential at some point $\vec{r}$ induced by magnetic moments of a finite medium of volume V is

$$\vec{A}(\vec{r}) = \int_V d\vec{A}(\vec{r}, \vec{r'})$$

where $d\vec{A}(\vec{r}, \vec{r'})$ is the element of vector potential at $\vec{r}$ due to magnetic moment at $\vec{r'}$. Using (49)

$$\vec{A}(\vec{r}) = \int_V \frac{\mu_0(d\vec{\mu} \times (\vec{r} - \vec{r'}))}{4\pi|\vec{r} - \vec{r'}|^3}$$

Using definition of magnetization

$$\vec{A}(\vec{r}) = \int_V \frac{\mu_0\vec{M} \times (\vec{r} - \vec{r'})}{4\pi|\vec{r} - \vec{r'}|^3} d^3r' \tag{51}$$

We can now similarly try to see how such magnetisation influences Maxwell equations, specifically the 4th Maxwell equation. Again, consider the case when no free currents are present - all currents are closed small currents causing magnetization. Then, since $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$$

The divergence of magnetization potential is

$$\nabla \cdot \vec{A} = \nabla \cdot \int_V \frac{\mu_0\vec{M}(\vec{r'}) \times (\vec{r} - \vec{r'})}{4\pi|\vec{r} - \vec{r'}|^3} d^3r' = \int_V \frac{\mu_0}{4\pi}\nabla \cdot \left(\frac{\vec{M}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}\right) d^3r'$$

Using (16)

$$\nabla\cdot\left(\frac{\vec{M}(\vec{r'})\times(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\right)=(\nabla\times\vec{M}(\vec{r'}))\cdot\left(\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)-\vec{M}(\vec{r'})\cdot\left(\nabla\times\left(\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)\right)$$

Since $\vec{M}$ only depends on $\vec{r'}$, $\nabla\times\vec{M}(\vec{r'})=\vec{0}$. Furthermore, as

$$\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}=\nabla\left(\frac{-1}{|\vec{r}-\vec{r'}|}\right)$$

we have, by (19)

$$\nabla\times\left(\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)=\nabla\times\left(\nabla\left(\frac{-1}{|\vec{r}-\vec{r'}|}\right)\right)=\vec{0}$$

Therefore

$$\nabla\cdot\vec{A}=0$$

And thus we are left with

$$\nabla\times\vec{B}=-\nabla^2\vec{A}=-\nabla^2\int_V\frac{\mu_0}{4\pi}\vec{M}(\vec{r'})\times\nabla'\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)d^3r'$$

Moving the laplacian inside of the integral

$$\nabla\times\vec{B}=-\int_V\frac{\mu_0}{4\pi}\nabla^2\left(\vec{M}(\vec{r'})\times\nabla'\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right)$$

We now strictly speaking need to prove another vector identity - we need to find the value of $\nabla^2(\vec{a}\times\vec{b})$. In cartesian components, choosing the x component

$$(\nabla^2(\vec{a}\times\vec{b}))_x=\nabla^2\left(a_yb_z-a_zb_y\right)=\frac{\partial^2}{\partial x^2}(a_yb_z-a_zb_y)+\frac{\partial^2}{\partial y^2}(a_yb_z-a_zb_y)+\frac{\partial^2}{\partial z^2}(a_yb_z-a_zb_y)$$

For each of these terms, we can do following operations

$$\frac{\partial^2}{\partial x^2}(a_yb_z-a_zb_y)=\frac{\partial}{\partial x}\left(a_y\frac{\partial b_z}{\partial x}+\frac{\partial a_y}{\partial x}b_z-a_z\frac{\partial b_y}{\partial x}-\frac{\partial a_z}{\partial x}b_y\right)=$$

$$=a_y\frac{\partial^2 b_z}{\partial x^2}+2\frac{\partial a_y}{\partial x}\frac{\partial b_z}{\partial x}+\frac{\partial^2 a_y}{\partial x^2}b_z-a_z\frac{\partial^2 b_y}{\partial x^2}-2\frac{\partial a_z}{\partial x}\frac{\partial b_y}{\partial x}-\frac{\partial^2 a_z}{\partial x^2}b_y$$

Summing all the terms

$$(\nabla^2(\vec{a}\times\vec{b}))_x=a_y\nabla^2b_z-a_z\nabla^2b_y+b_z\nabla^2a_y-b_y\nabla^2a_z+2(\nabla a_y\cdot\nabla b_z-\nabla a_z\cdot\nabla b_y)$$

We however cannot do much more simplification here. However, if vector $\vec{a}$ is constant, we obtain simple result

$$\nabla^2(\vec{a}\times\vec{b})=\vec{a}\times(\nabla^2\vec{b})$$

Hence in our case above

$$\nabla\times\vec{B}=-\int_V\frac{\mu_0}{4\pi}\vec{M}(\vec{r'})\times\left(\nabla^2\left(\nabla'\left(\frac{1}{|\vec{r}-\vec{r'}|}\right)\right)\right)d^3r'$$

Swapping the partial derivatives and using (29)

$$\nabla\times\vec{B}=\int_V\mu_0\vec{M}(\vec{r'})\times\nabla'(\delta(\vec{r}-\vec{r'}))d^3r'$$

Using (45) for each term of each component of $\vec{M}\times\nabla'(\delta(\vec{r}-\vec{r'}))$ leads to

$$\nabla\times\vec{B}=\mu_0\nabla\times\vec{M}$$

Hence, for a medium were there are both source currents and magnetization (but no changing electric field), we have

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \nabla \times \vec{M}$$

Transferring $\vec{M}$ term to the second side

$$\nabla \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{j}$$

We can now define

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

as magnetic field intensity. Then, the Maxwell equation becomes (without the electric component variation and after dividing through by μ_0)

$$\nabla \times \vec{H} = \vec{j}$$

2.5 Complete Maxwell Equations in Matter

We have now seen why Maxwell equations change form when inserted into media. It should be noted that we only illustrated this on a case of a static field, but the results are generalizable to the dynamical fields. For those, the Maxwell equations in matter become

$$\nabla \cdot \vec{D} = \rho \tag{52}$$

$$\nabla \cdot \vec{B} = 0 \tag{53}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{54}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \tag{55}$$

where

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

and

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

In the static case, the last two equations become

$$\nabla \times \vec{E} = \vec{0}$$

and

$$\nabla \times \vec{H} = \vec{j}$$

2.5.1 Linear Materials

An important approximation comes from the linear materials. In linear materials, both polarization and magnetization vectors are proportional to the applied field as

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{56}$$

and

$$\vec{M} = \chi_m \vec{H} \tag{57}$$

where χ_m and χ_e are magnetic and electric susceptibilities, respectively.
In this case

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

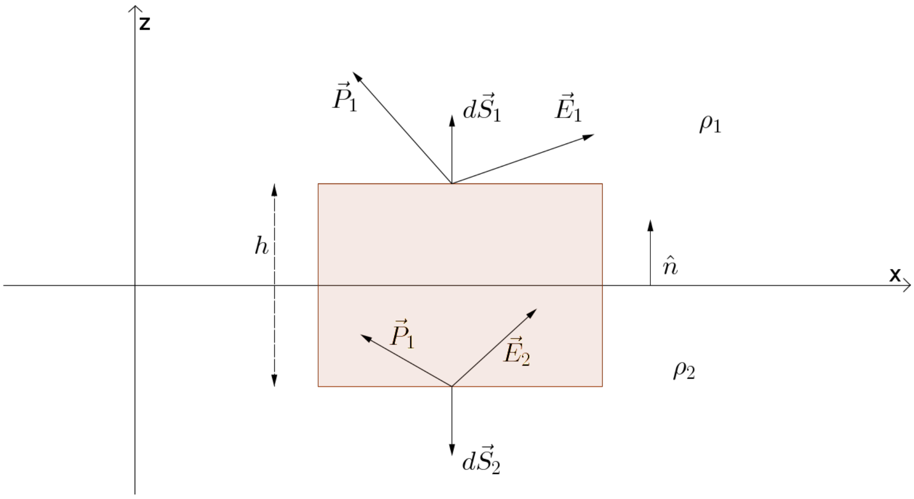
where ϵ is called the permitivity of the medium.
Similarly

$$\vec{B} = \mu_0 (\vec{H} + \chi_m \vec{H}) = \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H}$$

and μ is the permeability of the medium. These relationships can be then easily modified for anisotropic linear media by replacing scalars ϵ and μ with tensors/matrices.

2.6 Polarization Surface Charge Density

Now, consider a boundary of a medium with another medium, the cross-section of which is presented in the figure below. The xy plane forms the boundary (we are close to it so it appears planar), the electric field in material 1 is $\vec{E}_1$, in material 2 $\vec{E}_2$, and similarly for all other variables. The normal vector $\hat{n}$ is normal to the boundary and points towards medium 1.
We introduce a small box symmetrical around the boundary of height h, with faces of areas $|d\vec{S}_1| = |d\vec{S}_2| = dS$.



From the 1st Maxwell equation for $\vec{E}$ in vacuum with polarization from a medium (46)

$$\int_V \nabla \cdot \vec{E} d^3r = \int_V \left(\frac{\rho}{\epsilon_0} - \frac{\nabla \cdot \vec{P}}{\epsilon_0} \right) d^3r'$$

where V is the volume of the box. Using vector Gauss's law

$$\int_V \nabla \cdot \vec{E} d^3r = \oint_{\partial V} \vec{E} \cdot d\vec{S} = \int_V \frac{\rho}{\epsilon_0} d^3r - \oint_{\partial V} \frac{1}{\epsilon_0} \vec{P} \cdot d\vec{S}$$

where $d\vec{S}$ is a surface vector element of the box surface ∂V. If we shrink $h \to 0$ and take small dS, the surface integrals over the sides of the box vanish, and we are left with just

$$\oint_{\partial V} \vec{E} \cdot d\vec{S} = \vec{E}_1 \cdot d\vec{S}_1 + \vec{E}_2 \cdot d\vec{S}_2 = \frac{hdS}{2\epsilon_0}\rho_1 + \frac{hdS}{2\epsilon_0}\rho_2 - \frac{1}{\epsilon_0}\left(\vec{P}_1 \cdot d\vec{S}_1 + \vec{P}_2 \cdot d\vec{S}_2 \right)$$

Because h goes to 0, both volume charge density terms are much smaller than the polarization terms, assuming that ρ does not diverge anywhere. Then

$$\vec{E}_1 \cdot d\vec{S}_1 + \vec{E}_2 \cdot d\vec{S}_2 = -\frac{1}{\epsilon_0}\left(\vec{P}_1 \cdot d\vec{S}_1 + \vec{P}_2 \cdot d\vec{S}_2 \right)$$

Since the faces are opposite, $d\vec{S}_1 = dS\hat{n} = -dS(-\hat{n}) = -d\vec{S}_2$
So

$$\vec{E}_1 \cdot \hat{n} dS - \vec{E}_2 \cdot \hat{n} dS = \frac{1}{\epsilon_0}(\vec{P}_2 \cdot \hat{n} dS - \vec{P}_1 \cdot \hat{n} dS)$$

Dividing through by dS and factoring out $\hat{n}$ (since scalar product is distributive with respect to vector sum)

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = \frac{1}{\epsilon_0}(\vec{P}_2 - \vec{P}_1) \cdot \hat{n} \tag{58}$$

Therefore, if there is a discontinuity in polarizations, there occurs a discontinuity in electric fields as well. The meaning of this discontinuity can be further enlightened if we consider a different case - consider a situation where $\vec{P} = \vec{0}$, but ρ diverges on the boundary as $\rho = \rho_0(x, y, z) + \sigma(x, y)\delta(z)$, where ρ_0 does not diverge and σ is some surface charge density localised on the boundary of the surface ($\delta(z)$ is the Dirac delta).

Then, the 1st Maxwell equation is

$$\nabla \cdot \vec{E} = \frac{\rho_0 + \sigma\delta(z)}{\epsilon_0}$$

Repeating the integral as before

$$\int_V \nabla \cdot \vec{E} d^3r = (\vec{E}_1 - \vec{E}_2) \cdot \hat{n} dS = \int_V \frac{\rho_0}{\epsilon_0} d^3r + \int_V \frac{\sigma\delta(z)}{\epsilon_0} d^3r = \frac{hdS}{2\epsilon_0}\rho_{01} + \frac{hdS}{2\epsilon_0}\rho_{02} + \frac{\sigma}{\epsilon} dS$$

Again neglecting the volume charge density terms

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

Hence we can see that the polarization discontinuity creates the same effect as if there was some surface charge density σ. We usually than talk about bound surface charge density, as this density is only due to displacement of electric dipoles, which are, by definition, bound to the opposite charge in the pair. We usually denote the bound surface charge density as σ_b and by combining the results, above, we can see that

$$\sigma_b = (\vec{P}_2 - \vec{P}_1) \cdot \hat{n} \tag{59}$$

This is the so called bound surface charge density, or polarization surface charge density. Importantly, we can notice that if we rearrange (58)

$$\left(\vec{E}_1 + \frac{1}{\epsilon_0}\vec{P}_1 - \left(\vec{E}_2 + \frac{1}{\epsilon_0}\right)\right) \cdot \hat{n} = 0$$

Multiplying by ϵ_0

$$(\vec{E}_1\epsilon_0 + \vec{P}_1 - (\vec{E}_2\epsilon_0 + \vec{P}_2)) \cdot \hat{n} = (\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = 0$$

So, for non-divergent ρ (the free charge volume density), the dielectric displacement field is always continuous. If ρ is divergent on the surface by some free surface charge density σ_f (not the bound charge density, this density is due to real charges present in the medium being concentrated on the surface, as is the case for example for charged conductors), we then have

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \sigma_f \tag{60}$$

Which again follows from the same setup as above, and using (48) in form

$$\nabla \cdot \vec{D} = \rho = \sigma_f \delta(z)$$

2.7 Magnetization Surface Current Density

Similarly as above, the boundary conditions for fields not-continuous in magnetization will lead us to creation fo boundary sources, but this time surface current densities (with dimensions AL^{-1}, current per meter).
We will us a so called Amperian loop in the plane perpendicular to the boundary, as depicted in figure below.
Again, in the direction perpendicular to the boundary, the loop is of height h, which we will shrink towards 0, and small length dl in direction $\hat{l}$ parallel to the boundary. The oriented normal surface vector of the surface bounded by the loop is $\hat{n}_s$ (for surface inside the same plane as the loop), the normal boundary vector is $\hat{n}$ and thus, they are connected via relations

$$\hat{l} = \hat{n}_s \times \hat{n}$$

$$\hat{n}_s = \hat{n} \times \hat{l}$$

$$\hat{n} = \hat{l} \times \hat{n}_s$$

We now use the 4th Maxwell equation for static field to find the integral over the surface bounded by the loop with surface vector $\hat{n}_s$. The integral over the surface S is

$$\int_S (\nabla \times \vec{B}) \cdot \hat{n}_s dS = \int_S \left(\mu_0 \hat{j} + \mu_0 \nabla \times \vec{M}\right) \cdot \hat{n}_s dS$$

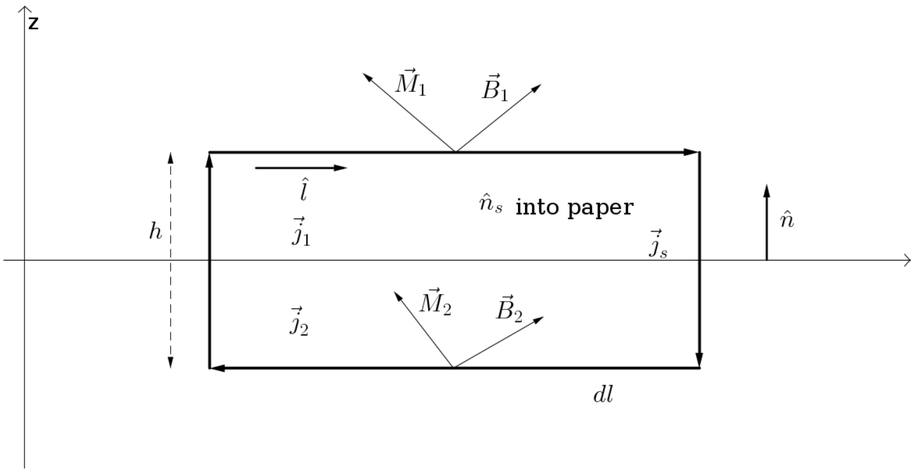
The integrals with curls can be transformed by Stokes' theorem into closed integrals over the loop, so we have

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot \hat{n}_s dS + \mu_0 \oint_{\partial S} \vec{M} \cdot d\vec{l}$$

Inspired by the previous case, we allow for non-divergent and divergent parts of the current density, as

$$\vec{j} = \vec{j}_f + \vec{j}_{sf}\delta(z)$$

where $\vec{j}_f$ is the free current density in the volume, and $\vec{j}_{sf}$ is the free surface current density, restricted to the surface of the boundary.



So, we have

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j}_f \cdot \hat{n}_s dS + \mu_0 \vec{j}_{sf} \cdot \hat{n}_s dl + \mu_0 \oint_{\partial S} \vec{M} \cdot d\vec{l} \tag{61}$$

For small h and dl, the integrals become

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \vec{B}_1(\vec{r}) \cdot \hat{l}dl - \vec{B}_1(\vec{r} + \hat{l}dl) \cdot \hat{n}\frac{h}{2} - \vec{B}_2(\vec{r} + \hat{l}dl) \cdot \hat{n}\frac{h}{2} - \vec{B}_2 \cdot \hat{l}dl + \vec{B}_2(\vec{r}) \cdot \hat{n}\frac{h}{2} + \vec{B}_1(\vec{r}) \cdot \hat{n}\frac{h}{2}$$

where $\vec{r}$ is the position vector along the $\hat{l}$ direction. If we choose $h << dl$, this becomes

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \vec{B}_1 \cdot \hat{l}dl - \vec{B}_2 \cdot \hat{l}dl = (\vec{B}_1 - \vec{B}_2) \cdot \hat{l}dl$$

By exactly analogous reasoning

$$\oint_{\partial S} \vec{M} \cdot d\vec{l} = (\vec{M}_1 - \vec{M}_2) \cdot \hat{l}dl$$

The integral over the surface becomes approximately ($h \to 0$)

$$\int_S \vec{j}_f \cdot \hat{n}_s dS = (\vec{j}_{f1} + \vec{j}_{f2}) \cdot \hat{n}_s \frac{h}{2} dl = 0$$

Hence, we have, by substituting into (61)

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{l}dl = \mu_0 \vec{j}_{fs} \cdot \hat{n}_s dl + \mu_0(\vec{M}_1 - \vec{M}_2) \cdot \hat{l}dl$$

The surface current density term can be trasformed as

$$\vec{j}_{fs} \cdot \hat{n}_s dl = \vec{j}_{fs} \cdot (\hat{n} \times \hat{l})dl$$

By rules for mixed vector product

$$\vec{j}_{fs} \cdot (\hat{n} \times \hat{l}) = \hat{l} \cdot (\vec{j}_{fs} \times \hat{n})$$

So, after dividing through by dl, the condition becomes

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{l} = \mu_0(\vec{j}_{fs} \times \hat{n}) \cdot \hat{l} + \mu_0(\vec{M}_1 - \vec{M}_2) \cdot \hat{l} \tag{62}$$

Now, we need to notice a useful identity that enables us to express the magnetisation term in such direction that it can be seen that this term creates the surface current density as well, but this time bound surface current density. So, consider

$$\hat{n} \times (\hat{n} \times (\vec{M}_1 - \vec{M}_2)) = \hat{n}(\hat{n} \cdot (\vec{M}_1 - \vec{M}_2)) - (\hat{n} \cdot \hat{n})(\vec{M}_1 - \vec{M}_2)$$

where I used standard Lagrange's triple vector formula. Since $\hat{n}$ is a unit vector

$$\hat{n} \times (\hat{n} \times (\vec{M}_1 - \vec{M}_2)) = \hat{n}(\hat{n} \cdot (\vec{M}_1 - \vec{M}_2)) - (\vec{M}_1 - \vec{M}_2)$$

Now, if we take the dot product with $\hat{l}$ of both sides

$$(\hat{n} \times (\hat{n} \times (\vec{M}_1 - \vec{M}_2))) \cdot \hat{l} = (\hat{n}(\hat{n} \cdot (\vec{M}_1 - \vec{M}_2))) \cdot \hat{l} - (\vec{M}_1 - \vec{M}_2) \cdot \hat{l}$$

But, since $\hat{l}$ and $\hat{n}$ are perpendicular, we obtain

$$(\hat{n} \times (\hat{n} \times (\vec{M}_1 - \vec{M}_2))) \cdot \hat{l} = -(\vec{M}_1 - \vec{M}_2) \cdot \hat{l}$$

Or

$$((\hat{n} \times (\vec{M}_1 - \vec{M}_2)) \times \hat{n}) \cdot \hat{l} = (\vec{M}_1 - \vec{M}_2) \cdot \hat{l}$$

Or, switching one more sign

$$(((\vec{M}_2 - \vec{M}_1) \times \hat{n}) \times \hat{n}) \cdot \hat{l} = (\vec{M}_1 - \vec{M}_2) \cdot \hat{l}$$

Inserting this into our equation

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{l} = \mu_0(\vec{j}_{fs} \times \hat{n}) \cdot \hat{l} + \mu_0(((\vec{M}_2 - \vec{M}_1) \times \hat{n}) \times \hat{n}) \cdot \hat{l}$$

Since the vector product and scalar product are both distributive over vector addition

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{l} = \mu_0((\vec{j}_{fs} + (\vec{M}_2 - \vec{M}_1) \times \hat{n}) \times \hat{n}) \cdot \hat{l} \tag{63}$$

Hence, we see that the magnetisation discontinuity creates the same effect as surface current density $\vec{j}_{bs}$, given as

$$\vec{j}_{bs} = (\vec{M}_2 - \vec{M}_1) \times \hat{n} \tag{64}$$

This is the predicted bound surface current density, due to the discontinuity in magnetisation. We should also notice that from (62), we can derive

$$(\vec{B}_1 - \mu_0\vec{M}_1 - (\vec{B}_2 - \mu_0\vec{M}_2)) = \mu_0(\vec{j}_{fs} \times \hat{n}) \cdot \hat{l}$$

Which can be reexpressed in terms of $\vec{H}$ as

$$(\vec{H}_1 - \vec{H}_2) \cdot \hat{l} = \mu_0(\vec{j}_{fs} \times \hat{n}) \cdot \hat{l} \tag{65}$$

And thus the only discontinuity in the H field is due to the free surface current density.

2.8 Other Boundary Conditions

Using the same setups as for previous two cases of boundary conditions but exchanging the electric and magnetic fields, the gaussian pillbox with 2nd Maxwell equation leads to

$$\int_V \nabla \cdot \vec{B} dV = \oint_{\partial V} \vec{B} \cdot d\vec{S} = (\vec{B}_1 - \vec{B}_2) \cdot \hat{n} dS = \int_V 0 dV = 0$$

So

$$\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n} \tag{66}$$

So, the normal component of the magnetic field is always continuous.
For the amperian loop setup, we can use the 3rd Maxwell equation, which for static fields is

$$\nabla \times \vec{E} = \vec{0}$$

So, the integral over the surface becomes

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{S} = \oint_{\partial S} \vec{E} \cdot d\vec{l} = (\vec{E}_1 - \vec{E}_2) \cdot dl\hat{l} = \int_S \vec{0} \cdot d\vec{S} = 0$$

So

$$\vec{E}_1 \cdot \hat{l} = \vec{E}_2 \cdot \hat{l} \tag{67}$$

Therefore, for static fields, the parallel component of the electric field is always continuous.
Therefore, we have a set of four boundary conditions - the normal component of the eletric displacement field $\vec{D}$ is discontinuous only by free surface charge densities as

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \sigma$$

where σ is the free surface charge density. The parallel component of the electric field is always continuous. The normal component of the magnetic field is always continuous as well, but the parallel component of the magnetic field intensity vector $\vec{H}$ is discontinuous by free surface current density as

$$(\vec{H}_1 - \vec{H}_2) \cdot \hat{l} = (\vec{j}_s \times \hat{n}) \cdot \hat{l}$$

3 Oscillating Fields in Matter

To understand the oscillating fields in matter, we first need to understand how free charges respond by movement to applied electromagnetic field. In many situations, the Ohm's law proves to be a very good approximation of behaviour of the free charges.

3.1 Ohm's Law

The Ohm's law states that the current density $\vec{j}$ of free charges that responds to the electric field $\vec{E}$ is determined as

$$\vec{j} = g\vec{E} \tag{68}$$

where g is the conductivity of the material (again, could possibly be a tensor for materials that conduct differently in different directions).
The conductivity can be estimated from the Drude model for conductivity. In this model, we assume that the force on the charges includes some velocity dependent friction term, so that the Newton's second law for the charge carriers is

$$m\frac{d\vec{v}}{dt} = q\vec{E} - \frac{m}{\tau}\vec{v}$$

where τ can be interpreted as the mean time between collisions of the charge carriers with the immobile matter elements (atoms, nuclei etc.). When the charges reach a terminal velocity $\vec{v}_d$, it follows that $\frac{d\vec{v}_d}{dt} = 0$ so

$$0 = q\vec{E} - \frac{m}{\tau}\vec{v}_d$$

So

$$\vec{v}_d = \frac{q\tau}{m}\vec{E}$$

Multiplying by the charge of the carrier q and the carrier density n, we then obtain the current density as

$$\vec{j} = qn\vec{v}_d = \frac{nq^2\tau}{m}\vec{E}$$

So, we can see that from this model

$$g = \frac{nq^2\tau}{m} \tag{69}$$

One direct application of Ohm's law is the charge dissipation for conductive materials. Take the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}$$

Using Ohm's law

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (g\vec{E})$$

for constant g

$$\frac{\partial \rho}{\partial t} = -g\nabla \cdot \vec{E}$$

Using 1st Maxwell equation in matter (assuming linear medium with constant permitivity)

$$\frac{\partial \rho}{\partial t} = -g\nabla \cdot (\frac{1}{\epsilon}\vec{D}) = -\frac{g}{\epsilon}\nabla \cdot \vec{D} = -\frac{g\rho}{\epsilon}$$

This is a standard differential equation, leading to

$$\rho = \rho_0(\vec{r})e^{-\frac{g}{\epsilon}t}$$

which leads to the decreasing value of ρ in the medium as time passes. But, since this is only valid inside the medium, not on the boundaries. So, all the charge that dissipates from the medium goes on the boundaries of the object.

3.2 Wave Equations in Conducting Matter

Now, we take the time dependent Maxwell equations in matter with the Ohm's law and linear isotropic materials approximations. Taking the curl of the 3rd Maxwell equation

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\nabla \times \frac{\partial \mu \vec{H}}{\partial} = -\mu \frac{\partial}{\partial t}\nabla \times \vec{H}$$

Using the identity for curl of a curl (22) and substituting for $\nabla \times \vec{H}$ from the 4th Maxwell equation

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\vec{j} + \frac{\partial}{\partial t}\vec{D}\right) = -\mu\frac{\partial}{\partial t}\left(\vec{j} + \epsilon\frac{\partial}{\partial t}\vec{E}\right)$$

Using 1st Maxwell equation for $\nabla \cdot \vec{E}$ and assuming that the volume charge density inside the materials is $\rho = 0$ (neutral medium, or a good conducting medium), we arrive at

$$\nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\vec{j} + \mu\epsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

Using Ohm's law

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} - \mu g\frac{\partial \vec{E}}{\partial t} = 0 \tag{70}$$

This is a wave equation same equation as for the vacuum waves, but we have an extra damping term, proportional to the conductivity g. We should therefore expect that conduction in materials causes the charges to act against the change in the field (similar principle as predicted by Lenz's law).
We should also notice that for insulating media where $g = 0$, this becomes exactly like the vacuum equation, but the speed of the wave is not c but $\frac{1}{\sqrt{\mu\epsilon}}$ (more on this later).
Similarly, taking the curl of the 4th equation first

$$\nabla \times (\nabla \times \vec{H}) = \nabla \times \vec{j} + \nabla \times \frac{\partial}{\partial t}\vec{D}$$

Using constitutive relations and Ohm's law (which is effectively also a constitutive relation)

$$\nabla(\nabla \cdot \vec{H}) - \nabla^2\vec{H} = \nabla \times g\vec{E} + \epsilon\frac{\partial}{\partial t}\nabla \times \vec{E}$$

Using $\nabla \cdot \vec{H} = \frac{1}{\mu}\nabla \cdot \vec{B} = 0$ and 3rd Maxwell equation

$$-\nabla^2\vec{H} = g\nabla \times \vec{E} + \epsilon\frac{\partial}{\partial t}\nabla \times \vec{E} = -\left(g + \epsilon\frac{\partial}{\partial t}\right)\frac{\partial \vec{B}}{\partial t} = -\mu g\frac{\partial \vec{H}}{\partial t} - \mu\epsilon\frac{\partial^2 \vec{H}}{\partial t^2}$$

Hence

$$\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\vec{H} - \mu g\frac{\partial \vec{H}}{\partial t} = 0 \tag{71}$$

This is exactly the same wave equation as (70), but for magnetic field intensity vector $\vec{H}$.
We can see that these wave equations are linear, therefore, we can use principle of superposition to build complicated wave patterns from easy patterns. So, we should first try whether the plane waves are the solutions for this equation.
Choosing the equation for $\vec{E}$ for example and plane wave

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

and substituting into the wave equation (70)

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} - \mu g\frac{\partial}{\partial t}\vec{E} = 0$$

We can still use relations (38) - (41), so we are left with

$$\left(-\vec{k}^2 - \mu\epsilon(-\omega^2)\right)\vec{E} - \mu g(-i\omega)\vec{E}$$

Dividing through by $\vec{E}$, we are left with a new dispersion relation

$$\vec{k}^2 = \mu\omega^2\left(\epsilon + i\frac{g}{\omega}\right) \tag{72}$$

We can see that this can be only satisfied if $\vec{k}$ is complex. We will explore this later. Now, when we know that we can make plane waves into solutions of these new wave equations, we need to inspect again the Maxwell equations in matter.
The 1st equation becomes

$$\nabla\cdot\vec{D} = i\epsilon\vec{k}\cdot\vec{E} = \rho_f$$

For neutral medium

$$\vec{k}\cdot\vec{E} = 0$$

so, $\vec{E}\perp\vec{k}$. The 2nd equation is

$$\nabla\cdot\vec{B} = \mu i\vec{k}\cdot\vec{H} = 0$$

So $\vec{k}\cdot\vec{H} = 0$ and $\vec{k}\perp\vec{H}$.
From the third equation

$$\nabla\times\vec{E} = i\vec{k}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\mu\vec{H}$$

which means that $\vec{H}\perp\vec{E}$ and $\vec{H}\perp\vec{k}$, so the relative positions of $\vec{H}$ and $\vec{E}$ are the same as for $\vec{B}$ and $\vec{E}$ in vacuum, with wavevector $\vec{k}$ still pointing in the direction of transport of energy.

3.3 Boundary Conditions for Oscillating Fields

For oscillating fields, which are time varying, we need to reformulate some of our boundary conditions. As first two Maxwell equations are independent of time, the boundary conditions associated with them, i.e. discontinuity of $\vec{D}$ as in (60) and continuity of $\vec{B}$ as in (66) remain unchanged. However, the 3rd and 4th Maxwell equations in matter include a time dependence, so their respective boundary conditions change. Furthermore, we need to ensure that the charge is conserved in this system, so the continuity equation will present another boundary condition. We start with the condition due to 3rd Maxwell equation.
Again, we use the same setup - an Amperian loop with long dimension parallel to the surface with parallel vector $\hat{l}$. The loop is oriented so that the oriented area element has direction $\hat{n}_s$ for which $\hat{n}_s = \hat{n}\times\hat{l}$. Taking the integral over the area of the loop of the 3rd Maxwell equation than is

$$\int_S (\nabla\times\vec{E})\cdot(\hat{n}_s dS) = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot(\hat{n}_s dS)$$

Using constitutive relation for $\vec{B}$ and assuming μ is constant

$$\int_S (\nabla\times\vec{E})\cdot(\hat{n}_s dS) = -\mu\int_S \frac{\partial\vec{H}}{\partial t}$$

We will only consider oscillating fields, for which we can consider $\vec{H}$ wavelike, so

$$\frac{\partial \vec{H}}{\partial t} = -i\omega \vec{H}$$

So, using Stokes' theorem to transform the integral of curl of $\vec{E}$

$$\oint_S \vec{E} \cdot d\vec{l} = i\omega\mu \int_S \vec{H} \cdot (\hat{n}_s dS)$$

For small dl and h, as defined before, this becomes

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l} dl + \frac{h}{2}\hat{n} \cdot (\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r} + d\vec{l}) - \vec{E}_2(\vec{r} + d\vec{l})) = dl \frac{h}{2} i\omega\mu\hat{n}_s \cdot \left(\vec{H}_1 + \vec{H}_2\right)$$

and when we choose $h << dl$, we then have

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l} = 0$$

$$\vec{E}_1 \cdot \hat{l} = \vec{E}_2 \cdot \hat{l}$$

Therefore, for oscillating fields, we still require that the parallel component of $\vec{E}$ is continuous on the boundary (in fact, this would apply for any $\vec{H}$ for which its time derivative is finite), which is exactly the same condition as before.
Similarly, for the 4th Maxwell equation in the same setup

$$(\vec{H}_1 - \vec{H}_2) \cdot \hat{l} = (\vec{j}_{fs} \times \hat{n}) \cdot \hat{l} + \frac{1}{dl}(-i\omega) \int_S \vec{D} \cdot d\vec{S}$$

If we again assume that $\vec{D}$ is finite here, then the integral goes to zero, and

$$(\vec{H}_1 - \vec{H}_2) \cdot \hat{l} = (\vec{j}_{fs} \times \hat{n}) \cdot \hat{l}$$

Using Ohm's law

$$\vec{j}_{fs} = (g_1 \vec{E}_1 \cdot \hat{j}_{fs})\hat{j}_{fs}$$

where $\hat{j}_{fs}$ is the unit vector in the direction of $\vec{j}_{fs}$, and we only take the component of $\vec{E}$ parallel to boundary, as $\vec{j}_{fs}$ is restricted to boundary. But, this must be also applicable from the perspective of the second medium, so

$$\vec{j}_{fs} = (g_2 \vec{E}_2 \cdot \hat{j}_{fs})\hat{j}_{fs}$$

But, since the parallel components of the $\vec{E}$ field are continuous over the boundary, $\vec{E}_2 \cdot \hat{j}_{fs} = \vec{E}_1 \cdot \hat{j}_{fs}$. But, since g_2 is generally different then g_1, we have a problem - our two equations are inconsistent.
The only way how to make them consisten is to say that we could not have written them in the first place, which could only be the case if we could not define the direction $\hat{j}_{fs}$, which is only true if $\vec{j}_{fs} = 0$.
This then means that

$$\vec{H}_1 \cdot \hat{l} = \vec{H}_2 \cdot \hat{l}$$

and the $\vec{H}$ field's parallel components are continuous.
The continuity equation is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}$$

we will allow for ρ to include a free surface charge density, so

$$\rho = \rho_f + \sigma_f \delta(z)$$

Now, imagine a Gaussian pillbox in the standard setup as described before. Doing the integral over this pillbox of the continuity equation leads to

$$\int_V \nabla \cdot \vec{j} dV = -\frac{\partial}{\partial t} \int_V (\rho_f + \sigma_f \delta(z)) dV$$

Using Gauss vector theorem and integrating over the Dirac delta function

$$\oint_{\partial V} \vec{j} \cdot d\vec{S} = -\frac{\partial}{\partial t} \left(\int_V \rho_f dV + \sigma_f dS \right)$$

For small dS and h
$$(\vec{j}_1 - \vec{j}_2)\cdot\hat{n}dS = -\frac{\partial}{\partial t}\left(\frac{h}{2}dS(\rho_{f1}+\rho_{f2}) + \sigma_f dS\right)$$
If we choose $h << \sqrt{dS}$, then
$$(\vec{j}_1 - \vec{j}_2)\cdot\hat{n} = -\frac{\partial}{\partial t}\sigma_f \tag{73}$$
Since we are considering oscillating fields, we can assume that if the σ_f is non-zero, it will be oscillating at frequency ω, same as the frequency of the fields $\vec{H}$ and $\vec{E}$, as
$$\sigma_f = \sigma_0 e^{-i\omega t}$$
So, the boundary condition becomes
$$(\vec{j}_1 - \vec{j}_2)\cdot\hat{n} = i\omega\sigma_f$$
Using Ohm's law
$$(g_1\vec{E}_1 - g_2\vec{E}_2)\cdot\hat{n} = i\omega\sigma_f$$
But, we already have a boundary condition involving $\vec{E}$ and σ_f, it is the condition on dielectric displacement field, which for linear materials is
$$(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = (\epsilon_1\vec{E}_1 - \epsilon_2\vec{E}_2)\cdot\hat{n} = \sigma_f$$
In order for these two conditions to be consistent, we require
$$(g_1\vec{E}_1 - g_2\vec{E}_2)\cdot\hat{n} = i\omega(\epsilon_1\vec{E}_1 - \epsilon_2\vec{E}_2)\cdot\hat{n}$$
Dividing by $i\omega$
$$\left(\frac{g_1}{i\omega}\vec{E}_1 - \frac{g_2}{i\omega}\vec{E}_2\right)\cdot\hat{n} = \left(\epsilon_1\vec{E}_1 - \epsilon_2\vec{E}_2\right)\cdot\hat{n}$$
Hence, using $\frac{1}{i} = -i$ and separating $\vec{E}_1$ from $\vec{E}_2$
$$\left(\epsilon_1 + i\frac{g_1}{\omega}\right)\vec{E}_1\cdot\hat{n} = \left(\epsilon_2 + i\frac{g_2}{\omega}\right)\vec{E}_2\cdot\hat{n} \tag{74}$$
This means that the normal component of the field $\left(\epsilon + i\frac{g}{\omega}\right)\vec{E}$ is continuous on the boundary.

3.3.1 Material Specific Boundary Conditions

Summarizing the independent boundary conditions in matter for time oscillating fields, we have
$$\vec{E}_1\cdot\hat{l} = \vec{E}_2\cdot\hat{l}$$
$$\vec{B}_1\cdot\hat{n} = \vec{B}_2\cdot\hat{n}$$
$$\vec{H}_1\cdot\hat{l} = \vec{H}_2\cdot\hat{l}$$
where $\vec{j}_s$ is the free surface current density. Finally
$$\left(\epsilon_1 + i\frac{g_1}{\omega}\right)\vec{E}_1\cdot\hat{n} = \left(\epsilon_2 + i\frac{g_2}{\omega}\right)\vec{E}_2\cdot\hat{n}$$
The main difference when compared to the static field boundary conditions is the last equation, which is also material dependent via the conductivities $g_{1/2}$. Therefore, it is useful to consider two extremes - an incidence on a non-conducting material, where $g_2 = 0$, and incidence on a perfect conductor, when $g_2 \to \infty$. For the case $g_2 = 0$, the equation (73) becomes equation
$$\frac{\partial}{\partial t}\sigma_f = 0$$
which leads to static surface charge density, which is consistent with boundary condition on $\vec{D}$ for static field, so the field $\vec{D}$ is only discontinuous if there is a static charge on the surface of the non-conductor, and for neutral non-conductor/dielectric, the $\vec{D}$ field is then continuous.
For $g_2 \to \infty$, from Maxwell 4th equation
$$\nabla\times\vec{H}_2 = \vec{j}_2 + \frac{\partial}{\partial t}\vec{D}_2 = (g_2 - i\omega\epsilon_2)\vec{E}_2$$

This curl is only finite if $\vec{E}_2 \to 0$, so $\vec{E}_2$ must be zero everywhere in the material. From the 3rd Maxwell equation

$$\nabla \times \vec{E}_2 = 0 = -\frac{\partial}{\partial t}\vec{B}_2 = i\mu\omega\vec{H}_2$$

which means that $\vec{H}_2 = 0$. Thus, in perfectly conductive materials, the electromagnetic fields cannot be non-zero. But, this condition destroys the argument for continuous parallel components of $\vec{H}$, so now, the condition on $\vec{H}_1$ is

$$\vec{H}_1 \cdot \hat{l} = (\vec{j}_{fs} \times \hat{n}) \cdot \hat{l}$$

The condition on $\vec{D}$ becomes simply

$$\vec{D}_1 \cdot \hat{n} = \sigma_f$$

3.4 Complex Wavevector

Usually, we denote the complex wavevector differently, say $\vec{\mathbf{k}}$, so that it is clear that its a complex number. The simplest model we can come up with for the complex wavevector is to be simply multiplied by some complex number z. We then usually choose z as

$$\vec{\mathbf{k}} = \vec{k}z = \vec{k}(1 + i\frac{\alpha}{|\vec{k}|}) = \vec{k} + i\vec{\alpha}$$

where $\vec{\alpha} = \alpha\frac{\vec{k}}{|\vec{k}|}$ is the imaginary part of the complex wavevector.
Then the dispersion relation is

$$\vec{\mathbf{k}}^2 = \mu\omega^2\left(\epsilon + i\frac{g}{\omega}\right)$$

where

$$|\vec{k}|^2\left(1 + i\frac{\alpha}{|\vec{k}|}\right)^2 = |\vec{k}|^2\left(1 + 2i\frac{\alpha}{|\vec{k}|} - \frac{\alpha^2}{|\vec{k}|^2}\right)$$

Denoting $|\vec{k}| = k$, we then obtain

$$k^2 + 2i\alpha k - \alpha^2 = \mu\omega^2\left(\epsilon + i\frac{g}{\omega}\right)$$

In order to satisfy equation in both real and imaginary part, we require

$$k^2 - \alpha^2 = \mu\epsilon\omega^2$$

and

$$2k\alpha = \mu\omega g$$

We can solve these equations for k and α. Starting from the imaginary part equation

$$\alpha = \frac{\mu\omega g}{2k}$$

So

$$k^2 - \frac{\mu^2\omega^2 g^2}{4k^2} = \mu\epsilon\omega^2$$

$$k^4 - \mu\epsilon\omega^2 k^2 - \frac{\mu^2\omega^2 g^2}{4} = 0$$

So

$$k^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{\mu^2\epsilon^2\omega^4 + \mu^2\omega^2 g^2}}{2}$$

Since k is real, we need to ensure that k^2 is positive for all values of ω, so we choose only the positive root, so

$$k^2 = \frac{\mu\epsilon\omega^2}{2}\left(1 + \sqrt{1 + \frac{g^2}{\epsilon^2\omega^2}}\right) \tag{75}$$

By substitution back to the equation for α

$$\alpha = \frac{\mu\omega g}{2}\frac{1}{k} = \frac{\mu\omega g}{2}\sqrt{\frac{2}{\mu\epsilon\omega^2}\left(1+\sqrt{1+\frac{g^2}{\epsilon^2\omega^2}}\right)^{-1}}$$

So

$$\alpha^2 = \frac{\frac{\mu g^2}{2\epsilon}}{\left(1+\sqrt{1+\frac{g^2}{\epsilon^2\omega^2}}\right)} \tag{76}$$

Usually, we denote

$$\beta = 1+\sqrt{1+\frac{g^2}{\epsilon^2\omega^2}}$$

So then

$$k^2 = \frac{\mu\epsilon\omega^2}{2}\beta$$

$$\alpha^2 = \frac{\mu g^2}{2\epsilon\beta}$$

Sometimes, we also use the complex index of refraction $\mathbf{n}$. The classical index of refraction is defined as

$$n = \frac{c}{v} = \frac{c}{\frac{\omega}{k}} = \frac{c}{\omega}k$$

where v is the phase speed in the medium. So, here it is defined by extension as

$$\mathbf{n} = \frac{c}{\omega}(k+i\alpha)$$

So that we have

$$\mathbf{n}^2 = \frac{c^2}{\omega^2}\vec{\mathbf{k}}^2 = \mu c^2(\epsilon + i\frac{g}{\omega}) \tag{77}$$

We can also use other classical analogue, and that is the relation for phase speed of the waves

$$v = \frac{1}{\sqrt{\epsilon\mu}}$$

To derive

$$n^2 = c^2\epsilon\mu$$

which can be used to introduce complex permitivity $\boldsymbol{\epsilon}$ as

$$\mathbf{n}^2 = c^2\mu\boldsymbol{\epsilon} \tag{78}$$

and thus, combining with the previous equation

$$\boldsymbol{\epsilon} = \epsilon + i\frac{g}{\omega} \tag{79}$$

This will be further explored later.
Because the wavenumber is complex, the plane wave becomes

$$\vec{E} = \vec{E}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{r}-\omega t)} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)+i^2\vec{\alpha}\cdot\vec{r}} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}e^{-\vec{\alpha}\cdot\vec{r}}$$

Hence, the wave behaves exactly like normal plane wave, only gets damped by an exponential factor as it travels inside the medium. Therefore, the phase speed of the wave is classicaly

$$v = \frac{\omega}{k}$$

The classical index of refraction of the wave is

$$n = \frac{c}{v} = \frac{c}{\omega}k = Re(\mathbf{n})$$

And, the phase speed can be also expressed as

$$v^2 = \frac{1}{\mu\epsilon} = \frac{1}{\mu Re(\boldsymbol{\epsilon})}$$

So, the real part of these complex variables relates to the plane wave nature of the waves, while the imaginary part relates to the damping part of the waves.

3.4.1 Material Dependent Complex Variables

Again, we have seen that most of the variables are material dependent, as all can be derived from $\vec{\mathbf{k}}$, which depends on β parameter, which depends on conductivity of the material. Hence, it is useful to again consider some extreme cases - dielectrics with $g = 0$ and perfect conductors with $g \to \infty$. Lets start with the dielectric case. The beta parameter becomes

$$\beta = 1 + \sqrt{1 + \frac{g^2}{\epsilon^2\omega^2}} = 1 + \sqrt{1} = 2$$

The real part of the wavevector becomes

$$k^2 = \frac{\mu\epsilon\omega^2}{2}\beta = \mu\epsilon\omega^2$$

which is the classical vacuum like dispersion relation
The complex part of the wavevector becomes

$$\alpha^2 = \frac{\mu g^2}{2\epsilon\beta} = \frac{\mu g^2}{4\epsilon} = 0$$

So, we the wavevector, is purely real, $\vec{\mathbf{k}} = \vec{k}$. Similarly the index of refraction

$$\mathbf{n} = \frac{c}{\omega}k = n$$

and permitivity is

$$\boldsymbol{\epsilon} = \epsilon + i\frac{g}{\omega} = \epsilon$$

Hence, we return to standard planewaves.
We expect very different behaviour for perfect conductors. For these, the β parameter becomes

$$\beta = 1 + \sqrt{1 + \frac{g^2}{\epsilon^2\omega^2}} \approx 1 + \sqrt{\frac{g^2}{\epsilon^2\omega^2}} = 1 + \frac{g}{\epsilon\omega} \approx \frac{g}{\epsilon\omega}$$

So, the real part of the wavevector is

$$k^2 = \frac{\mu\epsilon\omega^2}{2}\frac{g}{\epsilon\omega} = \frac{\mu\omega g}{2}$$

The imaginary part is

$$\alpha^2 = \frac{\mu g^2}{2\epsilon}\frac{\epsilon\omega}{g} = \frac{\mu g\omega}{2} = k^2$$

Hence, for perfect conductors

$$\vec{\mathbf{k}} = \vec{k} + ik\frac{\vec{k}}{|\vec{k}|} = \vec{k}(1+i)$$

The complex refractive index becomes

$$\mathbf{n} = \frac{c}{\omega}\mathbf{k} = \frac{c}{\omega}(1+i)k = (1+i)c\sqrt{\frac{\mu g}{2\omega}}$$

and the real and imaginary parts are also equal (we need to use first order definition, as higher order mixes the imaginary numbers, which breaks the model). For a good conductor, g is usually big, which means that $\mathbf{n}$ is also going to be very big.
The case is similar for the complex permitivity.

3.5 Drude Lorentz Model for Electric Susceptibility

In Drude-Lorentz model, we imagine that the electrons in the matter behave similarly as harmonic oscillators. We let the electrons to be coupled to the heavy nucleus by a spring with spring constant k and allow for some damping term to be present, so that the equation of motion of the electron is

$$m\frac{d^2\vec{r}}{dt^2} + m\gamma\frac{d\vec{r}}{dt} + k\vec{r} = q\vec{E}$$

where q is the charge of the electron (or other charge carrier, for which the model is applicable), given that the electron is much lighter than the nucleus, so we can neglect the motion of the nucleus induced by the field $\vec{E}$.
For oscillating field $\vec{E} = \vec{E}_0 e^{-i\omega t}$, the steady state will be reached when $\vec{r} = \vec{r}_0 e^{-i\omega t}$, where direction of $\vec{r}_0$ does not need to be parallel to $\vec{E}$. From the equation of motion we recover, using (41)

$$m(-\omega^2)\vec{r} - i\omega m\gamma\vec{r} + k\vec{r} = q\vec{E}$$

$$(-\omega^2 - i\omega\gamma + \frac{k}{m})\vec{r} = \frac{q}{m}\vec{E}$$

Since for classical simple harmonic oscillator, the natural frequency is $\omega_0 = \sqrt{\frac{k}{m}}$, we then can write

$$(\omega_0^2 - \omega^2 - i\omega\gamma)\vec{r} = \frac{q}{m}\vec{E}$$

But, as the electron gets further away from its nucleus, it creates a dipole moment

$$\vec{p} = q\vec{r}$$

In a volume dV, the total electric dipole moment is

$$d\vec{p} = ndVq\vec{r}$$

where n is the density of oscillating electrons in the matter. So, the polarization vector is

$$\vec{P} = \frac{d\vec{p}}{dV} = nq\vec{r}$$

Substituting from the motion equation of the electron

$$\vec{P} = nq\frac{q}{m}\vec{E}\frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)}$$

We can see that for a fixed ω, this is predicts a linear behaviour with $\vec{E}$, so we can determine the electric susceptibility as

$$\vec{P} = \epsilon_0\chi\vec{E}$$

So

$$\epsilon_0\chi\vec{E} = \frac{nq^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}$$

We therefore see that the susceptibility will be complex, so we will denote it now as $\boldsymbol{\chi}$ to highlight this fact. By removing the complex part from the denominator, we conclude that

$$\boldsymbol{\chi} = \frac{nq^2}{m\epsilon_0}\left(\frac{\omega_0^2 + \omega^2}{(\omega_0 - \omega^2)^2 + \omega^2\gamma^2} + i\frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right) \tag{80}$$

So, the permitivity is

$$\boldsymbol{\epsilon} = \epsilon_0(1 + \boldsymbol{\chi}) = \epsilon_0 + \frac{nq^2}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} + i\frac{nq^2}{m}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{81}$$

From our previous formalism, we had

$$\boldsymbol{\epsilon} = \epsilon + i\frac{g}{\omega}$$

and thus also

$$\vec{\mathbf{k}}^2 = \mu\omega^2\boldsymbol{\epsilon}$$

Therefore

$$\vec{\mathbf{k}}^2 = \mu\omega^2\left(\epsilon_0 + \frac{nq^2}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} + i\frac{nq^2}{m}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right)$$

3.5.1 Extreme Situations in Drude-Lorentz Model

Again, we will try to apply the model to some extreme conditions. One of these is called the thin plasma approximation, for which we assume that the electrons are essentially free, so we assume that $k \to 0$ and thus $\omega_0 \to 0$. Then

$$\boldsymbol{\epsilon} \to \epsilon_0 + \frac{nq^2}{m}\frac{-\omega^2}{\omega^4+\gamma^2\omega^2} + i\frac{nq^2}{m}\frac{\gamma\omega}{\omega^4+\gamma^2\omega^2}$$

We can also assume that in plasma, the electrons move without friction, so $\gamma \to 0$, and thus we are left with

$$\boldsymbol{\epsilon} \to \epsilon_0 - \frac{nq^2}{m}\frac{1}{\omega^2} = \epsilon_0\left(1 - \frac{nq^2}{m\epsilon_0\omega^2}\right) = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

where $\omega_p^2 = \frac{nq^2}{m\epsilon_0}$ is called the plasma frequency. Hence, we have that the permitivity is purely real and

$$\boldsymbol{\epsilon} = \epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{82}$$

This means that the wavevector is

$$\vec{\mathbf{k}}^2 = \mu\omega^2\boldsymbol{\epsilon} = \mu\epsilon(\omega^2 - \omega_p^2)$$

Hence, if $\omega < \omega_p$, there is an imaginary part of $\vec{\mathbf{k}}$ and the waves get strongly absorbed in the plasma. This can be also observed as strong reflection, but no transmittion of the waves on the plasma interface, which is sometimes used in radio signals bouncing from ionosphere.
Other limit we could take would be for $\omega << \omega_0$ in the original Drude-Lorentz expression and assume $\gamma << \omega_0$. If we take the first order approximation, we will only disregard terms of order $\frac{\omega^4}{\omega_0^4}$. Then, the permitivity becomes

$$\boldsymbol{\epsilon} = \epsilon_0 + \frac{nq^2}{m}\frac{\omega_0^2-\omega^2}{\omega_0^4\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{\omega_0^4}\right]} + i\frac{nq^2}{m}\frac{\gamma\omega}{\omega_0^4\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{\omega_0^4}\right]}$$

We can see that the imaginary part becomes very small. Also, the term $\frac{\gamma^2\omega^2}{\omega_0^4}$ is very small, so the permitivity becomes

$$\boldsymbol{\epsilon} \approx \epsilon_0 + \frac{nq^2}{m}\frac{\omega_0^2-\omega^2}{\omega_0^4\left(1-\frac{\omega^2}{\omega_0^2}\right)^2} = \epsilon_0 + \frac{nq^2}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2} = \epsilon_0 + \frac{nq^2}{m}\frac{1}{\omega_0^2-\omega^2}$$

Hence, we have only a real part again, so the index of refraction is also real only (in our approximation, $\omega << \omega_0$, so $\boldsymbol{\epsilon} > 0$). The index of refraction is

$$\mathbf{n}^2 = \mu c^2\boldsymbol{\epsilon} = \epsilon_0\mu c^2\left(1 + \frac{nq^2}{m}\frac{1}{\omega_0^2-\omega^2}\right)$$

We then see that the index of refraction increases as ω increases, which is also what we observe for many materials in nature (blue light is refracted more than red light, because it has higher frequency).
A specific approximation is the conductor approximation, where I will assume that the charges are light and almost not bound, but strongly scattered, so that effectively $m \to 0$, $k \to 0$, but ω_0 remains finite, and $\gamma \to \infty$, but $m\gamma$ remains finite. Then, the permitivity becomes

$$\boldsymbol{\epsilon} = \epsilon_0 + nq^2\frac{\omega_0^2-\omega^2}{m(\omega_0^2-\omega^2)^2 + m\gamma^2\omega^2} + inq^2\frac{\gamma\omega}{m(\omega_0^2-\omega^2)^2+m\gamma^2\omega^2} \approx$$

$$\approx \epsilon_0 + nq^2\frac{\omega_0^2-\omega^2}{(m\gamma\omega^2)\gamma} + inq^2\frac{\gamma\omega}{m\gamma^2\omega^2} \approx \epsilon_0 + i\frac{nq^2}{m\gamma}$$

Comparing this to the relation for complex permitivity

$$\boldsymbol{\epsilon} = \epsilon + i\frac{g}{\omega}$$

we find that in this limit, $\epsilon \to \epsilon_0$ and

$$g \to \frac{nq^2}{m\gamma}$$

which is a very similar expression to that in Drude conductivity model. We can therefore see to some extent what the parameter γ represents - a frequency of interactions with other parts of the matter that slow the electron down, as $\frac{1}{\gamma}$ corresponds to τ in Drude conductivity model.

4 Topics in Optics

We will now discover a few topics in optics of perfect neutral dielectrics. For these, the boundary conditions are

$$\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n}$$

where $\hat{n}$ is the unit vector normal to the boundary

$$\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$$

$$\vec{E}_1 \cdot \hat{l} = \vec{E}_2 \cdot \hat{l}$$

where $\hat{l}$ is any unit vector parallel to the surface, and

$$\vec{H}_1 \cdot \hat{l} = \vec{H}_2 \cdot \hat{l}$$

Since the medium is non-conductive and no free surface charge density can be therefore present. Hence, $\vec{D} \cdot \hat{n}$, $\vec{E} \cdot \hat{l}$, $\vec{B} \cdot \hat{n}$ and $\vec{H} \cdot \hat{l}$ are all continuous.
Furthermore, we will only limit our discusion to oscillating fields, for which we define

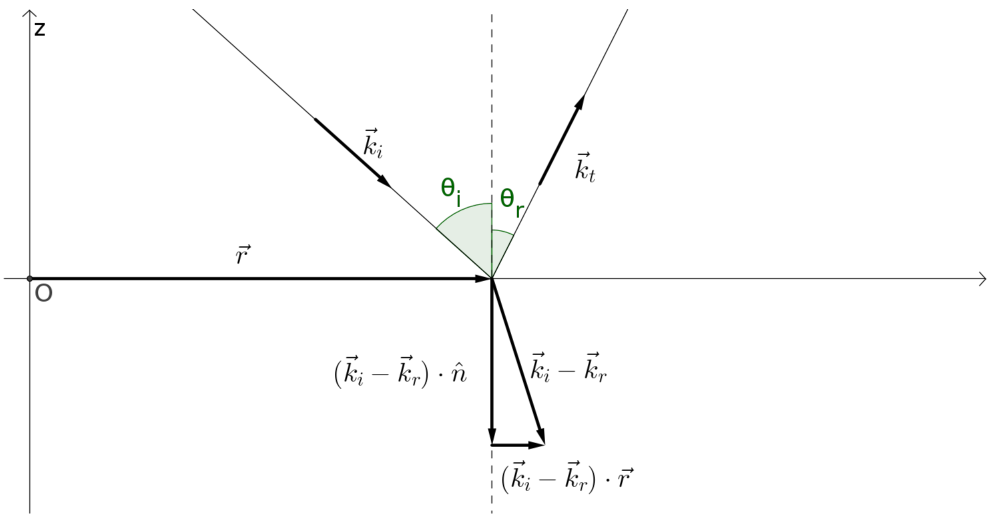
$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{H} = \vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

where $\vec{k} \perp \vec{H} \perp \vec{E}$, which can be shown from Maxwell equations.

4.1 Wavenumber Treatment of Boundaries

We will start by simply applying these boundary conditions without considering the orientation of the electric field/magnetic field intensity. When a wave is incident on a boundary, it can be generally transmitted and reflected. We will first discover the reflection case. The geometry is set so that z axis is pointing into the medium from which the wave is incident and the origin lies within the boundary. This is expressed in figure below



Here, the reflected wave travels with $\vec{k}_r$ and the incident with $\vec{k}_i$. There is still some transmitte wave with $\vec{k}_t$ in the other medium, only it is not drawn in the figure. Because the parallel component of $\vec{E}$ is continuous, we must have

$$(\vec{E}_i + \vec{E}_r) \cdot \hat{l} = \vec{E}_t \cdot \hat{l}$$

$$(\vec{E}_{i0} e^{i(\vec{k}_i\cdot\vec{r}-\omega t)} + \vec{E}_{r0} e^{i(\vec{k}_r\cdot\vec{r}-\omega t)}) \cdot \hat{l} = \vec{E}_{t0} e^{i(\vec{k}_t\cdot\vec{r}-\omega t)} \cdot \hat{l}$$

But, this must be true for all points along the boundary, which can be only satisfied if the arguments of the exponentials are equal, so

$$\vec{k}_i \cdot \vec{r} - \omega t = \vec{k}_r \cdot \vec{r} - \omega t = \vec{k}_t \cdot \vec{r} - \omega t \tag{83}$$

Hence, for the reflection case

$$(\vec{k}_i - \vec{k}_r) \cdot \vec{r} = 0$$

Comparing this to the image, we can see that the difference of the vectors is purely normal to the surface, which means that both $\vec{k}_i$ and $\vec{k}_r$ lie in the same plane perpendicular to the surface. This plane is called the plane of incidence, and enables us to the 2D geometry in otherwise 3D problem.
Since $(\vec{k}_i - \vec{k}_r)$ is normal to the surface, the vector product with $\hat{n}$ will be zero, i.e.

$$(\vec{k}_i - \vec{k}_r) \times \hat{n} = 0$$

This means that

$$|\vec{k}_i \times \hat{n}| = |\vec{k}_r \times \hat{n}|$$

and since $\hat{n}$ is a unit vector

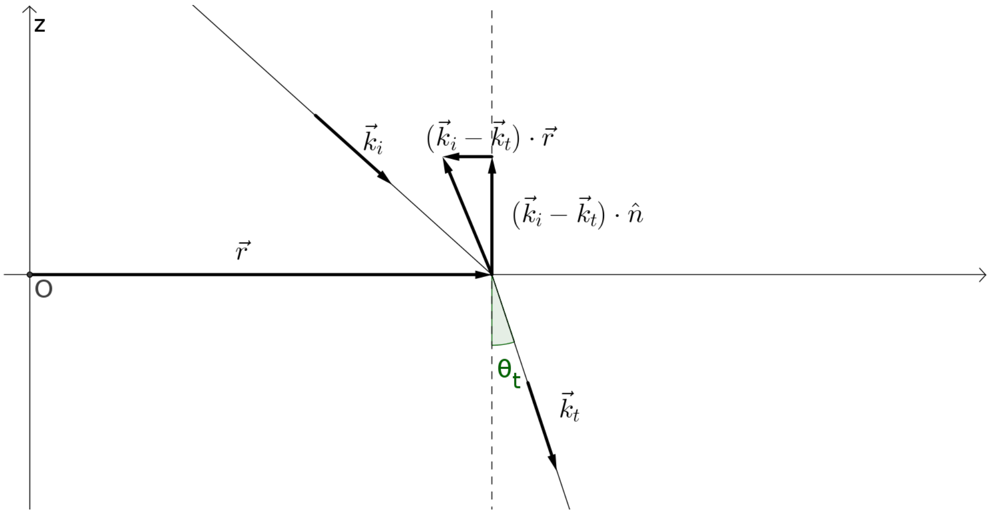
$$|\vec{k}_i| \sin\theta_i = |\vec{k}_r| \sin\theta_r$$

Since the incident and reflected waves are in the same medium, we have $|\vec{k}_i| = |\vec{k}_r|$, and thus

$$\sin\theta_i = \sin\theta_r$$

$$\theta_i = \theta_r \tag{84}$$

Similar line of reasoning can be used for the transmitted wave.



Again, from (83), we require

$$(\vec{k}_i - \vec{k}_t) \cdot \vec{r} = 0$$

So $\vec{k}_t$ also lies in the plane of incidence. Again, using the vector product

$$|\vec{k}_i| \sin\theta_i = |\vec{k}_t| \sin\theta_t$$

But, now the transmitted and reflected waves are in different media, so $\theta_t \neq \theta_r$. Using the definition of real index of refraction

$$n = \frac{k}{\omega} c$$

We then have, since frequency must be the same in both media

$$\frac{|\vec{k}_i|}{\omega} c \sin\theta_i = \frac{|\vec{k}_t|}{\omega} c \sin\theta_t$$

And thus

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{85}$$

These relations were determined just based on the wavevector properties, but we now have no information about the orientation of the $\vec{E}/\vec{H}$ fields in the transmitted and reflected waves. We also have now information about the relative magnitudes of these waves. To find this, we need to derive more general equations, called the Fresnel equations.

4.2 Fresnel Equations

For Fresnel equations, we will consider a linearly polarised EM waves, so that $\vec{E}_0$ is constant in time. Then, any wave incidence can be modelled as superposition of two wave polarisations - the normal polarisation and parallel polarization.

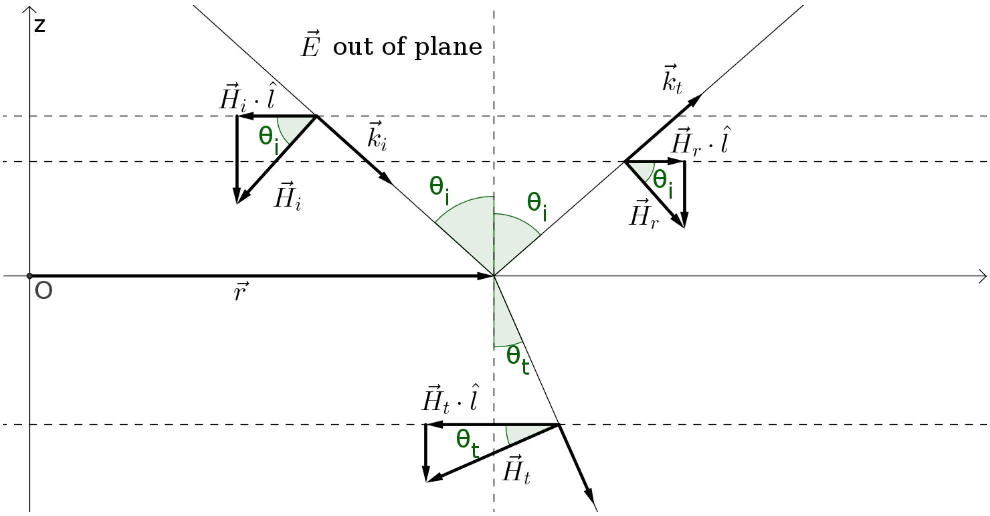
4.2.1 Normal polarization

For normal polarization, $\vec{E}_0$ is perpendicular to the plane of incidence and thus parallel to the surface of the boundary. Therefore, by boundary conditions

$$(\vec{E}_{i0} + \vec{E}_{r0}) \cdot \hat{l} = \vec{E}_{t0} \cdot \hat{l}$$

$$|\vec{E}_{i0}| + |\vec{E}_{r0}| = |\vec{E}_{t0}|$$

However, the $\vec{H}$ field is not entirely parallel to the surface - the situation is as presented in the figure below.



In the figure, I already used the general result that $\theta_r = \theta_i$. Snell's law will be used as well, only later. But, the parallel component of $\vec{H}$ has to be continuous for insulating media, so we must have

$$\cos\theta_i |\vec{H}_{i0}| - \cos\theta_i |\vec{H}_{r0}| = \cos\theta_t |\vec{H}_{t0}|$$

From the third Maxwell equation, we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

For wavelike $\vec{E}$, using (39), and using $\vec{B} = \mu \vec{H}$

$$i\vec{k} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

For wavelike $\vec{H}$, using (41)

$$i\vec{k} \times \vec{E} = i\mu\omega\vec{H}$$

$$\vec{k} \times \vec{E} = \mu\omega\vec{H}$$

But, we know that $\vec{k} \perp \vec{E}$, so

$$\vec{k} \times \vec{E} = |\vec{k}||\vec{E}|\hat{H} = \mu\omega\vec{H}$$

where $\hat{H}$ is the unit vector in direction of $\vec{H}$. Therefore, in any insulating neutral media with wavelike $\vec{E}$ and $\vec{H}$, we have

$$|\vec{k}||\vec{E}| = \mu\omega|\vec{H}| \tag{86}$$

Hence

$$|\vec{H}| = \frac{|\vec{k}|}{\mu\omega}|\vec{E}|$$

and thus

$$c|\vec{H}| = \frac{c}{\frac{\omega}{|\vec{k}|}\mu}|\vec{E}| = \frac{n}{\mu}|\vec{E}|$$

where n is the index of refraction. In our case, if we multiply the equation for parallel components of $\vec{H}$ by c, we obtain

$$\cos\theta_i(c|\vec{H}_{i0}| - c|\vec{H}_{r0}|) = \cos\theta_t c|\vec{H}_{t0}|$$

So, using our new identity, and since the reflected wave is in the same material as the incident one

$$\cos\theta_i(\frac{n_i}{\mu_i}|\vec{E}_{i0}| - \frac{n_i}{\mu_i}|\vec{E}_{r0}|) = \cos\theta_t\frac{n_t}{\mu_t}|\vec{E}_{t0}|$$

We can now substitute for $|\vec{E}_{t0}|$ from the equation for the parallel component of $\vec{E}$ derived above, so

$$\cos\theta_i\frac{n_i}{\mu_i}(|\vec{E}_{i0}| - |\vec{E}_{r0}|) = \cos\theta_t\frac{n_t}{\mu_t}(|\vec{E}_{i0}| + |\vec{E}_{r0}|)$$

Now, we can introduce the reflection coefficient r, which predicts the amplitude of the reflected wave via

$$|\vec{E}_{r0}| = r|\vec{E}_{i0}|$$

Using this definition, we have

$$\cos\theta_i\frac{n_i}{\mu_i}|\vec{E}_{i0}|(1-r) = \cos\theta_t\frac{n_t}{\mu_t}|\vec{E}_{i0}|(1+r)$$

Dividing through by $|\vec{E}_{i0}|$ and solving for r

$$\cos\theta_i\frac{n_i}{\mu_i}(1-r) = \cos\theta_t\frac{n_t}{\mu_t}(1+r)$$

$$\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t = r\left(\frac{n_t}{\mu_t}\cos\theta_t + \frac{n_i}{\mu_i}\cos\theta_i\right)$$

Hence

$$r_n = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \tag{87}$$

where the subscript n indicates that this is the normal incidence regime.
Similarly as we defined r, we can define transmission coefficient t as

$$|\vec{E}_{t0}| = t|\vec{E}_{i0}|$$

So, the parallel $\vec{E}$ component equation becomes

$$|\vec{E}_{i0}| + r|\vec{E}_{i0}| = t|\vec{E}_{i0}|$$

So

$$t = 1 + r$$

And so

$$t_n = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \tag{88}$$

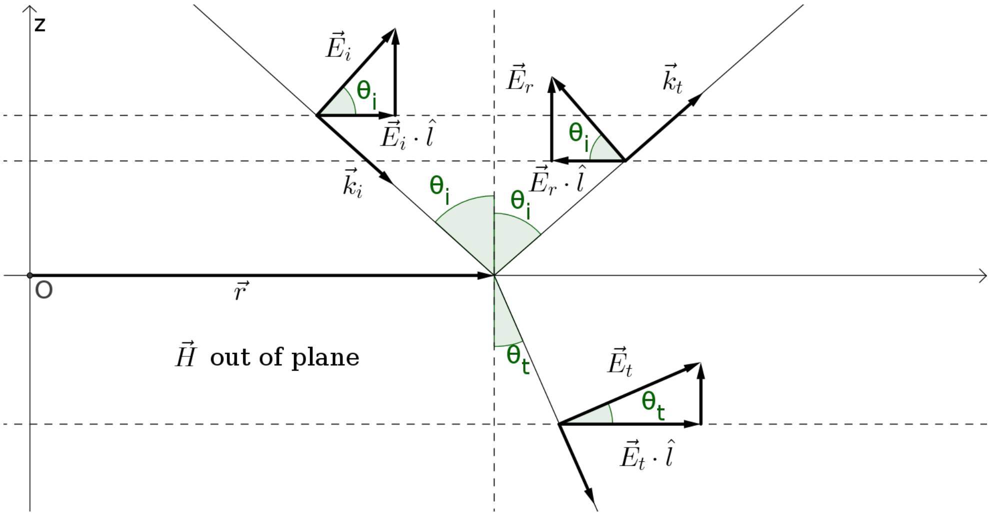
4.2.2 Parallel Polarization

In the case of parallel polarization, we can use similar reasoning, but the roles of $\vec{E}$ and $\vec{H}$ are now switched, as should be clear from the figure below.
Since we have $\vec{H}$ fully parallel to the surface, we must have

$$|\vec{H}_{i0}| + |\vec{H}_{r0}| = |\vec{H}_{t0}|$$

Multiplying by c, we obtain

$$c|\vec{H}_{i0}| + c|\vec{H}_{r0}| = \frac{n_i}{\mu_i}(|\vec{E}_{i0}| + |\vec{E}_{r0}|) = \frac{n_t}{\mu_t}|\vec{E}_{t0}|$$



Using the definition of r the reflection coefficient and t the transmission coefficient and dividing through by $|\vec{E}_{i0}|$

$$\frac{n_i}{\mu_i}(1+r) = \frac{n_t}{\mu_t}t$$

The second equation follows from the continuity of parallel component of $\vec{E}$, for which we require

$$\cos\theta_i|\vec{E}_{i0}| - \cos\theta_i|\vec{E}_{r0}| = \cos\theta_t|\vec{E}_{t0}|$$

Again, using t and r

$$\cos\theta_i(1-r) = \cos\theta_t t$$

Hence

$$\frac{n_i}{\mu_i}(1+r) = \frac{n_t}{\mu_t}\frac{\cos\theta_i}{\cos\theta_t}(1-r)$$

So

$$r\left(\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i\right) = \frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t$$

So

$$r_p = \frac{\frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t}{\frac{n_t}{\mu_t}\cos\theta_i + \frac{n_i}{\mu_i}\cos\theta_t} \tag{89}$$

We can use this to find t

$$t = \frac{n_i}{\mu_i}\frac{\mu_t}{n_t}(1+r)$$

So

$$t_p = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_t}{\mu_t}\cos\theta_i + \frac{n_i}{\mu_i}\cos\theta_t} \tag{90}$$

Therefore, we have two pairs of Fresnel equations, for parallel and normal polarization. If we than have a general linear polarization, we can split it into parallel and normal component, and use the Fresnel equations to find their reflected/transmitted amplitudes, and then sum the components back together to find the whole reflected/transmitted wave.

4.2.3 Combination with Snell's Law

To combine the Fresnel equations with Snell's law, we will assume that $\mu_i \approx \mu_t \approx \mu_0$, which is usually a quite good approximation (we strictly do not need to do that, but it makes the equations a lot easier without a significant loss of precision). Then, the Fresnel equation for normal reflection becomes

$$r_n = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

From the Snell's law, we know that

$$n_i\sin\theta_i = n_t\sin\theta_t$$

So, we can rewrite our equation as

$$r_n = \frac{\cos\theta_i - \frac{n_t}{n_i}\cos\theta_t}{\cos\theta_i + \frac{n_t}{n_i}\cos\theta_t}$$

Using Snell's law

$$r_n = \frac{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{\sin\theta_t\cos\theta_i - \cos\theta_t\sin\theta_i}{\sin\theta_t\cos\theta_i + \cos\theta_t\sin\theta_i} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{91}$$

We have therefore rewritten r in terms of just the transmission and incidence angles, which can be sometimes more useful form. However, probably most useful form is the predictive form, in which we only use the incidence angle and indices of refraction and we do not reference the transmission angle. To derive this form, start from the original expression for reflection with permeabilities assumed constant, so

$$r_n = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

Now, we express $\cos\theta_t$ in form of $\sin\theta_t$. Since θ_t ranges from 0 to $\frac{\pi}{2}$, we can write

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$$

Since Snell's law must apply, we have

$$\cos\theta_t = \sqrt{1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i}$$

So, we have

$$r_n = \frac{n_i\cos\theta_i - n_t\sqrt{1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i}}{n_i\cos\theta_i + n_t\sqrt{1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i}}$$

We can now define $n = \frac{n_t}{n_i}$ as the relative index of refraction and divide both the numerator and the denominator by n_i to get

$$r_n = \frac{\cos\theta_i - n\sqrt{1 - \frac{1}{n^2}\sin^2\theta_i}}{\cos\theta_i + n\sqrt{1 - \frac{1}{n^2}\sin^2\theta_i}}$$

By moving the n inside the square root, we obtain

$$r_n = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \tag{92}$$

From the relation $t = 1 + r$, we can obtain both the angular and the predictive form of the normal transmission coefficient as

$$t_n = 1 + \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{93}$$

Or

$$t_n = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \tag{94}$$

We can do a similar analysis for parallel reflection and transmission, starting with transmission

$$r_p = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t} = \frac{\frac{\sin\theta_i}{\sin\theta_t}\cos\theta_i - \cos\theta_t}{\frac{\sin\theta_i}{\sin\theta_t}\cos\theta_i + \cos\theta_t} = \frac{\sin\theta_i\cos\theta_i - \sin\theta_t\cos\theta_t}{\sin\theta_i\cos\theta_i + \sin\theta_t\cos\theta_t}$$

But

$$\sin\theta_i\cos\theta_i - \sin\theta_t\cos\theta_t = \sin\theta_i\cos\theta_i(\sin^2\theta_t + \cos^2\theta_t) - (\sin^2\theta_i + \cos^2\theta_i)\sin\theta_t\cos\theta_t =$$

$$= (\sin\theta_i\sin\theta_t)(\cos\theta_i\sin\theta_t) + (\sin\theta_i\cos\theta_t)(\cos\theta_i\cos\theta_t) - (\sin\theta_i\cos\theta_t)(\sin\theta_i\sin\theta_t) -$$

$$-(\cos\theta_i\sin\theta_t)(\cos\theta_i\cos\theta_t) = (\sin\theta_i\cos\theta_t - \cos\theta_i\sin\theta_t)(\cos\theta_i\cos\theta_t - \sin\theta_i\sin\theta_t) =$$

$$= \sin(\theta_i - \theta_t)\cos(\theta_i + \theta_t)$$

Similarly

$$\sin\theta_i\cos\theta_i+\sin\theta_t\cos\theta_t=\sin\theta_i\cos\theta_i(\sin^2\theta_t+\cos^2\theta_t)+(\sin^2\theta_i+\cos^2\theta_i)\sin\theta_t\cos\theta_t=$$

$$=(\sin\theta_i\sin\theta_t)(\cos\theta_i\sin\theta_t)+(\sin\theta_i\cos\theta_t)(\cos\theta_i\cos\theta_t)+(\sin\theta_i\cos\theta_t)(\sin\theta_i\sin\theta_t)+$$
$$+(\cos\theta_i\sin\theta_t)(\cos\theta_i\cos\theta_t)=(\sin\theta_i\cos\theta_t+\cos\theta_i\sin\theta_t)(\cos\theta_i\cos\theta_t+\sin\theta_i\sin\theta_t)=$$
$$=\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)$$

Hence

$$r_p=\frac{\sin(\theta_i-\theta_t)\cos(\theta_i+\theta_t)}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}=\frac{\sin(\theta_i-\theta_t)}{\cos(\theta_i-\theta_t)}\frac{\cos(\theta_i+\theta_t)}{\sin(\theta_i+\theta_t)}=\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}$$

So

$$r_p=\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_t+\theta_i)} \tag{95}$$

and

$$t_p=1+\frac{\tan(\theta_i-\theta_p)}{\tan(\theta_i+\theta_p)} \tag{96}$$

To find the predictive form, we start again from

$$r_p=\frac{n_t\cos\theta_i-n_i\cos\theta_t}{n_t\cos\theta_i+n_i\cos\theta_t}=\frac{n\cos\theta_i-\sqrt{1-\frac{1}{n^2}\sin^2\theta_i}}{n\cos\theta_i+\sqrt{1-\frac{1}{n^2}\sin^2\theta_i}}=\frac{n^2\cos\theta_i-\sqrt{n^2-\sin^2\theta_i}}{n^2\cos\theta_i+\sqrt{n^2-\sin^2\theta_i}}$$

So

$$r_p=\frac{n^2\cos\theta_i-\sqrt{n^2-\sin^2\theta_i}}{n^2\cos\theta_i+\sqrt{n^2-\sin^2\theta_i}} \tag{97}$$

and

$$t_p=\frac{2n^2\cos\theta_i}{n^2\cos\theta_i+\sqrt{n^2-\sin^2\theta_i}} \tag{98}$$

4.2.4 Special Cases

Two special situations can occur - first is that for $n>1$, it can happen that $n^2\cos\theta_i-\sqrt{n^2-\sin^2\theta_i}=0$. We can check that this happens at $n=\tan\theta_B$ where θ_B is a specific incidence angle

$$\frac{\sin^2\theta_B}{\cos^2\theta_B}\cos\theta_B-\sqrt{\frac{\sin^2\theta_B}{\cos^2\theta_B}-\sin^2\theta_B}=\frac{\sin^2\theta_B}{\cos\theta_B}-\sin\theta_B\sqrt{\frac{1-\cos^2\theta_B}{\cos^2\theta_B}}=\frac{\sin^2\theta_B}{\cos\theta_B}-\frac{\sin^2\theta_B}{\cos\theta_B}=0$$

In this case, we can see that $r_p=0$ and the reflected wave is fully polarized normal to the plane of incidence. This angle is called the Brewster angle.
Second situation occurs when $n<1$. Then, beyond certain θ_C, $\sqrt{n^2-\sin^2\theta_i}$ is no longer defined (square root of negative number). Here

$$\sin\theta_C=n$$

and at this critical angle, the light is fully reflected back into the material, and the situation is called the total internal reflection.
In case we are dealing with metals, the indices of refraction become complex, but the relations, particularly for the normal incidence, remains the same for the reflection and transmission coefficients.

4.2.5 Normal Incidence

For case of normal incidence $\theta_i=0$, we have

$$r_n=\frac{n_i-n_t}{n_i+n_t}=\frac{1-n}{1+n}$$
$$t_n=\frac{2}{1+n}$$
$$r_p=\frac{n_t-n_i}{n_t+n_i}=\frac{n-1}{n+1}$$
$$t_p=\frac{2n}{n+1}$$

4.2.6 Plasma External Reflection

Consider the thin plasma model, which predicts that

$$\boldsymbol{\epsilon} = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

This means that

$$n^2 = c^2 \mu \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \approx 1 - \frac{\omega_p^2}{\omega^2}$$

where I used $\mu \approx \mu_0$. This means that the index of refraction for frequencies higher than ω_p is always smaller than 1, so the internal refraction can occur in plasma for any material, including the vacuum. This peculiarity is called the total external reflection.